Variance Risk Dynamics, Variance Risk Premia, and Optimal Variance Swap Investments*

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ABSTRACT

With increasing appreciation of the fact that stock return variance is stochastic and variance risk is heavily priced, the industry has created a series of variance derivative products to span variance risk. The variance swap contract is the most actively traded of these products. It pays at expiry the difference between the realized return variance and a fixed rate, called the variance swap rate, determined at the inception of the contract. We obtain a decade worth of variance swap rate quotes at five maturities. With the data, we first exploit the information in both the time series and the term structure of the variance swap rates to analyze the return variance rate dynamics and market pricing of variance risk. We then study both theoretically and empirically how investors can use variance swap contracts across different maturities to span the variance risk and to revise their dynamic asset allocation decisions. We find that with the swap contract to span the variance risk, an investor increases her investment in the underlying stock. In addition, the investor’s indirect utility increases significantly when allowed to span the variance risk using variance swap contracts. Finally, an out-of-sample study confirms that the gains from including variance swaps into the portfolio mix are large.
The financial market is becoming increasingly aware of the fact that return variance on stock indexes is stochastic and the variance risk is heavily priced.\(^1\) Associated with this recognition is the development of a large number of variance-related derivative products. The most actively traded among them is the variance swap contract. The contract has zero value at inception. At maturity, the long side of the variance swap contract receives the difference between a standard measure of the realized variance and a fixed rate, called the variance swap rate determined at the inception of the contract. Although traditional derivative contracts such as calls, puts, and straddles have variance risk exposure, entering a variance swap contract represents the most direct way of achieving exposure to or hedging against variance risk.

Variance swap contracts on major equity indexes are actively traded over the counter. Accordingly, variance swap rate quotes on such indexes are now readily available from several broker dealers. In this paper, we obtain a decade worth of variance swap rate quotes from a major investment bank on the S&P 500 index at five fixed maturities from two months to two years. With the data, we first propose a class of models on variance risk dynamics and then estimate the variance dynamics and the market pricing of different sources of variance risk by exploiting the rich information embedded in both the time series and the term structure of the variance swap rate quotes. Furthermore, based on the estimated dynamics and market pricing, we also study both theoretically and empirically how investors can use variance swap contracts across different maturities to span the variance risk and to revise their dynamic asset allocation decisions.

Despite the well-appreciated importance of understanding variance risk dynamics and variance risk premia, it remains an unsettled issue on how to model and estimate variance risk dynamics and variance risk premia mainly because return variance is not directly observable. Previous literature mostly relies on the information in time series returns and option prices on the underlying security to make the inference. Yet, such inference is almost always joint inference with the underlying return dynamics. Misspecification on one leads to erroneous conclusions on the other. With variance swap rate quotes, we show that we can directly study and estimate the variance dynamics and variance risk premia without specifying the underlying return dynamics, and hence without interference from the potential misspecifications on the return dynamics.

Our model design and estimation show that we need two stochastic variance risk factors to explain the variation in the variance swaps at different maturities, with one factor controlling the instantaneous variance rate while the other controlling the central tendency of the variance rate movements. The variance rate

\(^{1}\text{See Engle (2004) for a recent review of variance risk and modeling, and Bakshi and Kapadia (2003), Carr and Wu (2004), and Bondarenko (2004) for evidence on the market price of variance risk.}\)
factor is much more transient than the central tendency factor under both the risk-neutral and the statistical measures, thus generating different loading patterns across the term structure from the two risk factors and different autocorrelation patterns for variance swap rates of different maturities. We also find that the market prices for both variance risk factors must be negative to match the upward-sloping mean term structure observed from the variance swap data.

With the estimated dynamics and market pricing, we consider a dynamic asset allocation problem, where an investor equipped with a CRRA utility function trades in the S&P 500 stock index, a riskless bond, and a series of variance swap contracts to maximize her utility on the terminal wealth. Compared to option contracts, variance swap contracts provide a more direct approach in spanning the variance risk. Since the variance swap is a linear contract in variance risk, by trading these contracts, the investor does not build up additional delta exposures to the underlying stock, as would be the case for strategies involving vanilla options.

We first derive the allocation decision in analytical form, and then calibrate the decision to the estimated variance dynamics. We find that by making the variance swap contract available to span the variance risk, an investor increases her investment in the underlying stock. In addition, the investor’s indirect utility increases significantly when allowed to span the variance risk using variance swap contracts. The certainty-equivalent cost of not using variance swap contracts increases with investment horizon and becomes especially large when the current variance level is low. When we perform an out-of-sample study to investigate the impact of variance swap investment on the overall performance of portfolio strategies, we find that incorporating variance swap contracts significantly increases the portfolio performance. For the three-year period starting January 1, 2003 and ending December 28, 2005, depending on the investment horizon, an investor with access to variance swap markets can outperform a strategy with only stock and bonds by 118% in terms of cumulative wealth and 36% in terms of Sharpe ratio. For example, a CRRA investor with an investment horizon of one week can generate a Sharpe ratio of 1.16 on her portfolio with variance swap contracts, compared to the Sharpe ratio of 0.85 with only stocks and bonds.

Our study is related to three strands of literature. The first strand includes all traditional studies that estimate the variance dynamics joint with the return dynamics (see Engle (2004) for a review). The second strand studies the market pricing of variance risk by comparing time-series returns and/or realized variances to options or option portfolios. Prominent examples include Bakshi and Kapadia (2003), Bondarenko (2004), Carr and Wu (2004), and Coval and Shumway (2001). By using the time series of variance swap
rates across different maturities in this paper, we infer both the variance risk dynamics and the market prices of difference sources of variance risk without resorting to the specification of the return dynamics. The third strand of the literature studies the asset allocation problem in the presence of derivative securities. Carr and Madan (2001) and Carr, Jin, and Madan (2001) focus on how to use vanilla options across different strikes to span the jump risk with random jump size, in the absence of stochastic variance. Complementing to their study, we focus on how to use variance swap contracts of different maturities to span the multi-dimensional stochastic variance risk, while excluding ourselves from jump risk. Liu and Pan (2003) analyze investments in vanilla options in the presence of both jumps and stochastic variance. In this case, the allocation to a vanilla option at a given strike and maturity is a result of mixed effects from spanning the jump risk and the stochastic variance risk. To disentangle the effects, they assume a constant jump size and hence effectively exclude themselves from the strike dimension analyzed in Carr and Madan (2001) and Carr, Jin, and Madan (2001).

The remainder of the paper is organized as follows. Section 1 introduces a class of affine stochastic variance models and shows how to price variance swaps under this setting and how to identify the variance dynamics and market prices of variance risks using both the time series and the term structure of variance swap rates. Section 2 presents the estimation results for the variance risk dynamics and variance risk premia. Section 3 derives the optimal portfolio allocation policies. Section 4 calibrates the allocation decisions to the estimated dynamics and analyzes how incorporating variance swap contracts alters an investor’s strategic decision and improves her welfare. Section 5 concludes.

1. Affine Market Models of Stochastic Variance

1.1. Basic setup

Formally, let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a complete stochastic basis and let \(\mathbb{P}\) be the statistical probability measure. Let \(\mathbb{Q}\) denote a risk-neutral measure that is absolutely continuous with respect to \(\mathbb{P}\). No arbitrage guarantees that there exists at least one such measure that prices all traded securities (Duffie (1992)). Let \(V_{t,T} \equiv \int_t^T v_s ds\) denote the aggregate return variance during the period \([t, T]\) with \(\tau = T - t\) denoting the length of the horizon.
We assume that the dynamics of the instantaneous variance rate $v_t$ is controlled by a $k$-dimensional Markov process $X$, which starts at $X_0$ and satisfies the following stochastic differential equation under the risk-neutral measure $Q$:

$$dX_t = \mu(X_t)dt + \Sigma^X(X)dB^X_t + (qdN^X(\lambda(X_t)) - \bar{\lambda}\lambda(X_t)dt),$$

(1)

where $\mu(X_t) \in \mathbb{R}^k$ denotes the instantaneous drift function, $B^X$ denotes a $k$-dimensional independent Brownian motion with $\Sigma^X(X)\Sigma^X(X)^T \in \mathbb{R}^{k \times k}$ being the symmetric and positive definite instantaneous covariance matrix, and $N^X$ denotes $k$ independent Poisson jump components with intensities $\lambda(X_t) \in \mathbb{R}^k$ and with the random jump magnitudes $q$ being a diagonal $(k \times k)$ matrix, characterized by its two-sided Laplace transform $\mathcal{L}_q(\cdot)$ and with $\bar{q} = \mathbb{E}^Q[q]$. The last two terms in equation (1) form a $k$-dimensional jump martingale.

To analyze the variance risk dynamics using variance swap rates, we adopt the affine framework of Duffie, Pan, and Singleton (2000) and model the term structure of variance swaps within the affine class.

**Definition 1** In affine stochastic variance models, the Laplace transform of the quadratic variation, $V_{t,T} = \int_t^T v_s ds$, under the risk-neutral measure $Q$ is an exponential-affine function of the state vector $X_t$:

$$\mathcal{L}_V(u) \equiv \mathbb{E}^Q[e^{-uV_{t,T}} | \mathcal{F}_t] = \exp\left(-b(\tau)^\top X_t - c(\tau)\right),$$

(2)

where $b(\tau) \in \mathbb{R}^k$ and $c(\tau)$ is a scalar.

The definition implicitly limits us to time-homogeneous models since the coefficients depend only on the horizon $\tau = T - t$, but not on the calendar time $t$. The following proposition presents a set of sufficient conditions for the affine definition in equation (2) to hold.

**Proposition 1** If under the risk-neutral measure $Q$, the instantaneous variance rate $v_t$, the drift vector $\mu(X)$, the diffusion covariance matrix $\Sigma^X(X)\Sigma^X(X)^T$, and the jump arrival rate $\lambda(X)$ of the Markov process $X$ are all affine in $X$, then the Laplace transform $\mathcal{L}_V(u)$ is exponential-affine in $X_t$. 

5
The above process specifications are directly adopted from Duffie, Pan, and Singleton (2000) on general asset pricing modeling. In particular, let the $\mathbb{Q}$-dynamics be defined as:

$$v_t = b_v^\top X_t + c_v, \quad b_v \in \mathbb{R}^k, c_v \in \mathbb{R},$$

$$\mu(X_t) = \kappa (\theta - X_t), \quad \kappa \in \mathbb{R}^{k \times k}, \theta \in \mathbb{R}^k,$$

$$\Sigma^X(X)\Sigma^X(X)^\top = \text{diag}[\alpha + \beta X_t], \quad \alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^{k \times k},$$

$$\lambda(X_t) = \alpha_\lambda + \beta_\lambda X_t, \quad \alpha_\lambda \in \mathbb{R}^k, \beta_\lambda \in \mathbb{R}^{k \times k}.$$  \hspace{1cm} (3)

We further constrain $\beta$ and $\beta_\lambda$ to be diagonal matrices. Given the above specification, the coefficients $\{b(\tau), c(\tau)\}$ for the Laplace transform in (2) are determined by the following ordinary differential equations:

$$b'(\tau) = ub_v - (\kappa + \bar{q}\beta_\lambda)^\top b(\tau) - \frac{1}{2} \beta \text{diag}[b(\tau)b(\tau)^\top] - \beta_\lambda (\mathcal{L}_q(b(\tau)) - 1),$$

$$c'(\tau) = uc_v + (\kappa\theta - \bar{q}\alpha_\lambda)^\top b(\tau) - \frac{1}{2} \alpha^\top \text{diag}[b(\tau)b(\tau)^\top] - \alpha_\lambda^\top (\mathcal{L}_q(b(\tau)) - 1),$$  \hspace{1cm} (4)

with the boundary conditions $b(0) = 0$, and $c(0) = 0$.

### 1.2. Pricing variance swaps

The terminal payoff of a variance swap contract is the difference between the realized variance over a certain time period and a fixed variance swap rate, determined at the inception of the contract. The variance difference is multiplied by a notional amount that converts the difference into dollar terms. Since the contract is worth zero at inception, no-arbitrage dictates that the value of the variance swap rate is equal to the risk-neutral expected value of the realized variance over the relevant time period. Formally, the time-$t$ value of an annualized variance swap rate with time-to-maturity $\tau$ is given by,

$$\text{VS}_{t,\tau} = \frac{1}{\tau} \mathbb{E}^\mathbb{Q}_t \left[ \int_{t}^{t+\tau} v_s ds \right].$$  \hspace{1cm} (5)

\footnote{We use the convention that $\text{diag}[v]$ maps the vector $v$ onto a diagonal matrix with elements from the vector $v$. For a matrix $M$, $\text{diag}[M]$ is the vector consisting of the diagonal elements of $M$.}
In the affine stochastic variance framework, the variance swap rate can be solved from the Laplace transform in equation (2):

\[
E_t^Q [V_{T-t}] = \left. \frac{-\partial E_t^Q [e^{-aV_{T-t}}]}{\partial a} \right|_{a=0} = L_V(u) \left( \left[ \frac{\partial b(\tau)}{\partial u} \right]^\top X_t + \frac{\partial c(\tau)}{\partial u} \right) \bigg|_{u=0} = B(\tau)^\top X_0 + C(\tau),
\]

which is affine in the current level of the state vector \(X_t\). Note that \(L_V(u)|_{u=0} = 1\) and the coefficients \(B(t)\) and \(C(t)\) are defined as the partial derivatives of \(b(\tau)\) and \(c(\tau)\) with respect to \(u\). Plugging the derivatives into the ordinary differential equations in (4) and setting \(u = 0\), we obtain a new set of ordinary differential equations that determine the coefficients of the variance swap rates:

\[
\begin{align*}
B'(\tau) &= b_v - (\kappa + q\beta_\lambda)^\top B(\tau) - \beta \text{ diag } [b(\tau)B(\tau)^\top] - \beta_\lambda \text{ diag } [\nabla L_q(b(\tau))B(\tau)], \\
C'(\tau) &= c_v + (\kappa\theta - \bar{q}\alpha_\lambda)^\top B(\tau) - \alpha^\top \text{ diag } [b(\tau)B(\tau)^\top] - \alpha_\lambda^\top \text{ diag } [\nabla L_q(b(\tau))B(\tau)],
\end{align*}
\]

with the boundary conditions \(B(0) = b(0) = 0\) and \(C(0) = 0\). With \(u = 0\), the ordinary differential equation for \(b(\tau)\) becomes,

\[
b'(\tau) = - (\kappa + q\beta_\lambda)^\top b(\tau) - \frac{1}{2} \beta \text{ diag } [b(\tau)b(\tau)^\top] - \beta_\lambda (L_q(b(\tau)) - 1).
\]

Starting at \(b(0) = 0\), we have \(b'(0) = 0\). Thus, \(b(t) = 0\), for all \(t = 0\), is a solution. The equations in (7) simplify to,

\[
\begin{align*}
B'(\tau) &= b_v - (\kappa + q\beta_\lambda)^\top B(\tau) - \beta_\lambda \text{ diag } [\nabla L_q(b(\tau))B(\tau)], \\
C'(\tau) &= c_v + (\kappa\theta - \bar{q}\alpha_\lambda)^\top B(\tau) - \alpha_\lambda^\top \text{ diag } [\nabla L_q(b(\tau))B(\tau)],
\end{align*}
\]

where \(\nabla L_q(b(0))\) denotes the gradient of \(L_q(b(\tau))\) with respect to \(b(\tau)\), evaluated at \(b(0) = 0\). Therefore,

\[
\nabla L_q(b(0)) = -\bar{q},
\]

and the equations in (9) reduce to

\[
B'(\tau) = b_v - \kappa^\top B(\tau), \quad C'(\tau) = c_v + B(\tau)^\top \kappa\theta.
\]
The two ordinary differential equations in (11) can be solved analytically.

**Proposition 2** In the affine stochastic variance framework as specified in (3), the variance swap rate at time \( t \) and time-to-maturity \( \tau \) is given by,

\[
VS_{t,\tau} = \frac{1}{\tau} \left[ B(\tau)^\top X_t + C(\tau) \right],
\]

with

\[
B(\tau) = \left( I - e^{-\kappa \tau} \right)^{1/2} b_v; \quad C(\tau) = \left( c_v + b_v^\top \theta \right) \tau - B(\tau)^\top \theta,
\]

where \( e^{(\cdot)} \) denotes the matrix exponential operator, defined as:

\[
e^M \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!}.
\]

From Proposition 2 we obtain the following corollary:

**Corollary 1** Under the affine stochastic variance framework defined in (2), the term structure of the return variance swap rate only depends upon the specification of the drift of the state vector, but does not depend upon the type and specification of the martingale component of these factors. Holding constant the long run mean and the reverting speed, the term structure remains the same whether the martingale component is a pure diffusion, a pure jump martingale, or a mixture of both.

**Proof.** This corollary follows readily by inspecting the solutions of the coefficients \( \{B(\tau), C(\tau)\} \) in equation (13) that determine the variance swap rate in equation (6). Note in particular that the parameters controlling the covariance matrix of the diffusion component \( (\alpha, \beta) \) and the parameters for the jump component \( (\alpha_\lambda, \beta_\lambda, \mathcal{L}_q(\cdot)) \) do not enter the solutions of the coefficients.

Corollary 1 implies that from the term structure of the return variance swap, one can identify the risk-neutral drift of the state vector that controls the dynamics of the return quadratic variation. Nevertheless, the innovation (martingale) specifications of the instantaneous variance rate play little role in determining the term structure of the return variance swap rate, although they affect the time-series behavior of the swap rates.
It is worth noting that although we use the affine specification to illustrate the corollary, the variance swap rate only depends on the risk-neutral drift of the instantaneous variance rate \( \mu(v) \) under any instantaneous variance rate dynamics:

\[
VS_{t, \tau} = \frac{1}{\tau} \mathbb{E}_t^Q \left[ \int_t^{t+\tau} v_s ds \right] = \frac{1}{\tau} \mathbb{E}_t^Q \left[ \int_t^{t+\tau} \mu(v_s) ds \right],
\]

(15)
as the expectation of the martingale component equals zero. This conclusion is a result of the linear relation between the variance swap rate and future instantaneous variance rates.

2. Estimating Variance Risk Dynamics and Premia Using Variance Swaps

We estimate the variance risk dynamics and variance risk premia using over-the-counter quotes on variance swap rates on the S&P 500 index. From a major broker dealer in New York City, we obtain daily closing quotes on variance swap rates with fixed time to maturities at two, three, six, 12, and 24 months starting January 10, 1996, and ending December 28, 2005, spanning ten years. To avoid the effect of weekday patterns on the dynamics estimation, we sample the data weekly on every Wednesday\(^3\) and estimate the models using the weekly data, which have 521 weekly observations for each of the five maturity series. The industry convention is to quote the variance swap rates in volatility percentage points.

2.1. Summary statistics of variance swap rates

Figure 1 plots the time series of the five variance swap rate series in the left panel and the term structure at each week in the right panel, with the bold solid line denoting the mean term structure. From the time series in the left panel, we observe that the variance swap started at relatively low rates, but experienced a spike during the 1997 Asian crisis, and another even larger spike during the hedge fund crisis in late 1998. The series witnessed another two spikes between 2001 and 2003, but otherwise have been declining to very low levels. Over the course of the past ten years, the variance swap rate level has varied greatly from as low as 10% to as high as 50%. The right panel shows that the mean term structure of variance swap rates is upward sloping, but at each date, the term structure can exhibit a wide variety of shapes, including downward sloping, upward sloping, as well as hump-shaped term structures. Hence, a successful model of variance

\(^3\)When Wednesday is a holiday, we use the quotes from the previous business day.
dynamics needs to capture not only the large variation in the volatility levels, but also the different shapes of the term structure.

Table 1 reports the summary statistics of the variance swap rates, both in levels and in weekly differences. The mean variance swap rates increase monotonically with maturities. The standard deviation declines as the maturity increases. The swap rates show positive skewness and, for short maturities, large excess kurtosis. The swap series are highly persistent and increasingly so at longer maturities. When we look at the weekly changes, the mean weekly changes are close to zero at all maturities. The standard deviations of the weekly changes also decline with increasing maturity. The weekly changes show small skewness, but large excess kurtosis.

2.2. Model design

Proposition 1 identifies a set of conditions that generates the affine stochastic variance class. From these conditions, we design both a one-factor and a two-factor model for the variance risk dynamics and compare their empirical performance in matching the time-series and term structure behaviors of the variance swap rates.

2.2.1. A one-factor variance rate dynamics

In the one-factor setting, we let the variance rate follow the square-root dynamics as in Heston (1993). Under the risk-neutral measure \( \mathbb{Q} \), the instantaneous variance rate dynamics are:

\[
\begin{align*}
    dv_t &= \kappa_v (\theta - v_t) \, dt + \sigma_v \sqrt{v_t} \, dB^v_t. 
\end{align*}
\]  

(16)

Comparing equation (16) to the general conditions in (3), we have \( b_v = 1, c_v = 0, b_v = 1, \alpha = 0, \beta = \sigma_v^2, \lambda = 0 \). Plugging these parameterizations into (13) and rearranging, we have the variance swap rate as,

\[
VS_{t, \tau} = \phi_v(\tau) v_t + (1 - \phi_v(\tau)) \theta, 
\]  

(17)
with

\[ \phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}. \]  

(18)

With a stationary risk-neutral variance rate dynamics \((\kappa_v > 0)\), the coefficient \(\phi_v(\tau)\) is between zero and one. Thus, the variance swap rate is a weighted average of the instantaneous variance rate \(v_t\) and its risk-neutral long-run mean \(\theta\). The weight depends on the maturity \((\tau)\) of the variance swap contract and the risk-neutral mean-reversion speed of the variance rate \((\kappa_v)\).

Holding the maturity fixed, as \(\kappa_v\) declines and the variance rate becomes more persistent, \(\phi_v(\tau)\) increases and the current variance rate \(v_t\) has a larger impact on the variance swap rate. In the limit \(\kappa \to 0\) and \(\phi_v(\tau) \to 1\), the variance swap rates of all maturities are equal to the instantaneous variance rate level. On the other hand, as the mean reversion speed increases, the long-run mean imposes a heavier weight on the variance swap rate. As \(\kappa \to \infty\) and the instantaneous variance rate shows zero persistence, the variance swap rate is always equal to the long-run mean.

Holding a fixed mean-reversion speed \(\kappa_v > 0\), the coefficient \(\phi_v(\tau)\) starts at one at \(\tau = 0\) and declines to zero with increasing maturity. Hence, the variance swap rate converges to the instantaneous variance rate as the maturity goes to zero and converges to the risk-neutral long-run mean as the maturity goes to infinity.

Taking expectations on both sides of equation (17) under the statistical measure \(\mathbb{P}\), we obtain the mean term structure of variance swap rates as:

\[ \mathbb{E}^\mathbb{P}[VS_{t, \tau}] = \phi_v(\tau)\theta_v^\mathbb{P} + (1 - \phi_v(\tau))\theta, \]  

(19)

which is essentially a weighted average of the statistical mean \(\theta_v^\mathbb{P} \equiv \mathbb{E}^\mathbb{P}[v_t]\) and the risk-neutral mean \(\theta\) of the instantaneous variance rate. Since \(\phi_v(\tau)\) declines monotonically with increasing maturity, the risk-neutral mean has an increasing weight at longer maturities. Therefore, to generate an upward-sloping mean term structure, we need the risk-neutral mean, which controls the level of the long-term swap rates, to be above the statistical mean, which controls the average level of the short-term swap rates. The difference between the two mean values dictates the sign of the variance risk premia.

Equation (17) also implies that under this one-factor setting, variance swap rates of all maturities show the same persistence. Thus, the observed increasing persistence at longer maturities (Table I) suggests that there are potentially multiple factors driving the variance rate dynamics.
2.2.2. A two-factor variance rate dynamics

We also estimate a two-factor variance rate dynamics, which satisfy the following risk-neutral dynamics,

\[
dv_t = \kappa_v (m_t - v_t) dt + \sigma_v \sqrt{v_t} dB^v_t, \\
dm_t = \kappa_m (\theta - m_t) dt + \sigma_m \sqrt{m_t} dB^m_t, \quad \mathbb{E}(dB^v_t dB^m_t) = 0, \tag{20}
\]

where the instantaneous variance rate \((v_t)\) reverts to a stochastic mean level \((m_t)\), which follows another square-root process. Analogous to Balduzzi, Das, and Foresi (1998) for interest rate modeling, we label \(m_t\) as the stochastic central tendency of the instantaneous variance rate.

Under this specification, the variance swap rates are given by,

\[
VS_{t, \tau} = \phi_v(\tau) v_t + \phi_m(\tau) m_t + (1 - \phi_v(\tau) - \phi_m(\tau)) \theta, \tag{21}
\]

with

\[
\phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}, \quad \phi_m(\tau) = \frac{1 + \frac{\kappa_m}{\kappa_v - \kappa_m} e^{-\kappa_v \tau} - \frac{\kappa_v}{\kappa_v - \kappa_m} e^{-\kappa_m \tau}}{\kappa_m \tau}. \tag{22}
\]

The variance swap rate in equation (21) is a weighted average of the instantaneous variance rate \(v_t\), its stochastic central tendency \(m_t\), and the risk-neutral long-run mean \(\theta\). The weight on the instantaneous variance rate is the same as in the one-factor case. The weight converges to one as the maturity goes to zero and converges to zero as the maturity goes to infinity. The weight on \(m_t\) also converges to zero as the maturity goes to infinity. Hence, the variance swap rate starts at the instantaneous variance rate level at zero maturity and converges to the long-run mean \(\theta\) as maturity goes to infinity. The stochastic central tendency factor plays a role in the intermediate maturities, with the weighting coefficient \(\phi_m(\tau)\) exhibiting a hump-shaped term structure.

Under this two-factor structure, swap rates at different maturities can show different degrees of persistence. In particular, if the central tendency factor is more persistent than the instantaneous variance rate, the short-term swap rate will become less persistent than the long-term swap rate, as we have observed from Table 1. Furthermore, as in the one-factor case, an upward sloping mean term structure also asks for the statistical mean of the variance rate and the central tendency to be lower than the risk-neutral mean \(\theta\).
2.2.3. Market prices of variance risks

For both models, we assume that the market price on each source of risk is proportional to the square root of the risk level:

\[ \gamma(B^v_t) = \gamma_v \sigma_v \sqrt{v_t}; \quad \gamma(B^m_t) = \gamma_m \sigma_m \sqrt{m_t}. \]  

Then, under the one-factor model, the statistical dynamics of the variance rate become,

\[ dv_t = \kappa^p_v \left( \theta^p_v - v_t \right) dt + \sigma_v \sqrt{v_t} dB^v_t, \]  

with \( \kappa^p_v = (\kappa_v - \gamma_v \sigma_v^2) \) and \( \theta^p_v = \frac{\kappa_v}{\kappa^p_v} \theta \). Under this model, to generate the observed upward-sloping mean term structure on variance swap rates, we need \( \theta^p_v < \theta \). Thus, the market price of the variance risk must be negative \( \gamma_v < 0 \). Under negative market price for the variance risk, the variance rate is more persistent under the risk-neutral measure \( \mathbb{Q} \) than under the statistical measure \( \mathbb{P} \): \( \kappa_v < \kappa^p_v \).

Under the two-factor model, the statistical dynamics of the variance rate become,

\[ \begin{align*}
    dv_t &= \kappa^p_v \left( \theta^p_v - v_t \right) dt + \sigma_v \sqrt{v_t} dB^v_t, \\
    dm_t &= \kappa^p_m \left( \theta^p_m - m_t \right) dt + \sigma_m \sqrt{m_t} dB^m_t,
\end{align*} \]  

with \( \kappa^p_m = \kappa_m - \gamma_m \sigma_m^2 \) and \( \theta^p_m = \frac{\kappa_m}{\kappa^p_m} \theta \). In this case, the long-run statistical mean of the variance rate becomes \( \theta^p_v = \frac{\kappa_v}{\kappa^p_v} \theta^p_m = \frac{\kappa_v}{\kappa^p_v} \frac{\kappa_m}{\kappa^p_m} \theta \). To generate the observed monotonically upward sloping mean term structure, we need \( \theta^p_v < \theta^p_m < \theta \). Therefore, the market prices on both sources of risks must be negative: \( \gamma_v < 0 \) and \( \gamma_m < 0 \), so that \( \kappa_v < \kappa^p_v \) and \( \kappa_m < \kappa^p_m \).

2.3. Estimation methodology

To estimate the variance dynamics, we cast the model into a state-space form, extract the conditional distributions of the state variables and the observed variance swap rates, and estimate the model parameters by maximizing the likelihood of the forecasting errors on the variance swap rates.
We build the state propagation equation based on the statistical dynamics of the variance rates. For illustration, we use the two-factor model and set \( X_t = [v_t, m_t]^\top \). We construct the state propagation equation based on the Euler approximation of the statistical dynamics in (25) as,

\[
X_t = A + \Phi X_{t-1} + \sqrt{Q_{t-1}} \epsilon_t,
\]

(26)

with \( \epsilon \) denoting a two-dimensional iid standard normal innovation vector, and

\[
A = (I - \Phi) \left( \kappa^p \right)^{-1} \begin{bmatrix} 0 \\ \kappa_m \theta \end{bmatrix}, \quad \Phi = e^{-\kappa^p \Delta t}, \quad \kappa^p = \begin{bmatrix} \kappa^p_v & -\kappa_v \\ 0 & \kappa^p_m \end{bmatrix}, \quad Q_{t-1} = \begin{bmatrix} \sigma^2_v \Delta t & 0 \\ 0 & \sigma^2_m \Delta t \end{bmatrix},
\]

with \( \Delta t = 1/52 \) being the weekly time interval of the discretization. The one-dimensional state propagation equation for the one-factor model is defined analogously.

The measurement equations are constructed based on the observed variance swap rate quotes:

\[
y_t = V S_{t, \tau}(X_t) + e_t, \quad \tau = 2, 3, 6, 12, 24 \text{ months.} \tag{27}
\]

where \( e_t \) denotes the measurement error. We assume that the measurement error is independent of the state vector and that the measurement error on each of the five series is also mutually independent but with distinct variance.

Since the state propagation equation is Gaussian linear and the measurement equations are also linear in the state vector, the Kalman filter provides the most efficient forecasts and updates on the state vector and the observed variance swap rates. We build the likelihood function on the forecasting errors of the variance swap rates. Let \( (\bar{y}_t, \bar{V}_t) \) denote the Kalman filter forecasts on the conditional mean and the conditional variance of the variance swap rates, the likelihood function is given by,

\[
L(y, \Theta) = -\frac{1}{2} \sum_{t=1}^N \left[ \log |\bar{V}_t| + \left( y_t - \bar{y}_t \right)^\top (\bar{V}_t)^{-1} \left( y_t - \bar{y}_t \right) \right], \tag{28}
\]

where \( \Theta \) denotes the set of model parameters and \( N = 521 \) denotes the number of weeks for the variance swap observation. Model parameters are estimated by maximizing the log likelihood function \( L(y; \Theta) \). The likelihood function on the forecasting errors is sensitive to the pricing relation and hence to the parameters that control the risk-neutral dynamics, but it is relatively insensitive to the statistical dynamics. To enhance
identification, given the estimated risk-neutral dynamics, we set the market prices \((\gamma_v, \gamma_m)\) by matching \(\theta^v_P\) to the sample mean of the extracted \(v_t\) series and matching \(\theta^m_P\) to the sample mean of the extracted \(m_t\) series.

2.4. Estimation results

Table 2 reports the summary statistics of the pricing errors, defined as the difference between the variance swap rate quotes and the model-implied values, both in volatility percentage points. The one-factor model fits the six-month variance swap almost perfectly, but the pricing errors increase at other maturities. In contrast, the performance of the two-factor model is more uniform across different maturities. The explained variations range from 88.54% to 99.99%. The root mean squared pricing errors range from practically zero to 81 basis points, no larger than the average bid-ask spreads for the over-the-counter variance swap rate quotes. The last row of Table 2 reports the maximized likelihood values for the two models. The two-factor model performs significantly better than the one-factor model in terms of the log likelihood values: \(-622\) for the one-factor model versus 1302.4 for the two-factor model. A formal likelihood ratio test rejects the one-factor model over any reasonable confidence level.

Exploiting the information in both the time series and the term structure of variance swap rates, we can accurately identify the variance risk dynamics and the market prices of different sources of variance risks, as shown by the parameter estimates and \(t\)-statistics in Table 3. Focusing on the two-factor variance dynamics specification, we observe that the risk-neutral persistence of the instantaneous variance rate \((\kappa_v = 4.1118)\) is much lower than that for the central tendency risk factor \((\kappa_m = 0.1281)\). To give more intuition, we define the half life of each series as the number of weeks for the autocorrelation of the series to decay to half of its weekly autocorrelation level, \(\text{Half-life (in weeks)} = \ln(\phi/2)/\ln(\phi),\) with \(\phi = \exp(-\kappa \Delta t)\) denoting the weekly autocorrelation of a series. Under the risk-neutral measure, the mean-reversion speed estimates imply a half life of about ten weeks for the instantaneous variance rate \(v_t\) and about five and a half years for the central tendency factor \(m_t\). The different risk-neutral persistence implies that the two risk factors have different impacts across the term structure of the variance swap rates. In particular, we can convert the risk-neutral persistence estimates into factor loading coefficients \(\phi_v(\tau)\) and \(\phi_m(\tau)\) according to (22). These coefficients measure the contemporaneous response of the variance swap term structure per unit shock to the two factors \(v_t\) and \(m_t\). Figure 2 plots the term structure of the two responses in the left panel, with the solid line denoting the response of \(v_t\) and the dashed line denoting the response of \(m_t\). We observe that the

\[4\text{Currently, the bid-ask spreads on variance swap rate quotes from major broker dealers average around 50-100 basis points.}\]
impact of the transient variance rate factor \((v_t)\) is mainly at short maturities. Its impact declines as maturity increases. On the other hand, the contribution of the persistent central tendency factor \((m_t)\) starts at zero, but increases progressively as the variance swap maturity increases.

Table 3 also shows that the market prices on both the variance rate and the central tendency factor are strongly negative. As we have analyzed in the previous section, negative market prices on both sources of risks are imperative to generate the observed upward sloping mean term structure on the variance swap rates. In the right panel of Figure 2, we plot the model-implied mean term structure (solid line) and compare it to the data (circles). Under the two-factor stochastic variance model, the mean term structure is given by,

\[
E^P[VS_{t,\tau}] = \phi_v(\tau)\theta^P_v + \phi_m(\tau)\theta^P_m + (1 - \phi_v(\tau) - \phi_m(\tau))\theta,
\]  

with

\[
\theta^P_v = \frac{\kappa_v}{\kappa_v - \gamma_v\sigma^2_v}\theta^P_v, \quad \theta^P_m = \frac{\kappa_m}{\kappa_m - \gamma_m\sigma^2_m}\theta^P_m.
\]  

The negative market prices on both sources of risks \((\gamma_v, \gamma_m)\) generate the following ranking on the three mean values: \(\theta^P_v < \theta^P_m < \theta\). This ranking, combined with the factor loading patterns \((\phi_v(\tau), \phi_m(\tau))\) shown in the left panel of Figure 2, yields a monotonically upward sloping mean term structure, as shown in the right panel of Figure 2. Under the parameter estimates in Table 3 we have \(\theta^P_v = 0.0473\) and \(\theta^P_m = 0.0591\), as compared to \(\theta = 0.0841\).

The negative market prices on the two sources of risks not only generate the upward sloping mean term structure, but also create differences between the risk-neutral and the statistical persistence for the two factors. Given the market price estimates, the mean-reversion speeds under the statistical measure \(P\) become \(\kappa^P_v = \kappa_v - \gamma_v\sigma^2_v = 5.1359\) and \(\kappa^P_m = \kappa_m - \gamma_m\sigma^2_m = 0.1824\), implying a half life of eight weeks for \(v_t\) and about four years for \(m_t\). The different statistical persistence for the two risk factors, combined with the different loading patterns across the term structure, generates the observed increasing statistical persistence for variance swap rates of longer maturities (Table I).

Our negative estimates for the variance risk premia are also in line with the conclusions in Carr and Wu (2004), who compare the 30-day realized variance with the 30-day variance swap rates to determine the variance risk premia on both stock indexes and individual stocks.
3. Optimal Portfolio Choice

The availability of variance swap contracts makes it easy for institutional investors to either hedge away variance risk or to achieve additional exposures in it. How does this availability alter an investor’s strategic allocation on the stock market? How does it influence the investor’s welfare? To answer these questions, we derive the optimal portfolio allocation for an institutional investor with access to bonds, stocks, and variance swaps. First, we describe the general form of the solution for a general affine $k$-factor model. Then, we focus on the one-factor and two-factor model specifications estimated in the previous section.

3.1. General framework

We consider a financial market consisting of $N+1$ basic tradable assets $(P_i)_{i=0,...,N}$ and a sufficient number of variance swaps to span the underlying stochastic variance risk. The instantaneous interest rate $r$ is constant. The asset $P_0$ is the riskless money market account. $(P_1,\ldots,P_N)$ denote the prices of $N$ risky assets or stocks, the $P$-dynamics of which are specified as,

$$dP_t = rP_t dt + \text{diag} [P_t] \Sigma^P (X_t) \left( \Sigma^P (X_t)^\top \gamma_P dt + dB^P_t \right),$$  \hspace{1cm} (31)

where $dB^P_t \in \mathbb{R}^N$ is a standard Brownian motion vector, $\Sigma^P (X_t)^\top \gamma_P \in \mathbb{R}^N$ is the market price, and $\Sigma^P (X_t) \Sigma^P (X_t)^\top$ denotes the instantaneous covariance matrix. In (31), we allow the return variance to be stochastic and we let the state vector $X_t$ control the variance dynamics. For $X_t$, we assume the following $P$-dynamics:

$$dX_t = \mu^P (X_t) dt + \Sigma^X (X_t) dB^X_t,$$  \hspace{1cm} (32)

where $dB^X_t$ is a $k$-dimensional independent $P$-Brownian motion.\footnote{For notational convenience, we use the same notation for the Brownian motions under $Q$ and $P$. No confusion should occur.} We further impose a constant correlation matrix between the Brownian motions $dB^P_t$ and $dB^X_t$,

$$\mathbb{E}[dB^P_t dB^X_t^\top] = \Lambda dt \in \mathbb{R}^{N \times k}.$$

To span the stochastic variance risk, we introduce a set of $k$ variance swap contracts with fixed expiry dates $[T_1, T_2, \cdots, T_k]$. Instead of directly modeling the investment in the zero-cost variance swaps, we con-
consider contracts that pay out the realized variance, the present value of which is linked to the variance swap rate by,
\[ \tilde{V}_S(t, T) ≡ e^{-r(T-t)}E^Q_t [RV_t, T] = e^{-r(T-t)}V_S(t, T). \] \hfill (33)

Under the statistical measure \( \mathbb{P} \), we can write the dynamics of the \( i \)th variance swap \( \tilde{V}_S(t, T_i) \) with fixed expiry date \( T_i \) as,
\[ \frac{d\tilde{V}_S(t, T_i)}{\tilde{V}_S(t, T_i)} = rdt + \frac{\nabla \tilde{V}_S(t, T_i)}{\tilde{V}_S(t, T_i)} \Sigma^X(X_t) \left( \Sigma^X(X_t)^\top \gamma_t dt + dB_t^X \right). \] \hfill (34)

Consider now an institutional investor who allocates her initial wealth \( W_0 > 0 \) in the \( N \) stocks as well as in \( K \) variance swaps. The investor’s wealth evolves according to
\[ \frac{dW_t}{W_t} = w^P_t \text{diag} [P_t]^{-1} dP_t + w^VS_t \text{diag} [\tilde{V}_S(t, T_i)]^{-1} d\tilde{V}_S(t, T_i) + w^r_t r dt, \] \hfill (35)
with \( w^P_t, w^VS_t, \) and \( w^r_t \) denoting the fractions of wealth invested in the stocks, the variance contracts, and the instantaneously riskless asset, respectively. Substituting in equations (31) and (34), we have the wealth dynamics as,
\[ \frac{dW_t}{W_t} = rdt + w^P_t \Sigma^P(X_t) \left( \Sigma^P(X_t)^\top \gamma t dt + dB_t^P \right) \]
\[ + w^VS_t \text{diag} [\tilde{V}_S(t, T_i)]^{-1} \nabla \tilde{V}_S(t, T_i) \Sigma^X(X_t) \left( \Sigma^X(X_t)^\top \gamma_t dt + dB_t^X \right), \] \hfill (36)
where \( \nabla \tilde{V}_S \in \mathbb{R}^{k \times k} \) denotes the Jacobian matrix \( \partial \tilde{V}_S / \partial X_t \).

The investor chooses the portfolio weights to maximize her utility of terminal wealth \( W_T \) at time \( T \). Assuming that the investor has a CRRA utility function with relative risk aversion coefficient \( \eta > 0 \), we can write the indirect utility function as
\[ J(t, W, X) = \sup_{(w^P_t, w^VS_t)} \mathbb{E} \left( \frac{W_T^{1-\eta} - 1}{1-\eta} \mid W_t = W, X_t = X \right), \quad \eta \neq 1. \] \hfill (37)
Note that for \( \eta \to 0 \), we get the log investor.
Proposition 3  If under the statistical measure $\mathbb{P}$, the drift vector $\mu^\mathbb{P}(X)$ and the diffusion covariance matrices $\Sigma^X(X)\Sigma^X(X)^\top$, $\Sigma^P(X)\Sigma^P(X)^\top$, $\Sigma^X(X)\Lambda\Lambda^\top\Sigma^X(X)^\top$, and $\Sigma^P(X)\Lambda\Sigma^X(X)$ are all affine in $X$, then the indirect utility function is exponential affine in $X$,

$$J(t,W,X) = \frac{e^{g(t,X)}W_1^{1-\eta} - 1}{1-\eta},$$

(38)

with

$$g(t,X) = b_g(t)^\top X_t + c_g(t),$$

(39)

where $b_g(t)$ and $c_g(t)$ solve a set of ordinary differential equations with boundary conditions $b_g(T) = 0$ and $c_g(T) = 0$, respectively.

Proof. The proof is given in Appendix A. ■

Combining the conditions in Proposition 3 with that in Proposition 2 we can state the following proposition.

Proposition 4 In a pure diffusion and affine stochastic variance framework and given the optimization problem (37), the optimal portfolio allocation for stocks and variance contracts is

$$W_P = \frac{1}{\eta} \left( \gamma_P + (\Sigma^P(X)^\top)^{-1}\Lambda\Sigma^X(X)^\top b_g(t) \right),$$

(40)

$$W_{VS} = \frac{1}{\eta} \left( B(\tau)^\top \right)^{-1} \text{diag} \left[ B(\tau)^\top X_t + C(\tau) \right] \left( \gamma_X + b_g(t) \right),$$

(41)

where $\tau$ denotes the vector of variance swap maturities with $B(\tau)$ and $C(\tau)$ being the affine variance swap rate coefficients given in (11).

From equation (40), we observe that the portfolio weight on the stock is deterministic and varies only with the investment horizon $T$. In contrast, the portfolio weight on the variance contract in (41) depends linearly on the current level of state vector $X_t$. 
3.2. Model specifications

Given the general solutions in Proposition 4, we analyze the optimal portfolio allocation for the one-factor and the two-factor stochastic variance models estimated in the previous section.

3.2.1. The one-factor stochastic variance model

Under the one-factor model, we focus on one stock index \( N = 1 \) with \( X_t = v_t \). Accordingly, the \( \mathbb{P} \)-dynamics become,

\[
\begin{align*}
\frac{dP_t}{P_t} &= (r + \gamma_P v_t) dt + \sqrt{v_t} dB_P^t, \\
dv_t &= (\kappa_v \theta - (\kappa_v - \gamma_v \sigma_v^2) v_t) dt + \sigma_v \sqrt{v_t} dB_v^t,
\end{align*}
\]

with \( \mathbb{E}[dB_P^t dB_v^t] = \rho dt \). In addition to the stock and the money market account, we let the investor engage in trading one variance contract \( \tilde{V}S_i(T_i) \) with time-to-maturity \( \tau = T_i - t \) to span the one-dimensional variance risk.

**Corollary 2** Consider the optimization problem (37). In the one-factor stochastic variance model defined in (42), the optimal portfolio weights in the stock \( w_P^p \) and the variance contract \( \tilde{V}S_i(T_i) \) are given as

\[
\begin{align*}
\quad w_P^p &= \frac{1}{\eta} (\gamma_P + \rho \sigma_v b_{gv}(t)), \\
\quad w_{VS}^i &= \frac{\phi_v(\tau) v_t + (1 - \phi_v(\tau)) \theta}{\eta \phi_v(\tau)} (\gamma_v + b_{gv}(t)),
\end{align*}
\]

with \( \phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau} \), and \( b_{gv}(t) \) satisfying the following ordinary differential equation,

\[
\quad b'_{gv}(t) = \frac{\eta - 1}{2\eta} (1 + \rho^2) \sigma_v^2 b_{gv}(t)^2 + \left( \kappa_v - \frac{\gamma_v \sigma_v^2}{\eta} + \frac{\eta - 1}{\eta} \gamma_P \rho \sigma_v \right) b_{gv}(t) + \frac{\eta - 1}{2\eta} (\gamma_P^2 + \gamma_v^2 \sigma_v^2),
\]

with the terminal condition \( b_{gv}(T) = 0 \).

**Proof.** The proof can be found in Appendix B.
We can solve the ordinary differential equation in (45) in closed form. To do so, we first define

\[
  k_0 = \frac{\eta - 1}{2\eta} \left( \gamma_p^2 + \eta^2 \sigma_v^2 \right),
  \]
\[
  k_1 = \kappa_v - \frac{\eta \sigma_v}{\eta} + \frac{\eta - 1}{\eta} \gamma_p \rho \sigma_v,
  \]
\[
  k_2 = \frac{\eta - 1}{2\eta} \left( 1 + \rho^2 \right) \sigma_v^2,
\]

and \( h = \sqrt{4k_0k_2 - k_1^2} \). Depending on the parameter values, we obtain two closed-form solutions:

\[
b_{gv}(t) = \begin{cases} 
- \frac{k_1}{2k_2} + \frac{h}{2k_2} \tan \left( -\frac{k_1}{2} t + \arccos \left[ \frac{d}{2\sqrt{k_0k_2}} \right] \right), & \text{if } \text{sign} k_1 = \text{sign} k_2 \\
- \frac{k_1}{2k_2} + \frac{h}{2k_2} \tan \left( -\frac{k_1}{2} t - \arccos \left[ \frac{d}{2\sqrt{k_0k_2}} \right] \right), & \text{if } \text{sign} k_1 = -\text{sign} k_2
\end{cases}
\]

Note that the constant coefficient \( c_g(t) \) in the indirect utility function satisfies the following ordinary differential equation:

\[
c'_g(t) = r (\eta - 1) - \kappa_v \theta b_{gv}(t), \quad \text{with } c_g(T) = 0,
\]

which can also be solved in closed form.

For comparison, we also derive the optimal portfolio strategy when the investor does not have access to variance swaps. The investor puts her initial wealth \( W_0 \) only in the riskless bond and the stock. In this case, her wealth evolves according to,

\[
\frac{dW_t}{W_t} = r dt + w_t^p \gamma_P \sigma_t dt + \sqrt{\sigma_t} dB_t^p,
\]

and her objective function reads,

\[
J(t, W, X) = \sup_{w^p} \mathbb{E} \left( \frac{W_T^{1-\eta} - 1}{1-\eta} | W_t = W, X_t = X \right).
\]

**Corollary 3** In the one-factor stochastic variance model defined in (42) and when the investor can only invest in the stock and the riskless bond, the optimal portfolio weight in the stock \( w^p_t \) is,

\[
w^p_t = \frac{1}{\eta} \left( \gamma_p + \rho \sigma_v b_{gv}(t) \right),
\]
where \( b_{gv}(t) \) and \( c_{g}(t) \) solve,

\[
\begin{align*}
    b'_{gv}(t) &= \frac{\eta - 1}{2 \eta} \rho^2 \sigma_v^2 b_{gv}(t)^2 + \left( \kappa_v - \gamma_v \sigma_v^2 + \frac{\eta - 1}{\eta} \gamma_P \rho \sigma_v \right) b_{gv}(t) + \frac{\eta - 1}{2 \eta} \gamma_P^2, \\
    c'_{g}(t) &= r (\eta - 1) - \kappa_v \theta b_{gv}(t),
\end{align*}
\]

(52) (53)

with \( b_{gv}(T) = c_{g}(T) = 0 \).

**Proof.** The proof can be found in Appendix B.

Comparing the Riccati equations in (45) and (52), we observe that the only difference lies in the constant term and in the parameters that multiply the second order term of \( b_{gv}(t) \). Therefore, for the Riccati equation in (52), we can write down the closed-form solution as in (47), but with

\[
    k_0 = \frac{\eta - 1}{2 \eta} \gamma_P^2, \quad k_2 = \frac{\eta - 1}{2 \eta} \rho^2 \sigma_v^2.
\]

Compared to the solution in (47), the parameters \( k_0 \) and \( k_2 \) now decrease by the amount \( \gamma_P^2 \sigma_v^2 (\eta - 1)/(2 \eta) \) and \( \sigma_v^2 (\eta - 1)/(2 \eta) \), respectively.

### 3.2.2. The two-factor stochastic variance model

Under the two-factor stochastic variance model, we have \( X_t = [v_t, m_t]^T \) while maintaining \( N = 1 \). The stock index dynamics under measure \( \mathbb{P} \) becomes,

\[
\begin{align*}
    dP_t / P_t &= (r + \gamma_P v_t) dt + \sqrt{\nu_t} dB_{P_t}^P, \\
    dv_t &= (\kappa_v m_t - (\kappa_v - \gamma_v \sigma_v^2) v_t) dt + \sigma_v \sqrt{\nu_t} dB_{v_t}^P, \\
    dm_t &= (\kappa_m \theta - (\kappa_m - \gamma_m \sigma_m^2) m_t) dt + \sigma_m \sqrt{m_t} dB_{m_t}^m,
\end{align*}
\]

(54)

with \( \mathbb{E}[dB_{P_t}^P dB_{v_t}^P] = \rho dt \) and \( \mathbb{E}[dB_{P_t}^P dB_{m_t}^m] = \mathbb{E}[dB_{v_t}^P dB_{m_t}^m] = 0 \).

Under this two-factor variance risk structure, we need two variance contracts with different maturities \( \tau_1 \) and \( \tau_2 \) to fully span the stochastic variance risk. The optimal allocation and indirect utility function coefficients can be solved explicitly:
Corollary 4  Consider the optimization problem (37). In the two-factor stochastic variance model defined in (54), the optimal portfolio weights in the stock and the two variance contracts $\bar{VS}_t(T_1)$ and $\bar{VS}_t(T_2)$ are given by,

\begin{align*}
  w^P_t &= \frac{1}{\eta} (\gamma_P + \rho \sigma_v b_{gv}(t)), \\
  w^{VS}_t &= \frac{1}{\eta} \begin{bmatrix}
    V S_t(\tau_1) \left( \phi_m(\tau_2)(\gamma_v + b_{gv}(t)) - \phi_v(\tau_2)(\gamma_m + b_{gm}(t)) \right) \\
    V S_t(\tau_2) \left( \phi_v(\tau_1)(\gamma_m + b_{gm}(t)) - \phi_m(\tau_1)(\gamma_v + b_{gv}(t)) \right)
  \end{bmatrix},
\end{align*}

with $\tau_1 = T_1 - t$, $\tau_2 = T_2 - t$, and

\begin{align*}
  \phi_v(\tau) &= \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}, \\
  \phi_m(\tau) &= \frac{1 + \frac{\kappa_m}{\kappa_v - \kappa_m} e^{-\kappa_v \tau} - \frac{\eta}{\kappa_v - \kappa_m} e^{-\kappa_m \tau}}{\kappa_m \tau}.
\end{align*}

Furthermore, $b_{gv}(t)$, $b_{gm}(t)$, and $c_g(t)$ are the solution to

\begin{align*}
  b'_{gv}(t) &= \frac{\eta - 1}{2\eta} \left( 1 + \rho^2 \right) \sigma_v^2 b_{gv}(t)^2 + \left( \kappa_v - \frac{\eta - 1}{\eta} \gamma_v \sigma_v^2 \right) b_{gv}(t) + \frac{\eta - 1}{2\eta} \left( \gamma_v^2 + \gamma_v \sigma_v^2 \right), \\
  b'_{gm}(t) &= \frac{\eta - 1}{2\eta} \sigma_m^2 b_{gm}(t)^2 + \left( \kappa_m - \frac{\eta - 1}{\eta} \gamma_m \sigma_m^2 \right) b_{gm}(t) + \frac{\eta - 1}{2\eta} \gamma^2_m \sigma_m^2 - \kappa_v b_{gv}(t), \\
  c'_g(t) &= r (\eta - 1) - \kappa_m b_{gm}(t),
\end{align*}

starting at $b_{gv}(T) = b_{gm}(T) = c_g(T) = 0$.

Proof. The proof can be found in Appendix C.

Compared to the solution to the one-factor model, the Riccati equation for $b_{gv}(t)$ does not change, since we assume zero correlation between the two state variables $v_t$ and $m_t$. Therefore, the function $b_{gv}(t)$ in the two-factor model can be expressed in closed-form according to equation (47).

For comparison, we again consider an investor who invests her initial wealth $W_0$ only in the stock and the bond, but does not have access to invest in variance swaps or other derivative instruments that would allow her to span the variance risk.
Corollary 5  Under the two-factor stochastic variance model defined in (54), and when the investor can only invest in the stock and the riskless bond, the optimal portfolio weight in the stock $w^*_t$ is,

$$w^*_t = \frac{1}{\eta} (\gamma v + \rho \sigma_v b_{gv}(t)).$$

Furthermore, $b_{gv}(t)$, $b_{gm}(t)$, and $c_g(t)$ solve,

$$b'_{gv}(t) = \frac{\eta - 1}{2\eta} \rho^2 \sigma_v^2 b_{gv}(t)^2 + \left( \kappa_v - \gamma_v \sigma_v^2 - \frac{\eta - 1}{\eta} \gamma \rho \sigma_v \right) b_{gv}(t) + \frac{\eta - 1}{\eta} \gamma^2,$$

$$b'_{gm}(t) = (\kappa_m - \gamma_m \sigma_m^2) b_{gm}(t) - \kappa_v b_{gv}(t),$$

$$c'_g(t) = r(\eta - 1) - \kappa_m \theta b_{gm}(t).$$

Proof. The proof can be found in Appendix C.

As with the one-factor stochastic variance model, the difference between a strategy involving variance swaps and a strategy that uses only stocks and bonds is reflected in different values for $b_{gv}(t)$. Again, when the correlation coefficient $\rho$ is negative, the investor with variance swaps in her portfolio optimally chooses a larger exposure to the stock market.

Using the estimated parameter values in Table 3 we compute the values of $b_{gv}(t)$ at different investment horizons and report them in Table 4. We observe that, due to the different parameter estimates, the value for $b_{gv}(t)$ in the two-factor model flattens out much faster with increasing investment horizon than in the one-factor model. Therefore, for $\rho < 0$, the investor increases her stock exposure with increasing investment horizon.

4. Empirical Analysis on Variance Swap Investments

In this section, we combine the theoretical results from Section 5 with the parameter estimates from Section 2 to analyze how the allocation to the stock index interacts with the allocation to the variance swap contracts under different risk aversion and investment horizons, and how the availability of the variance swap investment alters the investor’s utility.
4.1. Optimal portfolio weights

To compute the optimal portfolio weights, we take as given the estimated variance dynamics under the two-factor stochastic variance specification reported in Table 3. We fix the correlation parameter \( \rho = -0.7 \) and the stock risk premium coefficient \( \gamma_P = 1 \). Given the sample mean of the variance process \( v_t \), the choice of \( \gamma_P = 1 \) corresponds to an average equity risk premium of about 5%. Unless otherwise specified, we define the default case as an investment with a horizon of two years \( T - t = 2 \), a risk aversion of \( \eta = 2 \), and variance swap maturities of two months and two years \( \tau_1 = 2/12 \) and \( \tau_2 = 2 \). We further set the two state variables \( v_t \) and \( m_t \) to their long-run means.

The left panel of Figure 3 plots the optimal portfolio weights on the stock (solid line), the two-month variance swap (dashed line), and the two-year variance swap (dash-dotted line) as a function of the investor’s relative risk aversion \( \eta \). The dotted line is the optimal portfolio weight in the stock when the investor does not have access to the variance swaps. Confirming the results from our theoretical analysis, the plot shows that irrespective of her risk aversion, the investor increases her exposure to the stock index when she also has access to the variance swap contracts to span the stochastic variance risk. Given the estimated negative risk premium on variance risk, the investor goes short on both variance swap contracts and hence makes more money on the investment when the realized return variance is low. As the risk aversion increases, the investor takes smaller absolute positions in both the stock and the variance swap contracts. At low risk aversion, the investor shorts a larger fraction of her wealth on the short-term variance swap than on the long-term variance swap. The relative weight switches as the investor becomes more risk averse.

In the right panel of Figure 3, we plot the portfolio weights as a function of the investment horizon. The investment in the stock index does not vary much with the investment horizon. In contrast, the investments in the variance swap contracts change significantly with the investment horizon. When the investment horizon is short, the investor mainly invests in the short-term variance swap. As the investment horizon increases, the short investment in the long-term variance swap steadily increases whereas the short position in the short-maturity variance swap declines.
4.2. Economic values of variance swaps

The economic importance of the availability of variance contracts is a highly relevant issue for the appraisal of the variance swap markets. We measure this “importance” as utility costs that an investor bears, when she gives up the optimal strategy and follows instead a suboptimal strategy. We explore suboptimal strategies along two lines. First, we compute the economic costs of model misspecification by using the one-factor stochastic variance model instead of the two-factor model. Second, we compute the economic costs of not having variance swap contracts available to span the variance risk. Specifically, we compare three strategies:

\( S_0 \): Investing in bond, stock, and two variance swap contracts with time-to-maturity \( \tau_1 = 2/12 \) and \( \tau_2 = 2 \) under the estimated two-factor variance dynamics.

\( S_1 \): Investing in bond, stock, and one variance swap contract with time-to-maturity \( \tau_1 = 2/12 \) under the estimated one-factor variance dynamics.

\( S_2 \): Investing in bond and stock only under the estimated two-factor variance dynamics.

To assess the economic costs of the suboptimal strategies, we compute the monetary compensation \( c_i \), also called the certainty equivalent compensation, that makes an investor indifferent between a suboptimal strategy \( (S_i, i = 1, 2) \) and the optimal strategy \( (S_0) \). Formally, we solve,

\[
J^0(t, W, X) = J^i(t, W(1 + c_i), X),
\]

for \( c_i \), where \( J^0(t, W, X) \) is the value function for the optimal strategy \( S_0 \) and \( J^i(t, W, X), i = 1, 2, \) are the value functions for the suboptimal strategies. Since the value functions take the exponential affine form in \( (38) \) for all strategies, the utility costs \( c_i \) is equal to

\[
c_i = \exp \left( \frac{1}{1 - \eta} \left( g^0(t, X) - g^i(t, X) \right) \right) - 1,
\]

with \( g^0(t, X) \) and \( g^i(t, X), i = 1, 2, \) the \( g \)-functions for \( S_0 \) and the suboptimal strategies, respectively.

We report the certainty equivalent compensations under different investment horizons \( (T = 3/12, 1, 5) \) in Table 5 as annualized percentage costs per dollar investment. We base our calculations on the estimated parameter values in Table 3. For each investment horizon, we compare the economic costs for different values of \( v_t \) and \( m_t \). By \( \bar{v} \) and \( \bar{m} \), we denote the sample mean and by \( sd_v \) and \( sd_m \) the sample standard
deviation for the data sample spanning the period from January 10, 1996, to December 28, 2005 (521 observations for each series), extracted from the estimated two-factor models. For the calculations of the economic costs, we use the mean values of $v$ and $m$ as well as the values that are half a standard deviation and one standard deviation away from the sample mean.

To assess the economic costs of model misspecification, we compare strategies $S_1$ against $S_0$, i.e., the portfolio strategies under the one-factor and the two-factor stochastic variance specification. The economic costs increase as the investment horizon lengthens. At each fixed investment horizon, the costs are larger when the central tendency factor ($m_t$) takes on lower values. Overall, the economic costs are quite significant, ranging from 2% to 30%.

Comparing the investment strategy without access to variance swaps ($S_2$) to the optimal strategy with variance swaps $S_0$, we can infer the economic value of the variance swap market. The right hand side of Table 5 shows that the economic value of the variance swap contracts increases with the investment horizon. At a fixed investment horizon, the economic value increases at lower variance levels and lower central tendency levels. Overall, the economic value of using variance swap contracts can be as high as 30% at long investment horizons.

### 4.3. Historical performance of different investment strategies

In this section, we compare the historical out-of-sample performance of four different portfolio strategies, the optimal strategy with two variance swaps with maturities $\tau_1$ and $\tau_2$, a strategy involving one variance swap with maturity $\tau_1$ based on the one-factor variance risk dynamics ($S_1$), and a pure stock and bond strategy based on the two-factor variance risk dynamics ($S_2$). For the variance swap strategies, we start with the situation in which the investor uses the variance swap with the shortest maturity ($\tau_1 = 1/12$) and the longest maturity ($\tau_2 = 2$). As a second example, we use $\tau_1 = 1$ and $\tau_2 = 2$. Finally, we benchmark these strategies against a simple buy-and-hold market strategy, under which the investor puts all her wealth into the S&P 500 ($S_m$).

For the out-of-sample study, we proceed as follows. Our starting date for investments is January 1, 2003. On this day, we estimate variance risk models based on the sample period from January 10, 1996, to January 1, 2003. We estimate not only the variance risk dynamics and variance risk premia together with the current levels of $v_t$ and $m_t$, but also the risk premium coefficient $\gamma_p$ and $\rho$, the correlation between stock
price and variance changes. Based on the estimated model, we calculate the theoretical optimal portfolio weights for the four strategies for an investor with a relative risk aversion of two ($\eta = 2$) and with several different investment horizons. Once these weights are determined, we invest accordingly in the different markets using the current market prices. After one week, we re-estimate the variance dynamics as well as $\gamma_p$ and $\rho$ with the new (longer) set of historical data. We re-calculate the optimal weights and re-allocate the portfolio accordingly at the prevailing market prices. For the investment in the variance contracts, we assume that we hold their maturities fixed. After each week, we sell the variance contract with maturity $\tau - \Delta t$, $\Delta t = 1/52$, and enter a new variance contract with maturity $\tau$. Both transactions are done at the prevailing market prices. To determine the market price of the contract with maturity $\tau - \Delta t$, we use a simple linear interpolation scheme on the prevailing variance swap term structure.

Figure 4 plots the cumulative wealth paths for the four different investment strategies when the investment horizon is one week (left panel) and five years (right panel). Furthermore, for the variance swap maturities we assume $\tau_1 = 1/12$ and $\tau_2 = 2$. Under both investment horizons, the performance of the strategy with only stocks and bonds ($S_2$, dashed lines) is similar to the buy-and-hold market benchmark ($S_m$, dotted lines). For the short-term investor, the investment under the misspecified one-factor variance risk dynamics ($S_1$, dash-dotted lines) fails to outperform the market benchmark. Most importantly, the optimal investment under the two-factor variance risk specification ($S_0$, solid lines) significantly outperforms all other strategies under both investment horizons. The difference between $S_0$ and $S_2$ highlights the economic importance of incorporating variance swaps into the investment practice. The difference between $S_0$ and $S_1$ highlights the economic importance of specifying the right variance dynamics when performing the variance swap investments.

Table 6 reports the summary statistics on the weekly returns, as well as the cumulative wealth, from the four strategies under different investment horizons and with $\tau_1 = 1/12$, $\tau_2 = 2$. In addition to the myopic and the five-year investment horizon, we add a three year investment horizon that exactly matches the period of our out-of-sample study. The investor following a one-factor model involving an investment in the short-term variance swap ($S_1$) generates the lowest Sharpe ratio for all investment horizons. The highest Sharpe ratio arises from the optimal strategy $S_0$. For a three year time horizon, we can generate a Sharpe ratio of 99%, if we invest in stocks, bonds, and the two variance swaps. The cumulative wealth from $S_0$ is also
significantly higher than that from other strategies. This effect becomes more pronounced for longer time horizons. At the five-year investment horizon, the $S_0$ strategy outperforms the $S_2$ by 13% in terms of Sharpe ratio and 96% in terms of cumulative wealth. The optimal strategy $S_0$ also markedly outperforms the $S_1$ strategy in all cases. Overall, the results highlight the importance of not only using the variance swaps to span the variance risk, but also specifying the right variance dynamics.

For comparison, we also study the case when the investor substitutes her investment in the short-term variances swap with an investment in a variance swap with a longer maturity. For $\tau_1 = 1$, we find that with such a strategy, we can further improve on the cumulative wealth by the end of 2005. In Table 7, we see that the cumulative wealth, e.g., for the myopic investor following $S_0$ increases from 1.96 to 2.28. At the same time, the standard deviation of the strategy does not increase. Therefore, we observe an additional increase in the Sharpe ratio. For this investor, the Sharpe ratio is 36% higher than the Sharpe ratio if she would follow strategy $S_2$. Finally, for an investor with a time horizon of five years, the cumulative wealth ends up at 3.19, which is 118% above the final wealth of an investor following strategy $S_2$. In terms of Sharpe Ratio, these two strategies differ by 28%.

5. Concluding Remarks

Using a decade worth of weekly quotes on variance swap rates, we design and estimate models for the S&P 500 index return variance dynamics and market prices of variance risks. We find that we need two stochastic variance risk factors to explain the variation in the variance swaps at different maturities, with one factor controlling the instantaneous variance rate while the other controlling the central tendency of the variance rate movements. The variance rate factor is much more transient than the central tendency factor under both the risk-neutral and the statistical measures, thus generating different loading patterns across the term structure from the two risk factors and different autocorrelation patterns for variance swap rates of different maturities. We also find that the market prices for both variance risk factors must be negative to match the upward-sloping mean term structure observed from the variance swap data.

Embedding variance swaps into an optimal investment strategy, we find that with the variance swap contract to span the stochastic variance risk, an investor increases her investment in the underlying stock index. In addition, the investor’s indirect utility increases significantly when allowed to span the variance risk using variance swap contracts. We also find that a two-factor model is indispensable to optimally exploit the
term structure of variance. The economic costs of neglecting variance contracts for portfolio allocation and using the wrong model are both substantial. These costs are particularly high in a low volatility environment.

Our theoretical analysis is further supported by an out-of-sample study. For the three-year period starting January 1, 2003 and ending December 28, 2005, depending on the investment horizon, an investor with access to variance swap markets and exploiting the two-factor variance risk dynamics can outperform a strategy with stock and bonds only by 118% in terms of cumulative wealth and 36% in terms of Sharpe ratio.

Compared to traditional mean variance analysis, the modern financial industry has recognized the important impacts that stock price jumps and stochastic variance can have on an investor’s welfare. Accordingly, derivative securities such as options and variance swaps have been developed to span risks along these two dimensions. To simplify the problem and to gain a clearer picture of their separate effects, the academic literature either assumes constant volatility and focuses on how to choose options at different strikes to span the random jump risk in the stock price (Carr and Madan (2001), Carr and Wu (2002)), or assumes purely continuous dynamics (or jumps of fixed size) and focuses on how to choose options (Liu and Pan (2003)) or, as what we do in this paper, variance swap contracts at different maturities to span the stochastic variance risk. Integrating these two dimensions can be a challenging but interesting direction for future research.

Appendix

A. Proof of Proposition 3

The Hamilton-Jacobi-Bellman (HJB) equation for the optimization problem (37) follows as

\[
0 = \sup_{(w_t,w^S_t)} \left\{ J' + \mu P(X)^\top J_X + W \left( r + w^P \Sigma^P(X) \Sigma^P(X)^\top \gamma_P \right) J_w \\
+ W w^VST \frac{\nabla V_{S_t}}{V_{S_t}} \Sigma^X(X_t) \Sigma^X(X)^\top J_{XX} + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X)^\top J_{XX} \right] \\
+ \frac{1}{2} W^2 \left( w^P \Sigma^P(X) \Sigma^P(X)^\top w^P + w^VST \frac{\nabla V_{S_t}}{V_{S_t}} \Sigma^X(X_t) \Sigma^X(X_t)^\top \frac{\nabla V_{S_t}}{V_{S_t}} w^VS \right) J_{WW} \\
+ W \left( w^P \Sigma^P(X) \Lambda \Sigma^X(X)^\top + w^VST \frac{\nabla V_{S_t}}{V_{S_t}} \Sigma^X(X) \Sigma^X(X)^\top \right) J_{WX} \right\},
\]  

(A.1)
where $J_X, J_W, J_{WX},$ and $J_{WW}$ denote the first, second, and cross derivatives with respect to $X$ and $W$, and we write $\frac{\nabla VS}{VS}$ for $\text{diag} \left[ \nabla VS \right]^{-1} \nabla VS$ for notational convenience.

The first-order conditions are

$$
w^p = - \frac{J_W}{WJ_{WW}} \left( \gamma_p + \frac{1}{J_W} (\Sigma^P(X)^\top)^{-1} \Lambda \Sigma^X(X)^\top J_{WX} \right), \tag{A.2}
$$

$$
w^{VS} = - \frac{J_W}{WJ_{WW}} \left( \frac{\nabla VS}{VS} \right)^{-1} \left( \gamma_X + \frac{1}{J_W} J_{WX} \right). \tag{A.3}
$$

Plugging the optimal weights back into the HJB and rearranging, we get

$$
0 = J^\prime + r W J_W + \mu^P(X)^\top J_X + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X)^\top J_{XX} \right] + \frac{J_W}{2WJ_{WW}} \left[ \gamma_p^T \Sigma^P(X) \Sigma^P(X)^\top \gamma_p + \gamma_X^T \Sigma^X(X) \Sigma^X(X)^\top \gamma_X \right] \right]
- \frac{J_W}{2WJ_{WW}} \Sigma^X(X)^\top \left( \text{id}_k + \Lambda^\top \Lambda \right) \Sigma^X(X)^\top J_{WX}
$$

$$
- \frac{J_W}{WJ_{WW}} \left( \gamma_X^T \Sigma^X(X) \Sigma^X(X)^\top + \gamma_p^T \Sigma^P(X) \Lambda \Sigma^X(X)^\top \right) J_{WX}. \tag{A.4}
$$

From the homogeneity properties of the portfolio optimization problem in (37), the indirect utility function has the structure

$$
J(t, W, X) = \frac{W^{1-n}}{1-n} e^{\gamma(t, X)}, \tag{A.5}
$$

with the boundary condition $g(T, X) = 0$. Then, the HJB in (A.4) reduces to

$$
0 = g^\prime + r (1-n) + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X)^\top g_{XX} \right] + \left( \mu^P(X)^\top + \frac{1-n}{\eta} \left( \gamma_p^T \Sigma^P(X) \Sigma^P(X)^\top + \gamma_X^T \Sigma^X(X) \Sigma^X(X)^\top \Lambda \right) \right) g_X
$$

$$
+ \frac{1-n}{2\eta} \left( \gamma_p^T \Sigma^P(X) \Sigma^P(X)^\top + \gamma_X^T \Sigma^X(X) \Sigma^X(X)^\top \right) g_X + g_X \Sigma^X(X)^\top \left( \text{id}_k + \Lambda^\top \Lambda \right) \Sigma^X(X)^\top g_X \tag{A.6}
$$

By inspection of (A.6), we obtain the structural restrictions stated in the proposition. Note that if the investor has no access to the variance swap market, then the HJB in (A.6) would read

$$
0 = g^\prime + r (1-n) + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X)^\top g_{XX} \right] + \left( \mu^P(X)^\top + \frac{1-n}{\eta} \gamma_p^T \Sigma^P(X) \Lambda \Sigma^X(X)^\top \right) g_X
$$

$$
+ \frac{1-n}{2\eta} \left( \gamma_p^T \Sigma^P(X) \Sigma^P(X)^\top + g_X \Sigma^X(X)^\top \Lambda \right) \Lambda \Sigma^X(X)^\top g_X \tag{A.7}
$$

31
B. Proof of Corollary 2 and Corollary 3

In the one-factor model structure, we can write the indirect utility function as

\[ J(t, W, X) = J(t, W, v) = W_t^{1-\eta} e^{g(t, v)}. \]  \hspace{1cm} (A.8)

When the investor can invest in the variance contract, the function

\[ g(t, X) = b_gv(t)v_t + c_g(t), \]  \hspace{1cm} (A.9)

solves

\[ 0 = b'_g v(t)v_t + c'_g(t) + r(1 - \eta) + \left( \kappa_v - (\kappa_v - \gamma_v \sigma_v^2) v_t + \frac{1-\eta}{\eta} (\gamma_v \sigma_v^2 + \gamma_p \rho \sigma_v) v_t \right) b_g(t) \]
\[ + \frac{1-\eta}{2\eta} v_t \left( \gamma_p^2 + \gamma_v^2 \sigma_v^2 + (1 + \rho^2) \sigma_v^2 b_g(t)^2 \right). \]

If the investor has only access to the stock and the bond, we can use (A.7) to see that \( g(t, X) \) solves

\[ 0 = b'_g v(t)v_t + c'_g(t) + r(1 - \eta) + \left( \kappa_v - (\kappa_v - \gamma_v \sigma_v^2) v_t + \frac{1-\eta}{\eta} \gamma_p \rho v_t \right) b_g(t) \]
\[ + \frac{1-\eta}{2\eta} v_t \left( \gamma_p^2 + \rho^2 \sigma_v^2 b_g(t)^2 \right). \]

C. Proof of Corollary 4 and Corollary 5

We directly obtain the claimed results by assuming that the function \( g(t, X) \) has the form

\[ g(t, X) = b_gv(t)v_t + b_{gn}(t)m_t + c_g(t). \]  \hspace{1cm} (A.10)

Using the model specification in (54), we can derive the optimal weights from (A.2) and (A.3). To obtain the ordinary differential equations, we plug these weights into the HJB (A.6), for the case with variance swaps, and into (A.7) for the case without variance swaps, respectively.
References


Table 1  
Summary statistics of the variance swap rates

Entries report the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt) and weekly autocorrelation (Auto) of both the levels and weekly differences of the return variance swap rates on S&P 500 index at different maturities (in months). The variance swap rates are quoted in volatility percentage points. The quotes are from Banc of America Securities LLC weekly (every Wednesday) from January 10, 1996, to December 28, 2005, 521 observations per series.

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Table 2
Summary statistics of the pricing errors on the variance swap rates
Entries report the summary statistics of the model pricing errors on the variance swap rates, including the sample average (Mean), root mean squared error (Rmse), weekly autocorrelation (Auto), maximum absolute error (Max), and explained percentage variation ($R^2$), defined as one minus the variance of the pricing error to the variance of the original swap rate quotes. The pricing errors are defined as the difference between the variance swap rate quotes and the corresponding model-implied values, both in volatility percentage points. The last row reports the maximized log likelihood values for the two models.

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Table 3
Parameter estimates of affine stochastic variance models
Entries report the maximum likelihood parameter estimates and t-values (in parentheses) of the one-factor and the two-factor affine stochastic variance model. The estimation employs weekly data on variance swap rates at maturities of two, three, six, 12, and 24 months, from January 10, 1996, to December 28, 2005 (521 observations for each series).

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<th></th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$\kappa^p$</th>
<th>$\theta^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>One-factor model</td>
<td></td>
</tr>
<tr>
<td>$v_t$</td>
<td>0.1832</td>
<td>0.1100</td>
<td>0.2319</td>
<td>-3.8648</td>
<td>0.3910</td>
<td>0.0515</td>
</tr>
<tr>
<td></td>
<td>(5.68)</td>
<td>(19.89)</td>
<td>(21.00)</td>
<td>(-6.74)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Two-factor model</td>
<td></td>
</tr>
<tr>
<td>$v_t$</td>
<td>4.1118</td>
<td>-</td>
<td>0.3978</td>
<td>-6.4722</td>
<td>5.1359</td>
<td>0.0473</td>
</tr>
<tr>
<td></td>
<td>(34.39)</td>
<td></td>
<td>(36.01)</td>
<td>(-6.72)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_t$</td>
<td>0.1281</td>
<td>0.0841</td>
<td>0.1676</td>
<td>-1.9279</td>
<td>0.1824</td>
<td>0.0591</td>
</tr>
<tr>
<td></td>
<td>(9.38)</td>
<td>(27.54)</td>
<td>(42.06)</td>
<td>(-5.63)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4
Values for $b_{gv}(t)$
Entries report the value for $b_{gv}(t)$ at $t = 0$ as a function of the investment horizon $T$ in case of variance swap and no variance swap investments and for the one-factor and two-factor models with parameter values estimated in Table 3. In addition, we assume $\rho = -0.7$, $\eta = 2$, and $\gamma_p = 1$.

<table>
<thead>
<tr>
<th>Investment Horizon</th>
<th>One-factor model</th>
<th>Two-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without VS</td>
<td>With VS</td>
</tr>
<tr>
<td>$T = 2/12$</td>
<td>$-0.0406$</td>
<td>$-0.0734$</td>
</tr>
<tr>
<td>$T = 6/12$</td>
<td>$-0.1159$</td>
<td>$-0.2132$</td>
</tr>
<tr>
<td>$T = 1$</td>
<td>$-0.2154$</td>
<td>$-0.4067$</td>
</tr>
<tr>
<td>$T = 2$</td>
<td>$-0.3745$</td>
<td>$-0.7442$</td>
</tr>
<tr>
<td>$T = 5$</td>
<td>$-0.6441$</td>
<td>$-1.4771$</td>
</tr>
<tr>
<td>$T = 10$</td>
<td>$-0.7890$</td>
<td>$-2.1826$</td>
</tr>
<tr>
<td>$T \to \infty$</td>
<td>$-0.8318$</td>
<td>$-3.3062$</td>
</tr>
</tbody>
</table>
Table 5
Annualized economic costs of model misspecification and lack of variance risk spanning

Entries report the annualized economic costs of the suboptimal strategies $S_1$ (using a one-factor instead of a two-factor model) and $S_2$ (investments limited to stocks and bonds only) under different investment horizons and different values of the variance rate ($\nu_t$) and its central tendency ($m_t$). The calculations are based on the estimated values in Table 3 and assumptions of $\rho = -0.7$ and $\gamma_P = 1$. $\hat{\nu}$ and $\hat{m}$ denote the sample mean and $sd_v$ and $sd_m$ the sample standard deviation for the period from January 10, 1996, to December 28, 2005 (521 observations for each series). The certainty-equivalent compensations are in percentages of per dollar investment.

<table>
<thead>
<tr>
<th></th>
<th>$S_1$: One-factor model</th>
<th>$S_2$: Stocks and bonds only</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{m} - sd_m$</td>
<td>$\bar{m} - sd_m/2$</td>
</tr>
<tr>
<td>$\nu - sd_v$</td>
<td>11.20</td>
<td>9.81</td>
</tr>
<tr>
<td>$\nu$</td>
<td>9.55</td>
<td>8.17</td>
</tr>
<tr>
<td>$\nu + sd_v/2$</td>
<td>8.73</td>
<td>7.36</td>
</tr>
<tr>
<td>$\nu + sd_v$</td>
<td>7.91</td>
<td>6.56</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$T = 0.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu - sd_v/2$</td>
<td>15.67</td>
</tr>
<tr>
<td>$\nu$</td>
<td>15.60</td>
</tr>
<tr>
<td>$\nu + sd_v/2$</td>
<td>15.53</td>
</tr>
<tr>
<td>$\nu + sd_v$</td>
<td>15.46</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$T = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu - sd_v/2$</td>
<td>28.67</td>
</tr>
<tr>
<td>$\nu$</td>
<td>29.01</td>
</tr>
<tr>
<td>$\nu + sd_v/2$</td>
<td>29.35</td>
</tr>
<tr>
<td>$\nu + sd_v$</td>
<td>29.70</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$T = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu - sd_v/2$</td>
<td>28.67</td>
</tr>
<tr>
<td>$\nu$</td>
<td>29.01</td>
</tr>
<tr>
<td>$\nu + sd_v/2$</td>
<td>29.35</td>
</tr>
<tr>
<td>$\nu + sd_v$</td>
<td>29.70</td>
</tr>
</tbody>
</table>
Entries report the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt), the Sharpe ratios on the weekly returns, as well as the cumulative wealth, from alternative investment strategies. $S_0$ denotes the optimal strategy estimated with a two-factor variance dynamics and two variance swap contracts ($\tau_1 = 2/12, \tau_2 = 2$), $S_1$ denotes the strategy estimated based on a one-factor variance dynamics $\tau_1 = 2/12$, $S_2$ denotes the strategy estimated from a two-factor variance dynamics but without access to variance swap contracts. Finally, $S_m$ denotes the market benchmark of buying and holding S&P 500 index. The model estimates and the performance calculations are based on data from January 10, 1996, to the end of the sample period, updated weekly starting January 1, 2003, and ending December 28, 2005. To calculate the excess returns for the Sharpe ratio, we use the one month US Libor rates. The investor is assumed to have a relative risk aversion of two.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe Ratio</th>
<th>Cum. Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Investment Horizon: one week</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_0$</td>
<td>22.5266</td>
<td>21.2175</td>
<td>-0.0180</td>
<td>0.7433</td>
<td>0.9669</td>
<td>1.9656</td>
</tr>
<tr>
<td>$S_1$</td>
<td>10.8977</td>
<td>12.3496</td>
<td>0.0336</td>
<td>0.4029</td>
<td>0.7195</td>
<td>1.3867</td>
</tr>
<tr>
<td>$S_2$</td>
<td>12.2830</td>
<td>12.0564</td>
<td>0.1852</td>
<td>0.8747</td>
<td>0.8519</td>
<td>1.4456</td>
</tr>
<tr>
<td>$S_m$</td>
<td>12.6825</td>
<td>12.3025</td>
<td>0.5877</td>
<td>2.9638</td>
<td>0.8674</td>
<td>1.4630</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Investment Horizon: three years</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_0$</td>
<td>30.0048</td>
<td>28.2433</td>
<td>0.1127</td>
<td>1.0942</td>
<td>0.9911</td>
<td>2.4600</td>
</tr>
<tr>
<td>$S_1$</td>
<td>12.7874</td>
<td>15.0014</td>
<td>0.0547</td>
<td>0.6929</td>
<td>0.7183</td>
<td>1.4676</td>
</tr>
<tr>
<td>$S_2$</td>
<td>12.6327</td>
<td>12.4174</td>
<td>0.1790</td>
<td>0.8640</td>
<td>0.8553</td>
<td>1.4608</td>
</tr>
<tr>
<td>$S_m$</td>
<td>12.6825</td>
<td>12.3025</td>
<td>0.5877</td>
<td>2.9638</td>
<td>0.8674</td>
<td>1.4630</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Investment Horizon: five years</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_0$</td>
<td>35.1302</td>
<td>34.5392</td>
<td>0.0023</td>
<td>0.8684</td>
<td>0.9589</td>
<td>2.8688</td>
</tr>
<tr>
<td>$S_1$</td>
<td>14.1893</td>
<td>17.2352</td>
<td>-0.0006</td>
<td>0.4778</td>
<td>0.7066</td>
<td>1.5306</td>
</tr>
<tr>
<td>$S_2$</td>
<td>12.6578</td>
<td>12.4330</td>
<td>0.1787</td>
<td>0.8515</td>
<td>0.8563</td>
<td>1.4619</td>
</tr>
<tr>
<td>$S_m$</td>
<td>12.6825</td>
<td>12.3025</td>
<td>0.5877</td>
<td>2.9638</td>
<td>0.8674</td>
<td>1.4630</td>
</tr>
</tbody>
</table>
### Table 7

**Out-of-sample performance of alternative strategies.**

Entries report the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt), the Sharpe ratios on the weekly returns, as well as the cumulative wealth, from alternative investment strategies. $S_0$ denotes the optimal strategy estimated with a two-factor variance dynamics and two variance swap contracts ($\tau_1 = 1$, $\tau_2 = 2$), $S_1$ denotes the strategy estimated based on a one-factor variance dynamics $\tau_1 = 1$, $S_2$ denotes the strategy estimated from a two-factor variance dynamics but without access to variance swap contracts. Finally, $S_m$ denotes the market benchmark of buying and holding S&P 500 index. The model estimates and the performance calculations are based on data from January 10, 1996, to the end of the sample period, updated weekly starting January 1, 2003, and ending December 28, 2005. To calculate the excess returns for the Sharpe ratio, we use the one month US Libor rates. The investor is assumed to have a relative risk aversion of two.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe Ratio</th>
<th>Cum. Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Investment Horizon: one week</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_0$</td>
<td>27.5812</td>
<td>22.1122</td>
<td>0.0571</td>
<td>1.1870</td>
<td>1.1563</td>
<td>2.2874</td>
</tr>
<tr>
<td>$S_1$</td>
<td>12.5383</td>
<td>11.5204</td>
<td>0.0569</td>
<td>0.6001</td>
<td>0.9137</td>
<td>1.4567</td>
</tr>
<tr>
<td>$S_2$</td>
<td>12.2830</td>
<td>12.0564</td>
<td>0.1852</td>
<td>0.8747</td>
<td>0.8519</td>
<td>1.4456</td>
</tr>
<tr>
<td>$S_m$</td>
<td>12.6825</td>
<td>12.3025</td>
<td>0.5877</td>
<td>2.9638</td>
<td>0.8674</td>
<td>1.4630</td>
</tr>
</tbody>
</table>

| **Investment Horizon: three years** |         |         |        |        |              |             |
| $S_0$    | 34.7418 | 28.2764 | 0.1703 | 1.2363 | 1.1575       | 2.8356      |
| $S_1$    | 14.7850 | 13.7878 | 0.0751 | 0.8215 | 0.9264       | 1.5582      |
| $S_2$    | 12.6327 | 12.4174 | 0.1790 | 0.8640 | 0.8553       | 1.4608      |
| $S_m$    | 12.6825 | 12.3025 | 0.5877 | 2.9638 | 0.8674       | 1.4630      |

| **Investment Horizon: five years** |         |         |        |        |              |             |
| $S_0$    | 38.7332 | 33.6218 | 0.0698 | 0.9572 | 1.0922       | 3.1963      |
| $S_1$    | 16.8097 | 15.7487 | -0.0032 | 0.5707 | 0.9396       | 1.6558      |
| $S_2$    | 12.6578 | 12.4330 | 0.1787 | 0.8515 | 0.8563       | 1.4619      |
| $S_m$    | 12.6825 | 12.3025 | 0.5877 | 2.9638 | 0.8674       | 1.4630      |
Figure 1

Time series and term structure of the return variance swap rates.
The left panel plots the time series of the five return variance swap rates at maturities of two, three, six, 12, and 24 months. The right panel depicts the term structure of the return variance swap rate at each week, with the bold solid line denoting the mean term structure. Data are weekly from January 10, 1996 to December 28, 2005 (521 observations).
Figure 2
Factor loadings and the mean term structure of variance swap rates.
The left panel plots the contemporaneous response of the variance swap term structure to unit shocks on the variance rate $v_t$ (solid line) and the central tendency factor $m_t$ (dashed line). The right panel plots the mean term structure of the variance swap rate, with the circles denoting sample estimates from the data and the solid line denoting the values implied from the estimated two-factor stochastic variance model.
Figure 3
**Dependence of optimal investments on risk aversion and investment horizon.**
Lines denote the optimal portfolio weights on the stock (solid lines), a two-month variance swap (dashed lines), and a two-year variance swap (dash-dotted lines) as a function of the investor’s relative risk aversion (left panel) and the investment horizon (right panel). The dotted lines denote the optimal investment to the stock when the investor does not have access to the variance contracts. The portfolio weights are computed based on the two-factor variance dynamics estimates in Table 3 and the following assumptions: \( \rho = -0.7 \) and \( \gamma_p = 1 \). The investment horizon is two years for the left panel and the relative risk aversion is two for the right panel.
Figure 4
Cumulative wealth path of different investment strategies.
Lines denote the cumulative wealth path of different investment strategies for an investor with $\eta = 2$ and an investment horizon of one week (left panel) and five years (right panel). In each panel, the solid line denotes the optimal two-factor strategy ($s_0$ with $\tau_1 = 2/12$ and $\tau_2 = 2$), the dash-dotted line denotes the optimal one-factor strategy ($s_1$ with $\tau_1 = 2/12$), the dashed line denotes the two-factor strategy in the absence of variance swaps ($s_2$), and finally the dotted line denotes the buy-and-hold market benchmark ($s_m$).