8 Holt-Winters’ Exponential Smoothing with Seasonality

8.1 So far on Exponential Smoothing and Holt’s Linear model

In exponential smoothing, we’ve so far met:

Simple exponential smoothing: This model is good for non-seasonal data that is fairly level (no trend). It follows the equation

$$F_{t+1} = F_t + \alpha (y_t - F_t) = \alpha y_t + (1 - \alpha) F_t,$$

for $0 \leq \alpha \leq 1$.

For a longer range forecasts, it is assumed that the forecast function is flat:

$$F_{t+k} = F_{t+1}$$

Holt’s linear method/Double exponential smoothing: This model is good for non-seasonal data with a trend. It follows the equations

level \quad \quad L_t = \alpha y_t + (1 - \alpha)(L_{t-1} + b_{t-1});

\quad \quad b_t = \beta (L_t - L_{t-1}) + (1 - \beta)b_{t-1},

\quad \quad forecast \; F_{t+1} = L_t + b_t

for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$.

For a longer range forecasts, it is assumed that the forecast function follows the trend:

$$F_{t+k} = F_{t+1} = L_t + k b_t$$

8.2 Holt Winters’ Additive Model

What about an exponential smoothing for data with a trend and seasonal behaviour? Winters generalised Holt’s linear method to come up with such a technique, now called Holt Winters. A seasonal equation is added to Holt’s linear method equations:

level \quad \quad L_t = \alpha (y_t - S_{t-s}) + (1 - \alpha)(L_{t-1} + b_{t-1});

\quad \quad b_t = \beta (L_t - L_{t-1}) + (1 - \beta)b_{t-1},

seasonal \quad S_t = \gamma (y_t - L_t) + (1 - \gamma)S_{t-s}

forecast \; F_{t+k} = L_t + k b_t + S_{t+k-s},

where $s$ is the length of the seasonal cycle, for $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq 1$. The seasonal equation picks up differences between the current level and the data at that time in the seasonal cycle. This is then added to a forecast at the same point in the cycle.

We have to pick the values of $\alpha$, $\beta$ and $\gamma$. As with the other methods, we can use RMSE or MAPE to choose the best values.
8.1 Definition This particular model is known as the additive model (seasonality factor is added to forecast).

EXERCISE: what do these updating equations say?

8.2.1 Getting Initial Values

To get started, we need initial values of the level, trend and seasonality. To initialise the seasonality, at least one complete seasonal cycle \( y_1, y_2, \ldots, y_s \) is needed. We cannot make forecasts until one complete cycle has been observed. We therefore initialise \( L_s, b_s \) and \( S_1, S_2, \ldots, S_s \), then use the formulae above to make forecasts from time \( s \) onwards. It is in fact best to initialise the trend with 2 complete cycles. Sensible starting values are

\[
L_s = \frac{1}{s} \sum_{i=1}^{s} y_i; \\
b_s = \frac{1}{s} \left[ y_{s+1} - y_1 + \frac{y_{s+2} - y_2}{s} + \cdots + \frac{y_{2s} - y_s}{s} \right]; \\
S_i = y_i - L_s, \quad i = 1, \ldots, s.
\]

8.2.2 Holt Winter’s Additive Model Algorithm

Init:

\[
L_s = \frac{1}{s} \sum_{i=1}^{s} y_i \\
b_s = \frac{1}{s} \left[ y_{s+1} - y_1 + \frac{y_{s+2} - y_2}{s} + \cdots + \frac{y_{2s} - y_s}{s} \right] \\
S_i = y_i - L_s, \quad i = 1, \ldots, s.
\]

and choose \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \)

Compute:

- level \( L_t = \alpha (y_t - S_{t-s}) + (1 - \alpha) (L_{t-1} + b_{t-1}) \)
- trend \( b_t = \beta (L_t - L_{t-1}) + (1 - \beta) b_{t-1} \)
- seasonal \( S_t = \gamma (y_t - L_t) + (1 - \gamma) S_{t-s} \)
- forecast \( F_{t+1} = L_t + b_t + S_{t+1-s} \)

Until no more observation \( y_t \) are available and subsequent forecasts:

\[
F_{t+k} = L_t + k b_t + S_{t+k-s}
\]

Table 3: Holt Winter’s Additive Model Algorithm.
8.3 Holt Winters’ Multiplicative Model

An alternative Holt Winters’ model multiplies the forecast by a seasonal factor. Its equations are:

level \( L_t = \alpha \frac{y_t}{S_{t-s}} + (1 - \alpha)(L_{t-1} + b_{t-1}); \)
trend \( b_t = \beta(L_t - L_{t-1}) + (1 - \beta)b_{t-1}, \)
seasonal \( S_t = \gamma \frac{y_t}{L_t} + (1 - \gamma)S_{t-s} \)
forecast \( F_{t+k} = (L_t + kb_t)S_{t+k-s}, \)

where, as before, \( s \) is the length of the seasonal cycle, for \( 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1 \) and \( 0 \leq \gamma \leq 1 \). Initial values for \( L_s \) and \( b_s \) are the same as in the additive case, but the initial seasonal estimates are: \( S_1 = y_1/L_s, S_2 = y_2/L_s, \ldots, S_s = y_s/L_s. \)

8.4 Example

In the table on the next page are the first 14 months beer production data. Since the data have a 12 month seasonal cycle, we initialise \( L_{12}, b_{12} \) and \( S_1, \ldots, S_{12} \). Use the additive model formulae to calculate month 13 and 14’s level, trend and seasonality, and make a 1-step ahead forecast for months 13, 14 and 15. Use \( \alpha = 0.5, \beta = 0.3 \) and \( \gamma = 0.9 \).

<table>
<thead>
<tr>
<th>Month No.</th>
<th>Production</th>
<th>Level ( L_t )</th>
<th>Trend ( b_t )</th>
<th>Seasonal ( S_t )</th>
<th>Forecast ( F_t )</th>
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<tbody>
<tr>
<td>1</td>
<td>164</td>
<td>–</td>
<td>–</td>
<td>5.75</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>148</td>
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<td>–</td>
<td>–10.25</td>
<td>–</td>
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<tr>
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<td>–</td>
<td>–6.25</td>
<td>–</td>
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<tr>
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<td>144</td>
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<td>–</td>
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<tr>
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<td>155</td>
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<td>–</td>
<td>–3.25</td>
<td>–</td>
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<tr>
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<td>125</td>
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<td>–33.25</td>
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<tr>
<td>7</td>
<td>153</td>
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Figures 18.9 show the one-step ahead forecast for the complete data set. The RMSE is 13.6.
### 8.5 Conclusion

<table>
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<tr>
<th></th>
<th>No Seasonality</th>
<th>Additive Seasonality</th>
<th>Multiplicative Seasonality</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td><strong>Additive Trend</strong></td>
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<td></td>
<td></td>
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<tr>
<td><strong>Multiplicativ e Trend</strong></td>
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<td><strong>Damped Trend</strong></td>
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</tbody>
</table>

Figure 1: Trend and seasonality.
18.8 Comparing Forecasting Methods

18.9 Holt-Winters' Exponential Smoothing with Seasonality

Figure 28: Monthly Australian beer production (dashed line) and forecasts by Holt Winters' additive model (one step ahead for observed values), with $\alpha = 0.5$, $\beta = 0.3$ and $\gamma = 0.9$. 
Figure 29: Timeplot of residuals in fitting by Holt Winters’.