The representative agent of an economy with external habit-formation and heterogeneous risk-aversion

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Abstract

For the first time in the literature, we derive an analytic expression for the representative agent of a fairly general class of economies populated by agents with “catching up with the Joneses” preferences, but with heterogeneous risk-aversion. As Chan and Kogan (2002) show numerically, the representative agent has stochastic risk-aversion that moves countercyclically with the state variable. However, we show that the heterogeneity of risk-aversion is unlikely to be able to explain the empirical regularities -namely the variability of the Sharpe ratio- that Campbell and Cochrane (1999) explain in a model of a representative agent with stochastic risk-aversion.

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The consumption asset pricing literature, starting with Hansen and Singleton (1983), Mehra and Prescott (1985) and others, has tried with limited success to identify the fundamental factors that drive the level, variation, and cyclical movements of asset prices and conditional asset pricing moments. In particular, it seems difficult to reconcile the smoothness of aggregate consumption with the high volatility of asset returns and high average historical returns in excess of the risk free rate. In addition, the historical covariance between aggregate consumption growth and asset returns is very low. These two pieces of evidence imply that the unit price of risk required by investors is on average very high. Further evidence indicates that the unit price of risk also varies significantly across time. In particular, price dividend ratios exhibit high variability compared to dividends, and exhibit some forecasting power in predicting long-run stock returns, as found in Fama and French (1988), Campbell and Shiller (1988a,b) and Campbell (1991), among others. On the same note, the variance decomposition (see Cochrane (1992) ) of price dividend ratios reveals that almost all variation is attributed to varying future excess returns.

In parallel to the analysis of the variation of excess stock returns, a number of papers have studied stock market volatility, and they have found it to vary over the business cycle. This is shown in, for example, Bollerslev, Chou and Kroner (1992) or Ludvigson and Ng (2007). Several other papers have studied the contemporaneous relation between expected stock returns and conditional return volatilities and found conflicting results. The relation, however, seems to be weak but, more importantly, most of these studies indicate that the Sharpe ratio varies throughout the business cycle. In particular, it appears to increase considerably during recessions, and to fall during expansions.

It seems that at the heart of all these stylized facts there is a mean-reverting and counter-cyclically varying risk premium. Furthermore, it seems to vary more than the stock market volatility, giving rise to a counter cyclically varying Sharpe ratio. Campbell and Cochrane (1999) explain many of the mentioned asset pricing features in a model of a representative agent with external habit preferences and counter-cyclical variation in risk aversion. Chan and Kogan (2002) argue that such a variation in the risk aversion of the representative agent can be the result of the endogenous cross-sectional redistribution of wealth in an economy with multiple agents with heterogeneous risk-aversion parameters.

We study a model similar to that of Chan and Kogan (2002), although in discrete time. With a detailed study of the mechanics of the stationary equilibrium of our heterogeneous agents economy, we derive explicitly the time-varying risk-aversion of the representative agent. From the resulting expression, we can analyze the properties of the varying risk-aversion parameter of

the economy. We find that, although the counter-cyclical pattern of Campbell and Cochrane (1999) is accurate, many more assumptions are needed in order to have more than just a marginal effect on asset price dynamics. In particular, we show that enough time-varying risk aversion fails to obtain as the result of aggregation in an economy of rational agents with standard preferences and different risk-attitudes. Even when we inflate the level of heterogeneity and increase the risk in the economy to levels that gives us the ability to predict an average equity premium close to the average excess return of the past 75 years, we are not able to produce enough predictability. For the baseline model for which the level of heterogeneity is calibrated to fit the estimated distribution of Kimball, Sahm and Shapiro (2007) the underlying consumption risk is not enough to even predict an asset pricing behavior that is clearly different, either in patterns or in levels, from a homogeneous agents economy.

Chan and Kogan (2002) assume a highly persistent and slow moving external habit, as well as a particularly high level of heterogeneity in risk-aversion. Our representative agent formulation reveals that if either of these assumptions is missing, a substantially varying Sharpe ratio does not obtain. There are certain problems with such a slow moving state; (i) its effects will be seen over much longer periods than a business cycle and, (ii) it predicts a high variability in the risk-free rate which gives rise to a high-term premium as opposed to a high risk premium and (iii) it predicts higher persistence in price-dividend ratios than what we find in the data. Campbell and Cochrane (1999) also assume a slowly varying state variable, namely the surplus consumption (current aggregate consumption over the external habit), but with the additional feature of a highly time-varying conditional variance, that as we show is related to the risk-aversion of the economy. In this paper we let the persistence in the price-dividend ratio to guide us in the selection of the persistence of our habit process. The study of Campbell and Cochrane (1999) is very useful because it identifies the main features that a successful asset pricing model needs to have in order to explain all the aforementioned empirical facts. For risk-aversion heterogeneity to have an impact on prices we would need a different source of variability in the wealth distribution across agents. For example, in the overlapping generations model of Gärleanu and Panageas (2008), heterogeneity of risk aversion does have an impact because the re-allocation of wealth is considerably more drastic across time.

In this paper we also provide some positive results as to the relation of external habit to certain asset pricing facts. For example, we find that a model with external habit in the form of Abel (1990, 1999)\(^2\) can generate substantial variation in the price-dividend ratio that is unrelated to the variation of the risk aversion of the representative agent. At the same time, however, the price-dividend ratio has limited predictive power in forecasting future excess returns, unless the price of risk varies considerably.

\(^2\)Habit formation preferences have been extensively explored in the literature in various forms. Significant contributions include Gali (1994), Ryder and Heal (1973), Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991) and Hindy, Huang and Zhu (1997).
The literature on heterogeneity of risk aversion has a long tradition in finance. Dumas (1989) solves numerically a model with two agents, one of them with logarithmic utility. Wang (1996) considers also a two agent economy and concentrates on the dynamics of bond prices. Coen-Pirani (2004) focuses on the dynamics of wealth distribution among two agents with Epstein and Zin preferences. Bhamra and Uppal (2007) show that completing the market in an economy populated with heterogeneous agents might increase the stock price volatility substantially. Kogan, Makarov and Uppal (2007) show that in a two agent economy with borrowing constraints the Sharpe ratio can be high while at the same time having a low risk-free rate. In our paper we consider an arbitrary number of agents with “catching up with the Joneses” preferences where markets are dynamically complete. Using both analytical results as well as computing the exact equilibrium of several economies we find that in the absence of any frictions or incompleteness in the market the effect of heterogeneity is potentially minimal. In effect, representative agent economies can approximate well a certain family of heterogeneous agent economies.

The rest of the paper is structured as follows. In section 1 we describe the heterogeneous agents economy and solve for the competitive equilibrium. In section 2 we consider a representative agent economy that is homeomorphic in its pricing implications with the heterogeneous agents economy of section 1. We derive an expression for the stochastic risk aversion of the representative agent and analyze its properties. In section 3 we parametrize the distribution of agents and fit it to the distribution estimated by Kimball, Sahm and Shapiro (2007). Using the results of sections 1 and 2, we derive analytically the stochastic risk-aversion of the economy. In section 4 we assume a particular process for habit and examine theoretically the asset pricing behavior. An extensive quantitative analysis of the effects of heterogeneity is carried out in section 5. We conclude in section 6.

1 The Model

We consider a version in discrete time, but more general, of the infinite horizon endowment economy of Chan and Kogan (2002). We chose discrete time instead of continuous time in order to allow for more general specifications of the uncertainty of the economy that are yet numerically tractable. In our model, uncertainty is driven by an exogenous state that follows a time homogeneous Markov process. The exogenous state is perfectly observable to all agents in the economy. Financial markets are dynamically complete, in the sense that the equilibrium asset structure spans the one period ahead uncertainty at every possible state of nature. There is a single perishable good, and agents exhibit power utility preferences with external habit formation, in the style of the “catching up with the Joneses” preferences of Abel (1990). We present two versions of the model: In the first version the economy is populated by a number of different types of agents with possibly different coefficients of relative risk aversion; in the second version, we replace the heterogeneous agents with a representative agent with
a stochastic coefficient of relative risk aversion (as in Campbell and Cochrane 1999); we then introduce an expression for the stochastic risk aversion coefficient of the representative agent parameterized by the primitives of the multiple agent economy and derive the rule that makes the two economies equivalent.

As in Chan and Kogan (2002), catching up with the Joneses preferences are not only attractive from an economics point of view (there are some influential papers that assume this type of preferences, especially Campbell and Cochrane 1999), but they also yield a stationary equilibrium such that the wealth distribution follows a time-homogeneous Markov process. The problem of standard power utility preferences is that, in the limit, wealth is accumulated by the least risk-averse agent.

### 1.1 Aggregate Uncertainty

The single source of uncertainty in our economy is growth in the aggregate endowment. We denote aggregate endowment by $Y$, with $y_t = \log(Y_t)$. As it is customary, we model the dynamics of the logarithm of the growth process, which for now we assume to be normal iid,

$$y_t - y_{t-1} = \mu + \sigma \epsilon_t, \quad t \geq 0. \quad (1)$$

where $\epsilon_t \sim \text{i.i.d. } N(0, 1)$ and $y_0$ is given. This simple structure will allow us to compare our results to those of Campbell and Cochrane (1999) and Chan and Kogan (2002).

### 1.2 Financial Markets

We assume that financial markets are dynamically complete, that is, at any point in time the equilibrium asset structure locally spans the one period ahead uncertainty. To keep the model as simple as possible, we assume that there is only one dividend-paying asset, the market security (or simply market), and the risk-free asset. We denote the price of the market by $P^m$, and the dividend it pays by $D_t (d = \log(D))$. We assume that dividend growth is the result of the shock that drives aggregate endowment growth, and an independent normal shock,

$$d_t - d_{t-1} = \mu_d + \sigma_d \left( \varrho \epsilon_t + \sqrt{1 - \varrho^2} \epsilon_t^d \right), \quad (2)$$

where $\epsilon^d \sim \text{i.i.d. } N(0, 1)$ is independent of $\epsilon$, and $\varrho$ represents the correlation between dividend growth with consumption growth. This specification of the dividend growth nests the simple case in which the market security pays the totality of the aggregate endowment. In general, the dividend paid by the market is part of the aggregate endowment. The market asset is in positive net supply, with price given by its fundamental complete markets value. The log return on the market is denoted with $R^m$. There is also a riskfree security, with a return $R^f$. 

the risk-free rate. $R^e$ is the excess return of the market over the risk-free rate. In addition, we implicitly assume (we don’t need a formal characterization for our results) that there is a zero net supply contingent claim written on the aggregate endowment, so that markets are dynamically complete.

The process $p_t$ represents the “price” of the consumption good (or pricing kernel), so that the price of the market is the present value of future dividends priced at $p$. The one period risk free rate is derived from the price of a claim to a unit of consumption next period.

### 1.3 Heterogeneous Preferences with External Habit

There is a set of infinitely lived agents, $\Gamma$. For now we assume that $\Gamma$ is a compact set of positive values. All agents have the same type of time and state separable preferences,

$$U(c, X|\gamma) = \mathbb{E}_0 \sum_{t \geq 0} \delta^t u(c_t, X_t|\gamma),$$

where $\delta \in (0,1)$ is the common subjective discount factor and $c_t$ is consumption at time $t$. $X_t$, with $x_t = \log(X_t)$, is the external habit, common to all agents. The external habit is an indicator of contemporaneous and/or past aggregate consumption. We will discuss its specification later.

The running utility is drawn from the “catching up with the Joneses” literature and is given by,

$$u(c, X|\gamma) = \frac{c^{1-\gamma}X^{\gamma-\rho} - 1}{1 - \gamma}, \quad \forall \gamma \in \Gamma.$$

$\gamma$ is the coefficient of relative risk aversion of the agent, possibly different across agents, so that different types are characterized by their $\gamma$. Let $\tau$ be the inverse of $\gamma$, i.e., the coefficient of relative risk tolerance. Due to the homotheticity and time-separability properties of preferences, aggregation results hold for agents with the same type $\gamma$. Therefore, for our purpose we only need to specify the initial wealth distribution across the different types, which we denote by $\theta_0(\gamma)$. More precisely, $\theta_t(\gamma)$ denotes the proportion of wealth held by agents with type $\gamma$ at time $t$, and therefore,

$$\int_\Gamma \theta_t(\gamma) d\gamma = 1, \quad \forall t \geq 0.$$

Throughout the paper we will be using $\gamma$ and $\tau$ interchangeably, so that when we write $\theta(\gamma)$ or $\theta(\tau)$ we imply the same distribution.

The parameter $\rho$ is common to all agents and determines the relative effect that the external habit has on the marginal utility of each agent. The derivative of the marginal utility of
consumption with respect to the external habit is given by,
\[ \frac{\partial u_c(c, X|\gamma)}{\partial X} = (\gamma - \rho)c^{-\gamma}X^{\gamma-\rho-1}. \]
Since we would like to have a negative externality for all agents, we impose the restriction that \( \rho \leq \min_{\gamma \in \Gamma} \gamma \). With a negative externality, an increase in the level of habit increases the value that each agent places on consumption. We also note that the smaller the habit parameter is, the bigger is the effect of the habit on the marginal utility.

As noted by Chan and Kogan (2002), these preferences ensure that the curvature of the value function with respect to wealth is the same as that of the utility function with respect to consumption and the relative risk aversion w.r.t. wealth is still given by the parameter \( \gamma \); this is the case because the multiplicative external habit does not affect the curvature of the value function. However, Campbell and Cochrane (1999) assume a different utility specification whereby the external habit affects the risk aversion of the agent. Campbell and Cochrane (1999) specify the process for the so called surplus consumption ratio \( \frac{c-X}{X} \) instead of that of the habit. Hence, we can consider the Campbell and Cochrane (1999) model as a special case of ours in which there is only one type of agent with risk aversion coefficient \( \gamma \), \( \rho \) is equal to zero and \( x = \log X \) follows the process specified for the consumption surplus ratio of that paper. The risk-aversion of the representative agent in that particular case is always equal to \( \gamma \) unlike what is implied by the preference assumptions of Campbell and Cochrane (1999).

### 1.4 Financial Equilibrium

As it will become clear later in this section, it is convenient to introduce the following variable
\[ \omega_t = y_t - x_t, \tag{4} \]
which we will call endowment/habit ratio, for obvious reasons. The dynamics of \( \omega \) depend on the dynamics of \( y \), given by (1), and the specification -not yet provided- of \( x \). Our results hold for a large class of specifications of \( x \). We just need that \( x \) grows on average at the same rate as \( y \) and that \( x \) is a functional of current and/or past aggregate endowment \( y_s, s \leq t \). Therefore \( \omega \) is a stationary Markov process, and we treat \( \omega \) as our state variable. For example, in the continuous time model of Chan and Kogan (2001), \( x \) is a weighted average of past aggregate endowment, and the resulting \( \omega \) is a stationary Markov process. Finally, we denote by \( \bar{\omega} \) the unconditional average of the state which, in order to simplify notation, we assume is also the initial state of the economy.

Some of the following derivations are a discrete-time version of the results in Chan and Kogan (2002). We include them for completeness. The main difference with Chan and Kogan (2002)
is that they use as initial condition the weights the social planner gives to the utility of each type \((\gamma)\). Ideally, we would want to use as initial condition the wealth distribution across types, which has a clear economic interpretation. Furthermore, at the average state, characterized by \(\bar{\omega}\), the equilibrium distributions of wealth and consumption are almost identical.\(^3\) We then choose to use as initial condition the distribution of consumption in the average state, as a proxy for the distribution of wealth. As we show later, this allows us to derive the representative agent risk aversion function in closed form.

At any time \(t\), an agent type \(\gamma\) holds a positive proportion of the aggregate wealth \(\theta_t(\gamma)\). Since we have complete markets, the budget constraint of each agent can be expressed as a single intertemporal budget constraint. At the initial period the intertemporal budget constraint of an agent of type \(\gamma\) is

\[
\mathbb{E}_0 \sum_{t \geq 0} \delta^t p_t c_t(\gamma) \leq \theta_0(\gamma) p_0 (P_0^m + Y_0),
\]

where \((p_t, t \geq 0)\) is the equilibrium consumption price process. \(\mathbb{E}_t\) is the expectation operator conditional on the consumption growth process up to time \(t\) or, alternatively, conditional on the history of the endowment habit ratio since, for a given specification of the external habit process \(x\), we can infer the endowment process from the endowment habit ratio process. Therefore, it suffices to say that the information set is the endowment process and the initial value of the endowment habit ratio. Define also

\[
z_t = \log p_t + \rho x_t, \tag{5}
\]

which can be interpreted as a normalized and stationary pricing kernel. From now on we will refer to it simply as “the pricing kernel.” Given the pricing kernel process, the external habit process, and the initial price of the dividend-paying asset, agents optimally choose their consumption plan in order to maximize their utility. The following proposition characterizes the optimal consumption allocation as a proportion of the aggregate endowment, \(\alpha_t(\gamma) = c_t(\gamma)/Y_t\).

**Proposition 1.** The optimal consumption allocation of an agent type \(\gamma\) is characterized by,

\[
\alpha_t(\gamma) = \lambda(\gamma) \exp \left[ -\frac{z_t}{\gamma} - \omega_t \right], \tag{6}
\]

where \(\lambda(\gamma)^{1/\gamma}\) is the Lagrange multiplier of the intertemporal budget constraint, and is given

\(^3\)As we show later, we compute numerically and very efficiently the equilibrium, including the resulting distribution of wealth across types.
by,

\[
\lambda(\gamma) = \theta_0(\gamma) \frac{E_0 \sum_{t \geq 0} \delta^t p_t Y_t}{E_0 \sum_{t \geq 0} \delta^t p_t Y_t \exp \left[ -\frac{z_t}{\gamma} - \omega_t \right]}.
\]  

(7)

The optimality condition (6) is the same as equation (8) in Chan and Kogan (2002). However, in our case, the Lagrange multiplier \(\lambda(\gamma)\) is endogenously determined, given the initial wealth distribution. It is well known that there is one-to-one mapping between the equilibria resulting from each set of conditions, but this subtle point has important quantitative implications, as we will see later on, due to the fact that the distribution of wealth across types is a key factor to determine the equilibrium. For example if we increase the wealth held by the most extreme types, then the stochastic risk aversion of the economy is more volatile.

A financial equilibrium is a normalized pricing kernel process, \(\{z_t, t \geq 0; z_0 = 0\}\) and a set of consumption allocation processes of ratios of the aggregate endowment, \(\{\alpha_t(\gamma), t \geq 0; \gamma \in \Gamma\}\), such that consumption allocations satisfy the optimality conditions of the agents, and the consumption good market clears at all times. We have the following corollary.

**Corollary 1.** In equilibrium, the pricing kernel is a function of the endowment/habit ratio and the initial wealth distribution, and is characterized by the following equation,

\[
1 = \int_{\Gamma} \lambda(\gamma) \exp \left[ -\frac{z}{\gamma} - \omega \right] d\gamma,
\]  

(8)

where \(\lambda(\gamma)\) is given by (7).

From equation (8) it is not feasible to derive \(z\) in closed form. However, it can be computed numerically with very high accuracy after we discretize the distribution of types and the state variable, and solve a large system of equations (a similar method is provided in Judd, Kubler and Schmedders (2003), while our method is outlined in the Appendix B).

One alternative approach that allows us to derive an expression for \(z\) and use it solve for the risk aversion of the representative agent in closed form, is as follows. Instead of assuming the initial wealth distribution of types we assume that we know the initial (at the average state) consumption distribution. As we have argued before, at the average state is almost identical to the distribution of wealth (as we can verify using the numerical method sketched before). In addition, from an empirical point of view, the distribution of consumption across types is as easily observable as the distribution of wealth.
We introduce an auxiliary concept. For a given endowment/habit ratio $\omega$, we define the probability measure $\mathcal{P}_\omega(\tau)$ that assigns to agents of type $\gamma = 1/\tau$ probabilities equal to their equilibrium consumption share. Furthermore, let $\mathcal{E}_\omega$ denote the expectation operator under this probability measure. Then, from Proposition 1, this corollary follows,

**Corollary 2.** The probability measure at the average state is given by,

$$\mathcal{P}_\bar{\omega}(\tau) = \theta_0(\tau) \frac{\mathbb{E}_0 \sum_{t \geq 0} \delta^t p_t Y_t}{\mathbb{E}_0 \sum_{t \geq 0} \delta^t p_t Y_t e^{-\tau z_t - (\omega_t - \bar{\omega})}},$$  \hspace{1cm} (9)

and, therefore, the following relation holds,

$$\exp(\omega_t - \bar{\omega}) = \mathcal{E}_\bar{\omega} [\exp(-\tau z_t)], \quad \forall t \geq 0.$$  \hspace{1cm} (10)

We first note that $\mathcal{P}_\bar{\omega}$ is very close to $\theta_0(\tau)$, since the fraction on the right-hand side of (9) is close to one. This is due to the fact that, for any $\tau$, $e^{-\tau z_t - (\omega_t - \bar{\omega})}$ does not vary much in equilibrium and it is “centered” around one, its value at the average state.

In addition, the right-hand side of (10) is the moment generating function of $\tau$ and hence, for certain distributions it is known, and it is straightforward to derive $z$. However, in general, it is straightforward to approximate with high precision the pricing kernel by discretizing the distribution and using numerical integration methods. When agents are identical, the pricing kernel is linear in the state, $z(\omega) = -\gamma(\omega - \bar{\omega})$.

In the following lemma, we can analyze the properties of the pricing kernel with the help of the probability measure we just introduced.

**Lemma 1.** The pricing kernel is a continuous function of the state with a negative first derivative and positive second derivative,

$$z'(\omega) = -\frac{1}{\mathcal{E}_\omega(\tau)},$$  \hspace{1cm} (Z2)

$$z''(\omega) = \frac{\mathcal{E}_\omega(\tau^2) - \mathcal{E}_\omega(\tau)^2}{\mathcal{E}_\omega(\tau)^3}.$$  \hspace{1cm} (Z3)

Furthermore,

$$\lim_{\omega \to \pm \infty} z(\omega) = \mp \infty,$$

$$\lim_{\omega \to \pm \infty} z'(\omega) = -\min_{\gamma \in \Gamma} \gamma,$$

$$\lim_{\omega \to -\infty} z''(\omega) = 0,$$

$$\lim_{\omega \to +\infty} z'(\omega) = -\max_{\gamma \in \Gamma} \gamma.$$
As a natural extension of what happens in an economy populated by a single agent, equation (z2) shows that the slope of the pricing kernel is negative and equal to the inverse of the weighted average risk tolerance in the economy. If time was continuous $|\sigma z'(\cdot)|$ would correspond to the price of risk where $\sigma$ is the volatility of consumption growth. Bhamra and Uppal (2007) as well as Gârleanu and Panageas (2008) also show that the price of risk is determined by the weighted harmonic average of the risk-aversion of the agents. Furthermore, from (z3), the curvature of the pricing kernel depends on the dispersion of the risk tolerance types, which from (z2) implies that pricing kernel and average risk tolerance in the economy are more volatile when the dispersion of types is higher. This is due to the fact that the more variability there is in types, the more extreme the investment positions of the agents are, and this leads to bigger changes in the cross-sectional wealth distribution.

2 The Representative Agent Equivalent Economy

In this section we construct a representative agent economy with state-dependent risk-aversion parameter that is equivalent to the heterogeneous agent economy. We say that two economies are equivalent when they have the same aggregate endowment process, financial market structure and pricing kernel process. We will derive an expression for the stochastic risk-aversion parameter of the representative agent and study its properties. The risk-aversion of the representative agent provides a natural measure of risk-aversion in the heterogeneous agent economy.

2.1 Preferences and Equilibrium

In this economy, there is a representative agent with stochastic risk-aversion coefficient that depends on the state $\omega$. The representative agent has the following utility,

$$U_r(c, x) = \mathbb{E}_0 \sum_{t \geq 0} \delta^t \frac{c_1^{1-\gamma(\omega_t)} (X_t)^{\gamma(\omega_t)-\rho} - 1}{1 - \gamma(\omega_t)},$$

where the external habit is $X_t^\omega = X_t e^{\bar{\omega}}$, for all $t \geq 0$. With this change in the definition of the habit process, the stochastic risk-tolerance of the representative agent is equal to the average (using the probability measure $P_\omega$) risk-tolerance of the economy at the average state. With the standard definition, this would be true at the state $\omega = 0$. Since we would like the risk-aversion of the representative agent to be a good representation of the risk-aversion in the economy, this choice seems more appropriate. As in the heterogeneous agents economy, we require a negative externality from the habit, and for that we need the habit parameter to be always less than the risk-aversion parameter, i.e. $\rho \leq \min_\omega \gamma(\omega)$.

For a consumption price process $(p_t, t \geq 0)$, the representative agent maximizes the previous utility subject to the intertemporal budget constraint. Financial equilibrium exists when there
is a pricing kernel process \((z^r_t, t \geq 0; z^r_0 = 0)\) such that the consumption process \((Y_t, t \geq 0)\) is optimal for the representative agent. The following proposition states the result.

**Proposition 2.** The equilibrium pricing kernel of the representative agent economy is a function of the state, and is characterized by the following equation,

\[
1 = \exp [z^r_t + \gamma(\omega_t)(\omega_t - \bar{\omega})], \quad \forall t \geq 0.
\]  

(11)

For the representative agent economy to be equivalent to the heterogeneous agent economy, it suffices to have the same equilibrium pricing kernel processes in the two economies. Hence, we require that \(z(\omega) = z^r(\omega)\) for all \(\omega\). This gives rise to the following corollary.

**Corollary 3.** Let \(z(\omega)\) be the equilibrium pricing kernel of the heterogeneous agent economy. Then the representative agent economy is an equivalent economy if the stochastic risk aversion of the representative agent is the following continuous function of the state,

\[
\gamma(\omega) = \begin{cases} 
-\frac{z(\omega)}{\omega - \bar{\omega}}, & \omega \neq \bar{\omega} \\
-\frac{z'(\bar{\omega})}{\mathcal{E}_\omega(\tau)}, & \omega = \bar{\omega}
\end{cases}
\]  

(12)

We point out that \(\gamma(\bar{\omega})\) can be set to any value,\(^4\) but the choice in (12) makes the stochastic risk aversion function continuous at \(\bar{\omega}\). From the properties of the pricing kernel function we can derive certain properties of the risk-aversion function. We summarize them in the following corollary.

**Corollary 4.** The stochastic risk aversion function \(\gamma(\omega)\), characterized in (12), has the following first and second derivatives,

\[
\gamma'(\omega) = -\frac{1}{\omega - \bar{\omega}} \left[ \gamma(\omega) - \frac{1}{\mathcal{E}_\omega(\tau)} \right] < 0, \quad (\gamma 1)
\]

\[
\gamma''(\omega) = \frac{1}{\omega - \bar{\omega}} \left[ -2\gamma'(\omega) - 2z''(\omega) \right]. \quad (\gamma 2)
\]

Furthermore,

\[
\lim_{g \to \pm\infty} \gamma'(\omega) = 0, \quad \lim_{g \to \pm\infty} \gamma''(\omega) = 0
\]

\[
\lim_{g \to +\infty} \gamma(\omega) = \min_{\gamma \in \Gamma} \gamma, \quad \lim_{g \to -\infty} \gamma(\omega) = \max_{\gamma \in \Gamma} \gamma.
\]

\(^4\)In equation (11) we have the term \(\gamma(\omega_t)(\omega_t - \bar{\omega})\), therefore it does not matter what \(\gamma(\bar{\omega})\) is, since it is always multiplied by zero.

12
The negative first derivative of the risk-aversion function establishes the counter-cyclicality of risk aversion in the economy. The risk aversion of the economy moves from the highest level to the lowest as the endowment habit ratio goes from minus infinity to plus infinity. Intuitively, more risk-tolerant agents invest more in the stock market and, therefore, end-up wealthier in good states (high endowment/habit ratio) and poorer in bad states.

3 Agent Distribution and the Variation of $\gamma(\cdot)$

As we have explained before, it is straightforward to find the equilibrium pricing kernel numerically. However, for our analysis, it is convenient to make the following parametric assumption about $P_\omega(\tau)$: We assume that at the average state $\bar{\omega}$ the risk-tolerance is gamma distributed, with parameters $\kappa$ and $\theta$,

$$\tau(\bar{\omega}) \sim \text{Gamma}(\kappa, \theta),$$

so that $E_{\bar{\omega}}(\tau) = \kappa \theta$ and $V_{\bar{\omega}}(\tau) = \kappa \theta^2$, where $V$ denotes the variance. With this assumption, we can derive the pricing kernel, and hence the risk-aversion function, in closed form. Kimball, Sahm and Shapiro (2007), through survey responses, construct an empirical distribution of risk-tolerance, and then fit it to a log-normal distribution. The log-normal distribution does not have a moment-generating function, thus we choose the closely related gamma distribution. We fit the gamma distribution by minimizing the overall distance between the distribution of risk-aversion implied by the log-normal distribution and that implied by the gamma distribution. In this exercise we are making the implicit assumption that the wealth or the consumption share of an agent is independent of his/her relative risk aversion coefficient at the average or initial state. Figure 1 plots the cumulative and density functions of the log-normal distribution of Kimball, Sahm and Shapiro (2007), and our fitted gamma distribution. We note that they are very close to each other but the gamma distribution exhibits a fatter tail. This implies slightly higher mean and standard deviation of risk aversion. Let $\bar{\gamma}$ and $\bar{\nu}$ denote, respectively, the inverse of the average risk-tolerance and the standard deviation of risk-tolerance, both at the average state. The estimated parameters imply that $\bar{\gamma} = 5.17$ and $\bar{\nu} = 0.13$.

Given our parametric assumption about the cross-sectional consumption distribution of types we have the following corollary.

**Corollary 5.** Let us define $\eta = (\bar{\gamma} \bar{\nu})^2$. When the cross-sectional dispersion of types (risk-tolerance) at the average state is gamma distributed with mean $1/\bar{\gamma}$ and standard deviation $\bar{\nu}$, and weight for each type given by the consumption share, the equilibrium pricing kernel is

$$z(\omega) = \bar{\gamma} \exp\left[-(\omega - \bar{\omega})\eta\right] - \frac{1}{\eta}.$$  

\footnote{It is straightforward to verify numerically that the results are insignificantly different when we assume the log-normal distribution instead.}
We point out that as the level of heterogeneity in the economy $\bar{\nu}$ tends to zero, the pricing kernel tends to $-\bar{\gamma}(\omega - \bar{\omega})$, as was noted in the discussion of Corollary 2. We now can study the variability of the risk-aversion of the representative agent. First, we recall from (12) that the risk-aversion of the representative agent is equal to $\bar{\gamma}$. In addition, we define the following function,

$$h(\omega) = \frac{\gamma(\omega)}{\bar{\gamma}},$$

that is, the ratio of the risk-aversion coefficient of the representative agent in state $\omega$, given in (12), to the risk-aversion coefficient in the average state $\bar{\omega}$. It represents the coefficient of variation (we will call it “multiplier”) of risk-aversion in a given state, with respect to the average state. Using the result of corollary 5, we can express it in closed form. Figure 2 plots $h$ for three different values of $\bar{\nu}$, the value empirically estimated using the distribution of Kimball, Sahm and Shapiro (2007), twice this value, and one half this value; the value of $\bar{\gamma}$ is the one estimated given the distribution in the same paper. The function is plotted for deviations of the average state ranging between $-1$ to $+1$. As we will argue later, this kind of range is unrealistically large. It is more natural to expect that the deviations will be less than 0.5 in absolute value: a deviation of 0.5 implies that the aggregate consumption surplus ratio is higher than the average by around 65%. From figure 2 we observe that the possible variation in the coefficient of risk-aversion for the representative agent is very small. Unless we assume a level of heterogeneity twice as much as the estimated level, the risk-aversion of the representative agent is not expected to deviate from the average risk aversion by more than 20% at any time. Even for the most extreme case we consider, the risk-aversion of the economy doubles only when the economy is deep in recession. This plot is a first indication that the effect of risk-aversion heterogeneity on asset prices, and particularly on risk-premia, is potentially small.

In an economy with rational investors the risk-attitude of the representative agent can be time-varying due to two reasons. The first one, which drives the risk-aversion coefficient in this model, is the evolution of the cross-sectional wealth distribution. Unless the variation in the state vector that determines the cross-sectional wealth distribution is very high, the wealth reallocation across time cannot have a substantial effect on asset prices, as we have already seen. A second possible reason for time-variation in the risk-aversion coefficient of the representative agent is time-variation in the individual risk-aversion coefficients. In order to have a high variation in the risk-preferences of the representative agent we would also need the individual preferences to be moving together, and in the same direction. In particular the entire distribution needs to be moving up and down with the state variable.
4 Habit Process and Asset Prices

In order to solve for asset prices, we need to specify a process for the external habit. Following Chan and Kogan (2002) and a good part of the literature that uses “catching up with the Joneses” preferences, we assume that the habit is a weighted average of the previous habit and the previous aggregate endowment level,

\[ x_{t+1} = \lambda y_t + (1 - \lambda) x_t. \]  

(13)

From (13) the endowment/habit ratio, our the state variable, follows a mean-reverting process,

\[ \omega_{t+1} - \omega_t = -\lambda(\omega_t - \bar{\omega}) + \sigma \epsilon_{t+1}. \]  

(14)

The unconditional mean of the state variable is \( \bar{\omega} = \mu / \lambda \), and the unconditional volatility is \( \sigma_{\omega} = \sigma / \sqrt{\lambda(2 - \lambda)} \). The reversion rate parameter \( \lambda \) is particularly important in this model because it determines the likely “range” in which the state variable moves through the dependence of \( \sigma_{\omega} \) on \( \lambda \). The smaller \( \lambda \) is the bigger is the unconditional variance. Figure 2 shows that risk-aversion of the representative agent varies more the larger the “range” of the state variable is. Therefore, as \( \lambda \) decreases, the potential effect of agent heterogeneity on asset prices becomes larger. When the state of the economy moves further away from its average state the wealth allocation tends to be concentrated on the more or less risk-averse agents, depending on whether the deviation is negative or positive, respectively. Hence, with a smaller rate of reversion (equivalent to higher persistence in the state variable) largerreallocations of wealth and, therefore, variations in the risk-aversion of the economy, are possible. However, these large swings in risk-aversion require a long time. For example, Campbell and Cochrane (1999) use a reversion rate value of 0.13 which implies a half-life of around 5 years,6 while Chan and Kogan (2002) use values that imply half-lives of 12 years for their heterogeneous agents economy, and around 17 years for their single agent economy. Persistence in the state variable translates into persistence in the price-dividend ratio. It is natural therefore to select this parameter in order to match the price-dividend ratio persistence implied by the model to the persistence observed in the data.

4.1 The Stochastic Discount Factor

The fundamental price of an asset is the expected value of the discounted future dividends. Let \( M_{t+1} (m = \log(M)) \) denote the one-period stochastic discount factor between periods \( t \) and \( t + 1 \). Since \( p_t \) denotes the price of a unit of consumption in period \( t \), the stochastic discount

---

6The half-life is the time required for the deterministic version of the process to cover half of the distance to the unconditional mean. It is given by \(-\log(2) / \log(1 - \lambda)\)
factor is,

$$M_{t+1} = \delta \frac{P_{t+1}}{P_t}, \quad \forall t > 0.$$  

Using equation (5) and our other assumptions, we have the following corollary:

**Corollary 6.** Assume that the habit process is as in (13) and the cross-sectional distribution of types (risk-tolerance) with respect to their consumption share at the average state is gamma distributed, with mean $1/\bar{\gamma}$ and standard deviation $\bar{\nu}$. The equilibrium one period log stochastic discount factor is conditionally log-normally distributed,

$$m_{t+1} = \log(\delta) - \rho \lambda \omega_t - \frac{\bar{\gamma}}{\bar{\eta}} e^{-\bar{\eta}(\omega_t - \bar{\omega})} \left( 1 - e^{\lambda \eta(\omega_t - \bar{\omega}) - \sigma \eta \epsilon_{t+1}} \right).$$

When agents are homogeneous, i.e. $\bar{\nu} = 0$,

$$m_{t+1} = \log(\delta) - \rho \lambda \omega_t - \bar{\gamma}(\omega_{t+1} - \omega_t).$$

We observe that the stochastic discount factor of both the standard Lucas tree and the model of Campbell and Cochrane (1999) are particular cases of the stochastic discount factor given in corollary 6. Campbell and Cochrane (1999) assume homogenous agents, but their state variable, the consumption surplus ($x$ using our notation) satisfies some specific dynamics that, Chan and Kogan (2002) argue, might be obtained as the result of simpler dynamics and agents with heterogeneous risk-aversion.\textsuperscript{7} One of our objectives is to study this point further. For example, Campbell and Cochrane (1999) assume that their state variable is counter-cyclical, and that would explain the counter-cyclicality of the risk premium. In our model (as in Chan and Kogan 2002), the equilibrium risk-aversion of the representative agent turns out to be negatively related to the state of the economy ($\gamma'(\omega) < 0$). However, we have also showed that the possible variation of the risk-aversion of the representative agent (figure 2) is relatively modest for realistic parameter values, and it seems difficult to argue that it can explain the variation in the risk-premium observed in the data. We next elaborate further on this point.

The main driving forces of Campbell and Cochrane (1999) are: (i) the persistence in the consumption surplus ratio, which produces the persistence in price dividend ratios and the

\textsuperscript{7}In the special case of Campbell and Cochrane (1999) $x$ is the state-variable and not $\omega$, since it is Markov stationary, following the process,

$$x_{t+1} - x_t = -\lambda (x_t - \bar{x}) + \phi(x_t) \epsilon_{t+1},$$

where $\epsilon$ is the aggregate endowment growth innovation and $\phi(x)$ is some given function. Hence we have,

$$\omega_{t+1} - \omega_t = \mu - \lambda (x_t - \bar{x}) - \bar{\gamma} \left[ 1 + \phi(x_t) \right] \sigma \epsilon_{t+1}$$

. Since $\rho = 0$ in this special case we obtain the same stochastic discount factor.

16
variability of stock expected returns; (ii) the counter-cyclical conditional volatility of the state variable (their consumption surplus ratio or in our case of $x$). In Campbell and Cochrane (1999), the varying conditional volatility is chosen so that it fixes the risk-free rate at a certain level, and the entire variation in expected returns translates into variation in risk premia. In our model the persistence of the state variable is the result of assuming persistence in habit, but the varying conditional volatility of the stochastic discount factor is endogenous and related to the variation in risk-aversion. For simplification purposes, we assume $e^{-\sigma\eta\epsilon} \approx 1 - \sigma\eta\epsilon$ and introduce $\tilde{m}$, an approximation to the true stochastic discount factor,

$$
\tilde{m}_{t+1} = \log(\delta) - \mu\gamma(\bar{\omega}_{t+1}) + [\gamma(\bar{\omega}_{t+1}) - \rho] \lambda\omega_t + [\gamma(\omega_t) - \gamma(\bar{\omega}_{t+1})] (\omega_t - \bar{\omega})
$$

- $\gamma\sigma [1 + \phi(\omega_t)] \epsilon_{t+1}$

where

$$
\phi(\omega_t) = \eta(1 - \lambda)(\bar{\omega} - \omega_t)h(\bar{\omega}_{t+1}),
$$

and $\bar{\omega}_{t+1} = E_t[\omega_{t+1}]$. Since $\sigma\eta$ is a very small number, the approximation is indeed very good. $\tilde{m}$ is conditionally normally distributed with an endogenously varying conditional volatility equal to $\gamma [1 + \phi(\omega_t)] \sigma$. The conditional volatility of $\tilde{m}$ has the same form as the conditional volatility of the pricing kernel of Campbell and Cochrane (1999). The only difference is that the function $\phi(\omega)$ in Campbell and Cochrane (1999) is exogenously given, and it varies considerably more than ours, as we show next. In figure 3 we plot the function $\phi$ for three different levels of agent heterogeneity. The label of each line is the number that multiplies the value of the risk-tolerance standard deviation $\nu$ estimated by Kimball, Sahm and Shapiro (2007). To get the same persistence in the state variable as Campbell and Cochrane (1999) we set $\lambda = 0.13$. The sensitivity function $\phi$ in Campbell and Cochrane (1999) ranges in value from around 50 to 0. Clearly, the level of variation that can be generated endogenously in our economy is substantially smaller, implying that our economy will not be able to predict substantial variation in risk-premia. In addition, the level of the conditional volatility is quite small, and therefore this economy cannot predict the high equity premium observed in the data. Unless the consumption risk were substantially higher, and the level of heterogeneity in the economy significantly bigger than the estimate of Kimball, Sahm and Shapiro (2007), it is unlikely that a substantial part of the observed variation in risk-premia can be explained with risk-preference heterogeneity.

4.2 Asset Prices

The assets of interest are the risk-free bond, that pays a unit of consumption next period, and the infinitely lived market security, that pays the dividend process (2). The price of the
risk-free bond is

\[ P^f(\omega_t) = \mathbb{E}_t \left[ e^{m(\omega_t, \omega_{t+1})} \right]. \]

The price of the market security is increasing in the dividend, but the price-dividend ratio, that we denote \( PD \) is stationary,

\[ PD(\omega_t) = \mathbb{E}_t \left[ e^{m(\omega_t, \omega_{t+1}) + d_{t+1} - d_t} (PD(\omega_{t+1}) + 1) \right]. \]

In the appendix we explain how to compute the prices numerically.

The continuously compounded risk-free rate is the negative log of the bond price. Using (15) we can derive a good approximation,

\[ \tilde{r}_t^f = -\log(\delta) + \mu \gamma(\bar{\omega}_{t+1}) - [\gamma(\bar{\omega}_{t+1}) - \rho] \lambda \omega_t + [\gamma(\bar{\omega}_{t+1}) - \gamma(\omega_t)] (\omega_t - \bar{\omega}) - \gamma^2 \sigma^2 [1 + \phi(\omega_t)]^2. \]  

(17)

The model of Campbell and Cochrane (1999) explains the observed low volatility of the interest rate by assuming that the precautionary savings term is inversely proportional to the habit term. In fact the conditional volatility of the pricing kernel of their model is derived by making the risk-free rate constant. The habit term refers to the incentive to postpone consumption when consumption is high today with respect to habit. The precautionary savings term refers to the incentive to save less when the real risk in the economy tomorrow is low. In our model these two terms are the most significant and also inversely proportional to each other. However, the habit term dominates the precautionary savings term unless the level of heterogeneity in the economy is very high, and the fundamental risk of the economy is significantly higher than in reality. We show this next.

In (17), both \([\gamma(\bar{\omega}_{t+1}) - \rho] \) and \([\gamma(\bar{\omega}_{t+1}) - \gamma(\omega_t)] (\omega_t - \bar{\omega}) \) are always positive. The variability of the interest rate in (17) comes mostly from the habit term, \([\gamma(\bar{\omega}_{t+1}) - \rho] \lambda \omega_t \), increasing in \( \omega_t \) and, possibly, from the precautionary savings term \( \gamma^2 \sigma^2 [1 + \phi(\omega_t)]^2 \), decreasing in \( \omega_t \). The intertemporal substitution term \( \mu \gamma(\bar{\omega}_{t+1}) \) does not vary much compared to the other terms, even when the level of heterogeneity is high. The term \( [\gamma(\bar{\omega}_{t+1}) - \gamma(\omega_t)] (\omega_t - \bar{\omega}) \) varies even less since the change in \( \gamma(\omega_t) \) from \( t \) to \( t+1 \) is very small.

The habit term says that when consumption increases with respect to habit, agents want to save more in order to increase their future consumption, and this puts downward pressure on the interest rate. The precautionary savings term is quadratic in the “expected” risk aversion in the economy next period since \( \phi(\omega_t) \) is linear in \( \gamma(\bar{\omega}_{t+1}) \). The precautionary savings term’s variation depends on the risk of the economy as given by \( \sigma \) and the variability of \( \phi(\omega_t) \). We recall that \( \phi(\omega) \) depends on \( h(\omega) \), whose variability depends on both the level of heterogeneity
in the economy and the risk of the economy that drives the variation in $\omega$. When consumption increases with respect to the habit, the “expected” risk-aversion in the economy decreases, and this has a positive impact on the interest rate. Since these two terms work in opposite directions, the only way in which a decrease of the variability of the risk-free rate happens is if the precautionary savings part varies enough to offset part of the variation coming from the habit incentive. This is possible only when the risk is high and the level of heterogeneity in the economy is also high.

Using the expression for $\tilde{m}$ we can also approximate well the maximum possible Sharpe ratio in the economy. The maximum Sharpe ratio, that we denote $\overline{SR}$, is obtained when an asset (or portfolio of assets) is conditionally perfectly correlated with consumption growth. Since $\tilde{m}$ is conditionally normally distributed, the maximum Sharpe ratio, which is derived from the usual Euler equation on excess returns, takes the following simple form,

$$\overline{SR}(\omega_t) \simeq \sqrt{e^{\gamma^2 \sigma^2 [1+\phi(\omega_t)]^2} - 1}.$$  

From this expression we see how the function $\phi(\omega_t)$ relates directly to the variation in the price of risk. We also observe that the unconditional average of the Sharpe ratio is not affected substantially by the variation of $\phi$, and hence by the level of heterogeneity in the economy.

## 5 Quantifying The Effect of Agent-Heterogeneity

In this part of the paper we try to assess quantitatively up to what extent risk-aversion heterogeneity can explain the value of financial variables observed in the economy. In particular, we study the price of risk, the equity premium, the risk-free rate, the price dividend ratio and the conditional volatility of stock returns. The main economic implications of the model come from two key elements. First, the persistent habit, which is able to produce persistence in the price dividend ratio. Second, the endogenously generated varying conditional volatility of the stochastic discount factor due to the heterogeneity in risk-preferences. The main question we try to address with this model is whether risk-preference heterogeneity can produce enough variation in the price of risk so as to be able to explain the low variability in the risk-free rate, the variation in equity premia and the long-run predictability of excess returns. After we calibrate the heterogeneous-agent economy, we compare its predictions to a homogeneous-agent -but otherwise identical- economy. This exercise allows us to quantify the marginal effect resulting from the heterogeneity of risk-aversion. We next explain the methodology we follow to calibrate our model to real data.
5.1 Calibration Procedure

As we have discussed, the price impact of risk-aversion heterogeneity depends on several parameters. First and foremost, it depends on the level of heterogeneity, measured by the standard deviation of risk-tolerance at the average state. More heterogeneity induces agents to take more extreme positions, and this leads to higher variability in the cross-sectional wealth and consumption distributions, and hence higher variability in the risk-aversion of the representative agent of the economy. We parameterize these distributions using the empirical findings of Kimball, Sahm and Shapiro (2007). The second channel through which agent heterogeneity affects prices is the unconditional volatility of the state. If the state of the economy is very volatile, then there is more wealth re-distribution over time and, therefore, more volatility in the risk-aversion of the economy. The unconditional volatility of our state variable depends on the persistence parameter $\lambda$ and the volatility of consumption growth. To estimate the mean and standard deviation of aggregate consumption we use NIPA data on real consumption growth between 1930 and 2005. The persistence parameter is not directly observable, but can be selected so as to match the persistence it induces in price-dividend ratios: We estimate $\lambda$ by fitting the model implied price-dividend ratio autocorrelation function (up to lag 7) to the autocorrelation parameter found in the data. The price-dividend ratio data we use is the annual series of Boudoukh et al. (2007), which includes common share repurchases from cash flow statements.

Both the persistence parameter $\lambda$ and the habit parameter $\rho$ have a strong effect on the unconditional volatilities of the price-dividend ratio and the risk-free rate. In our calibration exercise we include both volatilities. We have already listed our source for the price-dividend ratio. For the (real) risk-free rate we take the yield of the 3-month Treasury bill after subtracting the realized inflation provided in NIPA. The risk-free rate volatility for the full sample is slightly over 3%. The post-war period, during which inflation was more predictable, the risk-free rate volatility drops to a bit less than 2%. For this reason, we put a smaller weight on the risk-free rate volatility in the calibration exercise.

The last parameter we need to calibrate is the subjective discount factor $\delta$. This parameter affects the average level of the price-dividend ratio, as well as the average risk-free rate. Since we are unable to fit both of them at the same time, following the emphasis of the literature, we exclude the average risk-free rate and focus on the average price-dividend ratio. This underscores the inability of the model to explain the average excess return found in the data. As we will see, in order to produce an excess return high enough, we need to assume a volatility of consumption growth significantly larger than the one estimated from the data. As we have explained, such an additional real risk amplifies the effect of risk-aversion heterogeneity. For that reason, in our calibration exercise we include the average excess return, but with a small weight.
The model solves for the price-dividend ratio, $pd$, and risk-free rate, $r^f$, endogenously. Then, for a given set of parameters, we take the observed consumption growth series and the estimated value for $\lambda$ and generate a time series for $\omega$. We use this time-series to compute the model-implied time-series for $pd$ and $r^f$. From the price-dividend time-series and the dividend growth time-series we derive the market returns.

We calibrate our model for three different sets of parameters that we henceforth call models (or parameterizations) 1, 2 and 3. These models differ in the parameter values that explain consumption growth and dividend growth, as well as the parameters that characterize the distribution of agents. Models 1 and 2 assume the distribution of agents estimated using the distribution of Kimball, Sahm and Shapiro (2007), with $\bar{\gamma} = 5.17$ and $\bar{\nu} = 0.13$. Models 1 and 3 assume the estimated consumption growth parameters, and that dividend and consumption are the same. Model 2 uses instead the estimated statistics of the dividend growth process, which displays a smaller average than that of consumption growth (2.40% to 3.12%) and substantially higher volatility (13.73% to 2.18%); its correlation with the consumption growth time-series is 0.09. The objective of the third model is to produce a sizable equity premium, similar to the one observed in the economy; we choose the parameters of this model with that objective in mind: we assume that volatility of consumption growth is 2.5 times the estimated level, and that the standard deviation of risk-tolerance is twice as high as that estimated using Kimball, Sahm and Shapiro (2007).

The parameters values of $\lambda$, $\rho$ and $\delta$ minimize the following objective,

$$J_j(\lambda, \rho, \delta) = [\mu_j(pd) - \mu(pd)]^2 + [\sigma_j(pd) - \sigma(pd)]^2 + \sum_{i=1}^{7} [acf_j(pd) - acf^i(pd)]^2$$

$$+ w_e[\mu_j(r^c) - \mu(r^c)]^2 + w_f \left[\sigma_j(r^f) - \sigma(r^f)\right]^2,$$  \hspace{1cm} (18)

for each model $j$. $\mu_j(x)$ and $\sigma_j(x)$ denote the model implied (for the observed consumption growth), average and standard deviation, respectively, of the time series of a variable $x$. $acf^i$ denotes the autocorrelation for lag $i$. $\mu$, $\sigma$ and $acf$ denote the data estimated values. In table 1 we collect the values of the parameters for each model both those assumed and those computed according to (18). The estimates for models 1 and 2 are almost identical. We estimate the persistence parameter $\lambda$ as explained before, and we find it to be 0.12, similar to the value in Campbell and Cochrane (1999). The persistence of the price-dividend ratio in the model implied time-series (not shown) is in fact slightly higher than the persistence in the data. Model 2 requires a slightly higher (in absolute value) habit parameter value in order to capture the unconditional volatility of the price-dividend ratio. Due to the low correlation of the dividend growth with consumption growth, the systematic risk of the market security is effectively lower in model 2 than in 1. In model 3 the resulting value for $\lambda$ is smaller (more persistence), while
5.2 Analysis

The main purpose of this section is to study whether time variation in the risk-aversion of the economy coming from agent heterogeneity can produce the asset pricing facts that Campbell and Cochrane (1999) focus on. These facts are the high and counter-cyclical equity premium, the high variability and persistence of the price-dividend ratio, the low volatility of the risk-free rate and the price-dividend ratio’s ability to predict long-run excess returns. We show first that under the baseline calibration of the model the impact of risk-aversion heterogeneity is negligible. The reason is twofold: First the consumption risk is small, and even with a persistent state variable, the wealth reallocation in the economy is minimal. Second, the level of risk-aversion heterogeneity as estimated by Kimball, Sahm and Shapiro (2007) is again quite small and the wealth reallocation does not lead to significant changes in the risk-aversion of the economy.

One might argue that the fundamental risk is higher than consumption risk and that maybe the level of heterogeneity is not well captured by the empirical study of Kimball, Sahm and Shapiro (2007). We increase both considerably and look at the new predictions. The model performs better when compared to the data, and heterogeneity has a noticeable effect on prices. However, the risk-free rate is still high on average and quite volatile, the greater part of the equity premium is still term-premium and the predictability of long-run excess returns does not come close to that found in the data.

In our analysis, we first study price and return unconditional statistics (table 2). For each model we generate two sets of results: (i) model statistics, that correspond to the long-run unconditional values as time tends to infinity; (ii) historical simulation statistics that correspond to the sample values estimated from the model implied time-series. We emphasize that the model implied time-series is generated using the model solutions of the pd-ratio and rf-rate and the historical data on consumption and dividend growth. These time-series are shown for each model in figures 4, 5 and 6 respectively.

We then consider autocorrelations of the pd-ratio and the excess return, as well as the pd-ratio predictability regressions (tables 3, 4 and 5). These statistics are the result of simulating the economy 1000 times. For each simulation we generate 75 years of annual artificial data which corresponds to the frequency and length of our real data. The reported statistics are the averages of the estimated values of each simulation. Finally we analyze the model implied conditional price and return statistics in figures 7, 8 and 9, each corresponding to one of the three parametrizations.
5.2.1 Pricing effect of risk-aversion heterogeneity

The top panel of table 2 compares unconditional price and return statistics of the data and model 1. In the heterogeneous economy, the variability, as well as the level of the price-dividend ratio, is fitted relatively well. The average market return is predicted at a bit more than 1% higher than what is found in the data. The model completely misses the average risk free rate, predicted at around 7.22%; the historical simulation average (explained before) is around 6% while the data average is less than 1%. This is the result of the inability of the model to generate enough conditional volatility for the stochastic discount factor that would explain both the high equity premium along with the low risk-free rate. The historical simulation captures the return variability but the model predicts a number around 3% lower than the data return variability. The risk-free rate, on the other hand, is more volatile in the model by a bit more than 1%. From the previous results, it is not surprising that the model predicts a very small average Sharpe ratio of around 0.11 while the data average is more than three times higher. The results from the homogeneous economy of the first parametrization are almost identical to these. The differences are negligible and, therefore, the two economies cannot be separated based on such data. The endogenous variation in the volatility of the stochastic discount factor (function $\phi$ and figure 3) generated by the variation in the risk-aversion of the economy is so small that it cannot be detected either in the risk-free rates, the risk premia, or the return volatilities.

Figure 4 shows the model 1 implied time-series of the $pd$-ratio and $r^f$-rate for both the heterogeneous agents and the homogeneous agents economies. From the theoretical and empirical results we have obtained so far it is no surprise that the two time-series are indistinguishable. The two economies are also indistinguishable when it comes to the autocorrelation coefficients shown in the top panels of tables 3 and 4. There is a slight difference in the predictability regressions in the top panel of table 5. The data predictability starts with a modest value of around 7% increases monotonically with horizon and reaches a level of 37%. For the homogeneous economy the predictability starts at 0.13% and goes up to 5.81%, while for the heterogeneous economy these numbers range from 0.15% to 5.87%.

In figure 7 we plot some conditional statistics. The state variable is shown within four standard deviations from the unconditional mean. The risk-aversion of the representative agent varies little, as it ranges from slightly less than 5 to only 5.4. The risk-aversion coefficient for the homogeneous economy is 5.17. For this reason, the curvature in the pricing kernel is almost unnoticeable. From lemma 1 we know that the slope of the pricing kernel is equal to the consumption-weighted harmonic average risk-aversion in the economy. Similarly, the price-dividend ratio, as well as the risk-free rate of the homogeneous and heterogeneous economies of model 1, are once more almost indistinguishable.
The curvature of the price-dividend ratio function is closely related to the conditional return volatility. When the price-dividend ratio is flatter around the expected future state, then the return volatility is smaller. Since heterogeneity in the economy makes the $pd$-ratio slightly flatter during expansions and slightly steeper during recessions, the conditional volatility of returns becomes slightly more counter-cyclical with heterogeneity. The reason why the $pd$-ratio behaves in this way is the following. When $\omega$ increases, the price of consumption today drops in relation to the future, and agents want to postpone consumption. Hence all asset prices increase. However, the discount rates decrease at a decreasing rate, since the risk-aversion of the economy decreases as the state of the economy improves, and this makes the $pd$-ratio flatter when the price of consumption is low.

Finally we look at the variation in the Sharpe ratio. For the homogeneous agent economy the variation is only due to the conditional correlation between the market returns and consumption growth. The maximum Sharpe ratio in an economy where consumption growth is iid normal is constant. For model 1, the Sharpe ratio of the heterogeneous economy becomes more counter-cyclical, but only marginally, since it decreases from around 0.12 to 0.105, reflecting again the small variability in the risk-aversion of the economy.

5.2.2 Abnormally high risk and level of heterogeneity

We next show that even when we multiply the consumption risk estimated from the data by two and a half times, and we double the level of heterogeneity found by Kimball, Sahm and Shapiro (2007) (parametrization 3), the heterogeneous agents economy is not able to offer significantly better predictions than the homogeneous agents economy. First, we consider the statistics (bottom panel of table 2), and we observe that heterogeneity increases the equity premium by about 0.5%. This parametrization manages to produce a sizeable equity premium of more than 6%, not very far away from the data, which is 7.7%. It also predicts a sizeable average Sharpe ratio of almost 0.3, while the average and volatility of the price-dividend ratio, as well as the volatility of the equity premium, are close to the true values. However, the homogeneous agents economy predicts an equity premium almost as high as the heterogeneous agents economy owing to the fact that the biggest part of the equity premium in this model is in fact term-premium rather than risk-premium. As Abel (1999) explains, the term-premium arises when the future discount rates are volatile, which is the case in our model. Furthermore, from the historical simulation statistics it might be argued that the homogeneous agents economy is as close, if not closer, to the data as the heterogeneous agents economy. With the exception of the average excess return and the volatility of the risk-free rate the rest of the statistics look marginally better for the homogeneous agents economy.

With respect to the model unconditional statistics, the average risk-free drops from 4.37% to 3.74% when we introduce heterogeneity. Even though this is a noticeable difference it is still
clearly different from the data. We emphasize again that the model is unable to fit the average price-dividend ratio and the average risk-free rate simultaneously, even for this parametrization. With this parametrization, heterogeneity also has a noticeable impact on the volatility of the risk-free rate. The estimated data variability of the risk-free rate is 3.8%, given that this sample includes periods of unusual volatility. For example the post war sample volatility of the risk-free rate is only around 2%. Hence, the 3.4% predicted by the heterogeneous agents economy of model 3 still seems high.

The main difference between the heterogeneous agents and the homogeneous agents economies relates to the variation in the price of risk, as shown in figure 9. While it is almost flat for the homogeneous agents economy, when we introduce heterogeneity it goes from around 0.6 down to almost 0.1. However, this quantity is not observable. What is observable, and is relevant to the variation of the equity premium is the long-run predictability of the the pd-ratio. The lower panel of table 5 shows that the heterogeneous economy does not explain a significantly greater level of predictability. For example the $R^2$ for the 7-year horizon is just less than 10% for the heterogeneous agents economy, while it is almost 6.5% for the homogeneous agents economy. In addition, in the lower panel of table 4 we observe that the heterogeneity does indeed have a noticeable impact on the counter-cyclicality of the excess returns, but still does not yield the level of mean-reversion shown in the data.

5.2.3 Additional findings

Figures 4, 5 and 6 show in the top panel the actual price-dividend ratio along with the model implied time series for both the heterogeneous and homogeneous economies. The first thing we observe is that the model implied time series of price-dividend ratio follows closely the true data. This stems from the high correlation between the time-series of $\omega$ generated from the true consumption growth process and the price-dividend ratio. The real evidence here is that positive consumption growth shocks are related to positive shocks in the price-dividend ratio, that persist for many years. The habit process assumed in this economy “captures” this behavior because of its persistence and its effect on the discount rates. When consumption is high in relation to habit the demand for financial assets increases and the subsequent increase in prices is expected to persist because it will take time for the habit to adjust to the increased consumption. The model, however, is unable to follow the true series during the period between 1930 and the first recession shown, as well as during the stock market boom of the 1990’s. The negative consumption growth of the 1930’s is not associated with an analogous decline in prices, while the prices during the 1990’s go up very fast, while consumption growth was around its average level. Despite this, it seems that a slowly moving habit process, along with a free utility parameter on habit, is able to fit the time variability in aggregate prices to some extent.

In practice, the stock market pays a dividend that has a low correlation with the aggregate
endowment. We take this into consideration in parametrization 2. We first observe from the unconditional statistics of table 2 and the model implied time-series in figures 4 and 5 that with a small change in the calibrated parameters this model is able to fit the data as well as model 1. This model falls apart in its long-run behavior relating to the equity premium.

6 Conclusion

Campbell and Cochrane (1999) consider a stochastic discount factor that can explain a number of well documented properties of asset prices. It is central to their model the assumption that risk-aversion is stochastic and counter-cyclical, however, for their model to be successful, they need risk-aversion to have a very high variation in a very broad range. Chan and Kogan (2002) show that a model with multiple agents with heterogeneous risk-aversion can produce the counter-cyclical pattern of the stochastic discount factor of Campbell and Cochrane (1999), as well as a varying Sharpe ratio.

In this paper we consider a model similar to that in Chan and Kogan (2002), we derive explicitly the risk-aversion coefficient of the representative agent and find that the variation required by the stochastic discount factor of Campbell and Cochrane (1999) is unlikely to be produced by such a model with reasonable (as observed in the economy) parameter values.

From our analysis, it appears that the sensitivity function of Campbell and Cochrane (1999) has to proxy for something other than the risk-aversion of the economy. A promising alternative would be to examine asset prices as they are formed on expectations or beliefs of investors about the underlying risks in the economy. The way they update these beliefs or the uncertainty they have about the conditional distribution might be able to explain why investors require such high compensation for every unit of ex-post risk they take and why the price of risk appears to vary so radically across time.

References


A Proofs

Below we provide the proofs of the propositions, lemmas, and corollaries found in the text. The preference assumption is that agents have time and state separable power preferences with external habit, and the running utility is given by

\[ u(c, X|\gamma) = \frac{c^{1-\gamma}X^{\gamma-\rho} - 1}{1-\gamma} \]

where \( c \) denotes consumption. We denote by \( \alpha \) the consumption proportion \( c/Y \). Agents’ types are characterized by the initial distribution of wealth \( \theta(\gamma) \), which is a density function over the set of types \( \Gamma \). The state variable of the economy is the aggregate endowment habit ratio \( \omega = y - x \), where \( y \) and \( x \) are the natural logs of \( Y \) and \( X \) respectively. We also define the pricing kernel \( z_t = \log(p_t) + \rho x_t \).

Proof of Proposition 1. Given the habit process \((X_t, t \geq 0)\), the consumption price process \((p_t, t \geq 0)\), and the initial total wealth \( W_0 \), the optimization problem of an agent of type \( \gamma \) with initial wealth allocation \( \theta(\gamma) \) is given by,

\[
\max_{(c_t(\gamma), t \geq 0)} L(\gamma) = \mathbb{E}_0 \sum_{t \geq 0} \delta^t u(c_t, X_t|\gamma) - \lambda(\gamma)^{-\gamma} \left[ \mathbb{E}_0 \sum_{t \geq 0} \delta^t p_t c_t(\gamma) - \theta_0(\gamma)p_0W_0 \right]
\]

Due to our preference assumptions, the optimal solution is interior, and is characterized by the first order condition,

\[
\left( \frac{c_t(\gamma)}{X_t} \right)^{-\gamma} = \lambda(\gamma)^{-\gamma} p_t X_t^\rho, \quad \forall t \geq 0.
\]

and the equality of the inter-temporal budget constraint. Using the definition of \( z_t \) and the consumption proportion \( \alpha \), we arrive at the optimality condition (6). We then substitute in the budget constraint, satisfied with inequality, the optimal consumption share from (6), to arrive at the equilibrium value for \( \lambda(\gamma) \).

Proof of Corollary 1. Market equilibrium is obtained when the optimal consumption share in each period as function of the type \( \gamma \) integrates to one. Given the optimality condition (6) it is evident that the pricing kernel is a function of the endowment habit ratio, \( z_t = z(\omega_t) \).

Proof of Corollary 2. The pricing kernel is normalized to have a value of zero at the average state, \( z(\bar{\omega}) = 0 \). The initial state is assumed to be the average state and hence expression (9) is easily derived from (6) at the initial state and (7). Since the equilibrium pricing kernel is a function of \( \omega \), so is the optimal consumption share as given by (6). At the average state it is given by

\[ \alpha(\bar{\omega}, \gamma) = \lambda(\gamma)e^{-\bar{\omega}}. \]
Using the previous result, we express \( \lambda(\gamma) \) in terms of \( \alpha(\bar{\omega}, \gamma) = \mathcal{P}_{\bar{\omega}}(1/\gamma) \) and substitute it in (8) to arrive at (10). We have replaced the coefficient of risk-aversion with the coefficient of risk-tolerance, \( \tau = 1/\gamma \).

\[ \square \]

**Proof of Lemma 1.** Differentiate expression (8) with respect to \( \omega \) to get

\[
0 = \int_{\Gamma} \left( -\frac{z'(\omega)}{\gamma} - 1 \right) \lambda(\gamma) \exp \left( -\frac{z(\omega)}{\gamma} - \omega \right) d\gamma
\]

and note that \( \mathcal{P}_{\omega}(\tau) = \lambda(1/\tau) \exp (-\tau z(\omega) - \omega) \) to derive (z2). Differentiate again with respect to \( \omega \) to get,

\[
0 = \int_{\Gamma} \left( -\frac{z''(\omega)}{\gamma} + \frac{z'(\omega)^2}{\gamma^2} + 2 \frac{z'(\omega)}{\gamma} + 1 \right) \mathcal{P}_{\omega}(1/\gamma)d\gamma
= -z''(\omega)E_{\omega}(\tau) + z'(\omega)E_{\omega}(\tau^2) - 2z'(\omega)E_{\omega}(\tau) + 1.
\]

Using the result for \( z'(\omega) \) and rearranging we get the second derivative of \( z \). Since the numerator is a variance term then the second derivative is positive.

For the rest of the results we assume explicitly that the set \( \Gamma \) has only positive values and is bounded above and below. Now, since the term \( E_{\omega}\left[\exp(-\tau z(\omega))\right] \) is strictly positive and finite for all finite and positive values of \( z(\omega) \), from (10) it has to be that \( z \) tends to minus and plus infinity, as \( \omega \) tends to plus and minus infinity, respectively. Let \( \gamma_{\text{max}} \) denote the maximum \( \gamma \) of the set \( \Gamma \), and consider the following ratio of optimal consumption shares,

\[
\frac{\alpha(\omega, \gamma_{\text{max}})}{\alpha(\omega, \gamma)} = \frac{\lambda(\gamma_{\text{max}})}{\lambda(\gamma)} \exp \left[ -z(\omega) \left( \frac{1}{\gamma_{\text{max}}} - \frac{1}{\gamma} \right) \right]
\]

Now since \( \lim_{\omega \to -\infty} z(\omega) = +\infty \) then from the previous ratio, we have to conclude that

\[
\lim_{\omega \to -\infty} \alpha(\omega, \gamma) = 0, \quad \forall \gamma < \gamma_{\text{max}}.
\]

Hence, the consumption distribution collapses to a spike at the maximum \( \gamma \). Similarly, when the state tends to plus infinity, the distribution collapses to the minimum \( \gamma \). Finally, since in either case the variance tends to zero, then the second derivative tends to zero as well.

\[ \square \]

**Proof of Proposition 2.** Given the habit process \( (X_t^r = X_t e^{\bar{\omega}}, t \geq 0) \), the consumption price process \( (p_t^r, t \geq 0) \) and the initial total wealth \( W_0^r \), the optimization problem of the representative agent is given by,

\[
\max_{(c_t, t \geq 0)} \mathcal{L}_r = \mathbb{E}_0 \sum_{t \geq 0} \delta^t u_r(c_t, X_t^r) - \lambda_r \left[ \mathbb{E}_0 \sum_{t \geq 0} \delta^t p_t^r c_t - p_0^r W_0^r \right]
\]

30
where
\[ u_r(c_t, X^r_t) = \frac{c_t^{1-\gamma(\omega_t)}(X^r_t)^{\gamma(\omega_t)-\rho}}{1 - \gamma(\omega_t)}, \]
\( \gamma(\omega) \) is the stochastic risk-aversion of the representative agent, and \( \omega \) is as before. Then, the representative agent optimally consumes the aggregate endowment, and the first order condition is given by,
\[ \left( \frac{Y_t}{X_t} \right)^{-\gamma(\omega_t)} e^{\omega[\gamma(\omega_t)-\rho]} = \lambda_r e^{z^r_t}, \]
where \( z^r_t = \log(p^r_t) + \rho x_t \). To normalize \( z^r \) to be zero at the average state the Lagrange multiplier has to equal \( e^{\rho \bar{\omega}} \).

**Proof of Corollary 3.** From the definitions of \( z^r \) and \( z \), the two pricing kernels are equal in all states iff the consumption processes \( p^r \) and \( p \) are equal. Therefore, the asset price processes of the two economies are equal if \( z^r \equiv z \). Hence, from Proposition 2 we must have that,
\[ \gamma(\omega) = -\frac{z(\omega)}{\omega - \bar{\omega}}, \quad \forall \omega \neq \bar{\omega}. \]
For continuity, we use l'Hôpital's rule to define \( \gamma(\omega) \) at the average state. \( \square \)

**Proof of Corollary 4.** Since
\[ \gamma(\omega) = -\frac{z(\omega)}{\omega - \bar{\omega}}, \]
the first and second derivatives are obtained using the first two derivatives of \( z \) from Lemma 1. In order to prove that \( \gamma'(\omega) \) is negative, we need to show that the expression in the square brackets \( \gamma(\omega) + z'(\omega) \geq 0 \) when \( \omega \geq \bar{\omega} \). From the convexity of function \( z \) and the fact that \( z(\bar{\omega}) = 0 \), we have that
\[ -z(\omega) + z'(\omega)(\omega - \bar{\omega}) > 0 \]
The result then obtains after dividing by \( (\omega - \bar{\omega}) \), and noting that the inequality switches when \( \omega < \bar{\omega} \). The limits of \( \gamma(\omega) \), \( \gamma'(\omega) \) and \( \gamma''(\omega) \) are straightforward applications of the limit properties of \( z \). \( \square \)

**Proof of Corollary 5.** If the risk-tolerance at the average state is gamma distributed with parameters \( (\kappa, \vartheta) \), from the moment generating function we know that,
\[ \mathbb{E}_{\bar{\omega}} [\exp(-\tau z(\omega))] = [1 + \vartheta z(\omega)]^{-\kappa}. \]
Using the equilibrium condition (10) and rearranging, we get,
\[ z(\omega) = \frac{\exp\left(-\frac{\omega - \bar{\omega}}{\kappa} \right) - 1}{\vartheta} \]
\[ \gamma \] denotes the inverse of the average risk-tolerance, and therefore is equal to \((\kappa \theta)^{-1}\), and \(\nu^2\) denotes the risk-tolerance variance, given by \(\kappa \theta^2\). Furthermore, we use \(\eta = (\gamma \nu)^2\) to obtain the result.

**Proof of Corollary 6.** From the definition of \(z\) we have that,

\[ m(\omega, \omega') = \log(\delta) + z(\omega') - z(\omega) - \rho(x' - x). \]

The habit process is given by,

\[ x' = \lambda y + (1 - \lambda)x \]

and therefore \(x' - x = \lambda \omega\). The result is then obtained once we substitute in the expression for \(z\) derived in Corollary 5. When \(\nu = 0\) then it is obvious from (10) that \(z(\omega) = -\gamma(\omega - \bar{\omega})\).

**B Computational Approach**

We outline below the numerical method used to compute the price-dividend ratio and the risk-free rate as functions of the state. The price-dividend ratio of the market security that pays a dividend stream that grows according to (2) is given by

\[ PD(\omega) = \delta E_{\omega} \left[ \exp \left( z(\omega') - z(\omega) - \rho \omega + \mu_d + \sigma_d \epsilon_d + \sigma_d \sqrt{1 - \rho^2} \epsilon_d \right) (PD(\omega') + 1) \right] \]

where \(\epsilon\) and \(\epsilon_d\) are independent and normally distributed variables. The distributional assumptions made here are not required, in general. The method can be adapted to any arbitrary distribution. Let the law of motion for \(\omega\) be some known function,

\[ \omega' = L(\omega, \epsilon). \]

Let \(\tilde{\delta} = \delta \exp (\mu_d + \frac{1}{2} \sigma^2 \omega (1 - \sigma^2))\), and also let,

\[ N(\omega, \epsilon) = \exp \left[ z(L(\omega, \epsilon)) - z(\omega) - \rho \lambda \omega + \sigma_d \epsilon \right]. \]

Then the price-dividend equation can be rewritten as follows:

\[ PD(\omega) = \tilde{\delta} \int_{-\infty}^{+\infty} \varphi(\epsilon) N(\omega, \epsilon) [PD(L(\omega, \epsilon)) + 1] d\epsilon, \]

where \(\varphi(\epsilon)\) is the standard normal density. We then discretize the state variable in a set of \(n\) values \(\Omega\) and compute the \(n \times n\) transition matrix \(\Pi = [\pi(\omega, \omega')]\) based on \(L\) and \(\varphi\). Let \(PD\) be the vector of price-dividend ratios and let \(N\) be the \(n \times n\) matrix computed from \(N(\omega, \epsilon)\). The price-dividend ratio equation can now be written simultaneously for all states in \(\Omega\) in the
following matrix form:

\[ \mathbf{PD} = \delta [\Pi \circ \mathbf{N}] (\mathbf{PD} + \mathbf{1}_n), \]

where \( \circ \) denotes the element-wise matrix multiplication and \( \mathbf{1}_n \) is the \( n \)-dimensional unit vector. After a matrix inversion the price-dividend ratio vector is computed according to,

\[ \mathbf{PD} = \delta [\mathbf{I}_n - (\Pi \circ \mathbf{N})]^{-1} \mathbf{1}_n, \]

where \( \mathbf{I}_n \) is the \( n \)-dimensional identity matrix.

The price of the risk-free bond that pays a unit of consumption the next period is computed in a similar fashion. Let

\[ M(\omega, \epsilon) = \delta \exp \left( z(L(\omega, \epsilon)) - z(\omega) - \rho \lambda \omega \right) \]

and let \( \mathbf{M} \) denote the square matrix for all the combination of values in the set \( \Omega \). Then the vector of bond price values is computed by

\[ \mathbf{P}^f = (\Pi \circ \mathbf{M})\mathbf{1}_n. \]

The risk-free rate is then computed as

\[ R^f(\omega) = 1/PD(\omega) - 1. \]

For the simulation part of our results we also approximate the price-dividend ratio function, as well as the risk-free rate function, with cubic-splines. The data used to estimate the piece-wise polynomial coefficients were the computed vectors \( \mathbf{PD} \) and \( \mathbf{P}^f \) and the vector of arguments \( \Omega \). For our computational results we use a specific assumption for the state variable that gives the following law of motion,

\[ \omega' - \omega = -\lambda(\omega - \bar{\omega}) + \sigma \epsilon, \]

where \( \sigma \) is the volatility of consumption growth. Hence the state variable is unconditionally normally distributed. The set \( \Omega \) used was an equidistant grid of 251 values where the minimum and the maximum values are 8 standard deviations, left and right from the mean, respectively.
The parameters with * were chosen by minimizing an objective function that measures the squared errors in certain quantities between the data and those of the model implied time-series. The quantities were the mean, standard deviation and autocorrelation function of the price-dividend ratio, the standard deviation of the risk-free rate and the mean equity premium. The model implied time-series of the price-dividend ratio and the risk-free rate were generated by using the true consumption and dividend growth series.

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Table 2: Price and return statistics

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<td>Model</td>
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The historical simulations were generated by using the true consumption and dividend growth series. The homogeneous economy of each parametrization sets the level of heterogeneity to zero by setting $\nu = 0.$
Table 3: Autocorrelations of price-dividend ratio

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<td>Homog.ec. 1</td>
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The autocorrelation function for each model was estimated by averaging the sample estimates of 1000 simulations of the same length as the data.

Table 4: Autocorrelations of excess returns

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<td>-0.02</td>
<td>-0.01</td>
<td>-0.02</td>
<td>-0.01</td>
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<tr>
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<td>-0.02</td>
<td>-0.01</td>
<td>-0.01</td>
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<tr>
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<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
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<tr>
<td>Heter.ec. 3</td>
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<td>-0.03</td>
<td>-0.03</td>
<td>-0.02</td>
<td>-0.01</td>
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<td>Homog.ec. 3</td>
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<td>-0.01</td>
<td>-0.02</td>
<td>-0.01</td>
<td>-0.01</td>
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</table>

The autocorrelation function for each model was estimated by averaging the sample estimates of 1000 simulations of the same length as the data.
Table 5: Long-run predictability regressions

<table>
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<tr>
<th>Horizon</th>
<th>Data</th>
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<tr>
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<td>Homogeneous ec.</td>
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<tr>
<td></td>
<td>coef.</td>
<td>std.er.</td>
<td>coef.</td>
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<tr>
<td></td>
<td></td>
<td>$R^2$</td>
<td>coef.</td>
</tr>
<tr>
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<td>(0.05)</td>
<td>6.86</td>
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<tr>
<td>2</td>
<td>-0.30</td>
<td>(0.10)</td>
<td>11.29</td>
</tr>
<tr>
<td>3</td>
<td>-0.41</td>
<td>(0.12)</td>
<td>18.66</td>
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<tr>
<td>5</td>
<td>-0.61</td>
<td>(0.16)</td>
<td>29.07</td>
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<tr>
<td>7</td>
<td>-0.82</td>
<td>(0.21)</td>
<td>36.82</td>
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</table>

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<table>
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<td>coef.</td>
<td>std.er.</td>
<td>coef.</td>
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<td></td>
<td>$R^2$</td>
<td>coef.</td>
</tr>
<tr>
<td>1</td>
<td>-0.02</td>
<td>(0.07)</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>-0.00</td>
<td>(0.12)</td>
<td>1.22</td>
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<tr>
<td>3</td>
<td>0.01</td>
<td>(0.17)</td>
<td>2.38</td>
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<tr>
<td>5</td>
<td>0.01</td>
<td>(0.25)</td>
<td>4.24</td>
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<tr>
<td>7</td>
<td>0.00</td>
<td>(0.30)</td>
<td>5.68</td>
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</tbody>
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<table>
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<td></td>
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</tr>
<tr>
<td></td>
<td>coef.</td>
<td>std.er.</td>
<td>coef.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R^2$</td>
<td>coef.</td>
</tr>
<tr>
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<td>-0.08</td>
<td>(0.05)</td>
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</tr>
<tr>
<td>2</td>
<td>-0.10</td>
<td>(0.09)</td>
<td>3.50</td>
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<tr>
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<td>-0.12</td>
<td>(0.13)</td>
<td>4.98</td>
</tr>
<tr>
<td>5</td>
<td>-0.16</td>
<td>(0.18)</td>
<td>7.53</td>
</tr>
<tr>
<td>7</td>
<td>-0.20</td>
<td>(0.22)</td>
<td>9.67</td>
</tr>
</tbody>
</table>

Regressions were run with log price-dividend ratio as the predictive variable and $j$-period realized excess log returns on the left hand side,

$$r_{t+j}^e = \beta_0 + \beta_1 pd_t + \varepsilon_{t+j}.$$

The regression estimates for each model are the averages of the sample estimates of 1000 simulations of the same length as the data. The standard errors were corrected using a GMM procedure and the Newey-West weighting scheme.
Given the estimated risk-tolerance distribution $\log(\tau) \sim N(-1.84, 0.73)$, by Kimball, Sahm and Shapiro (2007) we fit the gamma distribution by minimizing the distance between the two distribution functions,

$$\min_{\kappa, \theta} \int_{0}^{+\infty} [F_1(\tau) - F_2(\tau)]^2 d\tau$$

where $F_1$ and $F_2$ are the log-normal and gamma distribution functions. The integral was approximated numerically. The estimated parameters where $\kappa = 2.2474$ and $\theta = 0.0860$. 
Figure 2: Variation in the representative agent risk aversion.

Function $h$ is the multiplier of the risk-aversion of the representative agent as the economy deviates from the average state. For the three plots we multiply the estimated standard deviation of risk-tolerance with the corresponding number while the average risk-tolerance is the one estimated.

Figure 3: Variation in the conditional volatility of $\tilde{m}$

The conditional volatility of the approximate pricing kernel is,

$$\tilde{\gamma} [1 + \phi(\omega)] \sigma.$$ 

For the three plots we multiply the estimated standard deviation of risk-tolerance with the corresponding number while the average risk-tolerance is the one estimated. Parameter $\lambda$ was set to 0.13 for these plots.
Figure 4: Data and model 1 implied time series of price-dividend ratio and risk-free rate

The model implied time-series of the price-dividend ratio and the risk-free rate were generated by using the true consumption and dividend growth series.
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Figure 7: Model parametrization 1

The range of the state variable is four standard deviations around its mean.
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