Asset Pricing with Dynamic Margin Constraints *

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Abstract

This paper studies asset pricing implications of endogenously determined time-varying margin requirements in an exchange economy with heterogeneous investors. It demonstrates that margin constraints reduce the risk-free rate and the volatility of returns, but increase the risk premium, the market price of risk, and the value of risky assets. Many of these results are surprisingly robust to the form of margin requirements and their dependence on market conditions, as well as to the possibility of random jumps in the level of constraints. The hedging demand of potentially constrained investors amplifies the effects.

Keywords: exchange economy, heterogeneous investors, margin requirements, general equilibrium

JEL classification: G11, G12

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Margin debt is pervasive in the financial industry and used by many investors who build leveraged portfolios. To protect themselves against losses caused by adverse price movements, lenders often impose margin requirements which put limits on portfolio leverage. The tightness of margin constraints may be regulated legislatively or determined by an agreement between borrower and lender. For instance, Regulation T of the Federal Reserve sets initial margin requirements on equity investments, i.e. it restricts the amount of credit that investors can get from brokers for purchasing stocks. Since January 1974 the initial margin requirement is 50%. Besides that, brokerage firms can impose additional restrictions on how leveraged their customers can be. The margin requirements imposed by prime brokerage firms are of paramount importance for sophisticated investors like hedge funds who typically structure loans as repurchase agreements (repos).

In many circumstances, the tightness of margin constraints is not fixed, but determined by current market conditions. For example, the terms of repo agreements between hedge funds and prime brokerage firms substantially vary over time since the level of margin requirements is an outcome of continuous bilateral negotiations. Typically, prime brokers are permitted to tighten margin requirements (modify the collateral haircut) at their own discretion and even without advance notice. When markets are strained or excessively volatile, prime brokers are more likely to use their discretionary right to adjust margin requirements. For exchange-traded derivatives, margins are also often adjusted in response to changing market conditions. For example, CME Group uses Standard Portfolio Analysis of Risk Performance (SPAN) methodology to compute margins. The major variable in this approach is volatility of each futures market. Typically, the margin requirements become tighter when volatility increases.

Although time variation of margin requirements is a common phenomenon, the equilibrium asset pricing implications of such variation have received little attention in the academic literature. This paper intends to fill the gap. The main objective of my analysis is to study the impact of margin requirements that are endogenously determined by market conditions, which in turn are determined by optimal behavior of investors who take into account the time variation in margin requirements. As a laboratory for analysis, I consider a pure exchange economy with heterogeneous investors who have CRRA preferences with different levels of risk aversion. In contrast to many existing models with heterogeneous investors assuming that at least one type of investors has logarithmic preferences, I do not impose restrictions on the levels of risk aversion and account for the hedging demand produced by time-varying constraints. In the equilibrium, the agents who are less risk averse borrow from more risk averse agents but their leverage cannot exceed a certain level, i.e. they face margin constraints. In the main analysis, margin requirements are deterministically linked to endogenously determined current market conditions (e.g., the volatility of returns). Since market conditions vary across states, margin constraints are also state-dependent. In addition, I consider a version of the model in which margin requirements can experience exogenous and unhedgeable jumps.

The paper contains several results. First, I demonstrate that constraints significantly change the equilibrium: in the economy with margins the risk-free rate and the volatility of returns are lower than in the unconstrained economy, whereas the expected excess returns, the market price of risk,
and asset valuations are higher. Moreover, I conclude that these effects are very robust and hold for a wide class of margin constraints whose levels are linked to current market conditions.

It is particularly interesting that portfolio constraints decrease the volatility of returns when they bind, although the heterogeneity of agents can produce the volatility of returns exceeding the volatility of fundamentals. I show that the volatility reduction is a very general effect whose key mechanism is quite intuitive: binding margin constraints effectively reduce the heterogeneity of investors and curb the excessive volatility produced by optimal risk sharing. This result implies that an increase in return volatility which is often attributed to binding margin constraints cannot be produced by margin constraints themselves even if they are dynamical, and additional market imperfections should be in play. For example, the volatility of returns can increase due to the presence of “disutility of trading” (Aiyagari and Gertler, 1999), asymmetry of information (Brunnermeier and Pedersen, 2009), stochastic proportion of less risk averse investors in the economy (Kupiec and Sharpe, 1991), or stochastic enforcement of margin constraints (Rytchkov, 2008).

Other effects documented in the paper also admit intuitive economic explanations. In particular, the increase in expected excess returns and the market price of risk produced by constraints can be attributed to the redistribution of the risky asset from less risk averse investors to more risk averse investors who require higher risk premium. The decrease in the risk-free rate is caused by the reduction in the demand for credit: since margin constraints limit the leverage of less risk averse investors, they borrow less, and this drives the risk-free rate downward. Because the decrease in the risk-free rate is more pronounced than the increase in the risk premium, the total cost of capital decreases driving the value of the risky asset up. Thus, in contrast to a widespread belief, binding portfolio constraints in a general equilibrium without additional market imperfections increase asset prices, not decrease them.

Next, I explore the impact of hedging demand of constrained investors on the equilibrium properties. To this purpose, I compare the economy with its modified analog in which less risk averse investors are replaced by myopic investors who do not anticipate changes in investment opportunities. I demonstrate that without margin constraints the hedging demand does not affect the equilibrium. However, hedging of time-varying investment opportunities by constrained investors may strongly affect the equilibrium, magnifying the impact of constraints. This result emphasizes the importance of the deviation from the widely used assumption that constrained investors have logarithmic preferences, which produces only a myopic demand. I also show that the hedging demand matters only when margins actually bind and there is almost no anticipation effect (except for the price-dividend ratio for which it is quite small).

In the main version of the model, the relation between margin constraints and market conditions (e.g., the volatility of returns) is deterministic. Since volatility is a continuous function of the state variable, margin requirements also change continuously. However, in practice changes in margin requirements are often discrete and unpredictable. To assess the sensitivity of my results to incorporating these realistic features, I consider an extension of the model in which the level of constraints is determined by an additional binary state variable evolving as a Markov chain. Since there is no additional tradable instrument, the market in the model with stochastic margins is incomplete even for uncon-
strained investors. My analysis of the extended model shows that the possibility of discrete changes in constraints does not invalidate previous conclusions. In particular, although jumps in constraints produce jumps in returns and the state variable, investors only weakly adjust their portfolios leaving equilibrium statistics almost unaffected. The main cause of this insensitivity is that the presence of margin constraints has only a small effect on the value of the risky asset, so the jumps in returns produced by shifts in constraints are also quite small and almost ignored by rational investors.

This paper is related to a vast literature exploring the structure of equilibrium in pure exchange economies with rational investors who have heterogeneous preferences (e.g., Berrada, Hugonnier, and Rindisbacher, 2007; Chan and Kogan, 2002; Cvitanic and Malamud, 2010; Dieckmann and Gallmeyer, 2005; Dumas, 1989; Kogan, Makarov, and Uppal, 2007; Longstaff and Wang, 2008; Wang, 1996; Weinbaum, 2009; Xiourros and Zapatero, 2010) and/or heterogeneous beliefs (e.g., Basak, 2000, 2005; Bhamra and Uppal, 2010; Chen, Joslin, and Tran, 2011; Cvitanic, Jouini, Malamud, and Napp, 2009; David, 2008; Detemple and Murthy, 1994; Dieckmann, 2010; Dieckmann and Gallmeyer, 2005; Dumas, Kurshev, and Uppal, 2009; Kogan, Ross, Wang, and Westerfield, 2006; Li, 2007; Yan, 2008; Zapatero, 1998).  

It also extends the literature studying the impact of constant portfolio constraints in the general equilibrium framework (e.g., Basak and Croitoru, 2000; Basak and Cuoco, 1998; Detemple and Murthy, 1997; Gallmeyer and Hollifield, 2008; Garleanu and Pedersen, 2011; Pavlova and Rigobon, 2008; Shapiro, 2002). The majority of existing papers assume that constrained investors have logarithmic preferences. Although this assumption often delivers tractability, it limits the scope of the analysis making it impossible to explore the impact of hedging motives on the properties of the equilibrium. The exception is Chabakauri (2010), who extends the analysis to models with constrained investors having arbitrary levels of risk aversion.

One of the main contributions of this paper is the analysis of time-varying endogenous margin constraints. Although such constraints are common in financial markets, they are surprisingly underexplored in the academic literature. Basak and Shapiro (2001) study the optimal portfolio policy and equilibrium implications of restrictions on the value-at-risk over a given finite horizon which can be interpreted as a form of dynamic portfolio constraints. Danielsson, Shin, and Zigrand (2004) and Danielsson and Zigrand (2008) examine the impact of the value-at-risk constraint imposed over a one-period horizon. Danielsson, Shin, and Zigrand (2011) and Prieto (2011) explore the equilibrium effect of constraints having the form of restrictions on the volatility of portfolio returns. Yuan (2005) constructs a one-period rational expectation model in which borrowing constraint is a function of price.

It is interesting to compare my results with conclusions of several other papers exploring the link between margin constraints and the volatility of stock returns. Using an OLG framework with heterogeneous agents, Kupiec and Sharpe (1991) show that the relation between constant margin requirements and stock price volatility is ambiguous and depends on the source of variation in the aggregate risk-bearing capacity. In particular, if the proportion of more risk averse investors exogenously increases,

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1Several studies including Danielsson, Shin, and Zigrand (2011), Kyle and Xiong (2001), Xiong (2001) examine models in which one type of investors is characterized by an aggregate demand curve.
and stochastically changes over time, the imposition of constant margin requirements can increase the volatility of returns. However, to arrive at this conclusion Kupiec and Sharpe (1991) need to assume that the risk-free rate is exogenously fixed. Chowdhry and Nanda (1998) show how in a two-period model constant margin requirements can produce market instability, which is understood as a multiplicity of prices clearing the market. One of the important assumptions made by Chowdhry and Nanda (1998) is the existence of storage technology, which is equivalent to the exogeneity of the risk-free rate. Aiyagari and Gertler (1999) introduce the “disutility of trading” in the preferences of one group of agents and demonstrate that in this setting even constant margin requirements can amplify fundamental shocks and increase the volatility of stock prices. Effectively, constraints increase the agents’ degree of risk aversion and induce destabilizing trading by investors who want to avoid forced selling when margin constraints bind. Brunnermeier and Pedersen (2009) build a four-period model in which the tightness of margin constraints is endogenously determined within the model and show that the asymmetry of information between leveraged traders (“speculators”) and financiers can cause margin constraints to have a destabilizing effect on asset prices. Thus, in the presence of constant or time-varying margin requirements the volatility of returns can be higher than in the corresponding unconstrained economy, but only if there are other market imperfections and often only in a partial equilibrium setting assuming, for example, the exogeneity of the risk-free rate.

There is a long-lasting discussion in the empirical literature about the relation between margin requirements mandated by the Federal Reserve and stock market volatility. Many authors claim that there is no robust evidence in favor of margin requirements as an effective policy instrument (e.g., Moore, 1966; Schwert, 1989; Hsieh and Miller, 1990). This point of view is challenged by Hardouvelis (1990), who argues that historically higher margin requirements are associated with lower stock market volatility. This conclusion is partially reaffirmed by Hardouvelis and Theodossiou (2002) who demonstrate that there is a negative causal relation between tightness of margins and volatility in bull markets and no relation in bear markets. Similarly, using data from the Tokyo Stock Exchange, Hardouvelis and Peristiani (1992) show that an increase in margin requirements is followed by decline in volatility of daily returns. My theoretical results can provide an explanation to these findings.

The rest of the paper is organized as follows. Section I presents the main model featuring endogenously determined margin constraints and describes the structure of the equilibrium. Sections II provides the analysis of the model and contains main numerical results of the paper. Section III is devoted to an extension of the model in which margin constraints can experience jumps. Section IV concludes by summarizing the main results. Appendix A collects all proofs and Appendix B provides details on numerical techniques used in the paper.

I. Model

In this section, I lay out the main model of the paper. This is a continuous-time pure exchange economy with an infinite horizon and heterogeneous investors. The key component of the model is state-dependent margin constraints whose level is endogenously determined by equilibrium conditions.
A. Assets

There are two assets in the economy. The first one is a risk-free asset in zero net supply with the rate of return $r_t$ determined in equilibrium. The other asset is risky, and an aggregate supply of this asset is normalized to 1. The risky asset produces a consumption good paid as a dividend $D_t$. The flow of the dividend follows a geometric Brownian motion

$$
\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t, \quad (1)
$$

where $B_t$ is the standard Brownian motion defined on a probability space $(\Omega, F, P)$. The drift $\mu_D$ and volatility $\sigma_D$ are constant. In general, the price of the risky asset $S_t$ is a function of the current dividend $D_t$ and state variables. It is convenient to introduce the cum dividend excess return on one dollar invested in the risky asset:

$$
dQ_t = \frac{dS_t + D_t dt}{S_t} - r_t dt.
$$

In general, the stochastic differential equation for $Q_t$ can be written as

$$
dQ_t = \mu_Q dt + \sigma_Q dB_t, \quad (2)
$$

where $\mu_Q$ is the risk premium and $\sigma_Q$ is the instantaneous volatility of returns. Both $\mu_Q$ and $\sigma_Q$ are functions of state variables and determined by equilibrium conditions.

B. Agents

There are two types of agents in the model, and the agents of one type (type A) are more risk averse than the agents of the other type (type B) (following Garleanu and Pedersen (2011) they can be dubbed as Averse and Brave). There is an infinite number of identical investors of both types, and they all behave competitively. All investors are rational, use all available information, and choose portfolio policies and consumption streams that maximize their utility functions. It is assumed that both types of investors have standard CRRA preferences

$$
U_{it} = E_t \left[ \int_t^\infty e^{-\beta_t u(C_{is})} ds \right], \quad u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma}, \quad (3)
$$

but in general may differ in terms of the coefficient of risk aversion $\gamma$ and the time preference parameter $\beta$: $\gamma = \gamma_A$, $\beta = \beta_A$ for type A investors and $\gamma = \gamma_B$, $\beta = \beta_B$ for type B investors, where $\gamma_A > \gamma_B$. Given the dividend process from Eq. (1), this expected utility is uniformly bounded if

$$
\beta > \max \left( 0, (1-\gamma)(\mu_D - \frac{1}{2}\gamma \sigma_D^2) \right),
$$

and this condition should be satisfied for both types of investors.

The heterogeneity of investors plays a crucial role in the model. Type B investors can be thought of as hedge funds or sophisticated proprietary traders who are professional risk-takers and who actively invest in the risky asset. However, when type B investors need to sell a part of their portfolio (there
may be several reasons for that including binding margin constraints), investors of type A take the other side of the trade. Thus, type A investors are buyers of last resort. It is convenient to associate them with pension funds, sovereign funds, or large individual investors. Trading and risk sharing between type A and type B investors is responsible for the majority of interesting effects identified in the model.

C. State variable

Following the literature on equilibrium asset pricing models with heterogeneous agents, I introduce a relative consumption share of one type of investors as a state variable. Specifically, define \( s_t \) as a share of type B investors in the aggregate consumption:

\[
    s_t = \frac{C_t^B}{C_t^A + C_t^B} = \frac{C_t^B}{D_t},
\]

where \( C_t^A \) and \( C_t^B \) are aggregate equilibrium consumption streams of type A and type B investors, respectively. Later it will be shown that this is indeed the only variable needed to describe the state of the economy. In particular, the interest rate \( r_t \) is a function of \( s_t \): \( r_t = r(s_t) \). Similarly, the expected excess returns on the risky asset \( \mu_Q(s_t) \) and the volatility of returns \( \sigma_Q(s_t) \) are functions of \( s_t \) only, and these functions are determined in the equilibrium. In general, the dynamics of the state variable \( s_t \) can be represented as

\[
    ds_t = \mu_s(s_t)dt + \sigma_s(s_t)dB_t, \tag{4}
\]

where the functions \( \mu_s(s_t) \) and \( \sigma_s(s_t) \) are also determined by the equilibrium conditions.

D. Margin constraints

The key feature of the model is margin constraints imposed on type B investors. If existing investment opportunities are attractive, such investors short the risk-free asset and invest borrowed money in the risky asset. Capital providers put restrictions on how leveraged the position of type B investors can be (i.e., they set margin requirements). The financing process as well as the contracting between borrowers and lenders is not specified in the model. Instead, I assume that at each moment \( t \) the weight of the risky asset \( \omega_{it} \) in the portfolio of a potentially constrained investor \( i \) should satisfy the following state-dependent margin requirement:

\[
    \omega_{it} \leq \bar{\omega}(s), \tag{5}
\]

where \( \bar{\omega}(s) > 1 \) for all \( s \in [0,1] \). The exact functional form of \( \bar{\omega}(s) \) is endogenous and determined by equilibrium conditions. To make the model well specified, it is assumed that the tightness of margin requirements is (exogenously) linked to some (endogenous) characteristics of market equilibrium. Since in practice margin constraints are often determined by the volatility of stock returns, in the numerical analysis I will assume that

\[
    \bar{\omega}(s) = \bar{m} \left( \frac{\sigma_D}{\sigma_Q(s)} \right)^\alpha, \tag{6}
\]
where $\alpha$ and $\bar{m}$ are exogenous parameters. The constraint in the form of Eq. (5) covers several special cases. When $\alpha = 0$, the portfolio constraint is constant. In the case with $\alpha = 1$ it is an approximation to margin requirements set by the value-at-risk (VaR) approach (Danielsson, Shin, and Zigrand, 2004; Rytchkov, 2008) and effectively it is a constant constraint imposed on the maximal permitted volatility of investor’s wealth. Also, I will explore the case $\alpha = 2$ which corresponds to margin requirements that are more sensitive to the volatility of the asset than the requirements set by the VaR rule. This type of margin constraints will be dubbed “aggressive”. It also will be assumed that $1 < \bar{m} < \gamma_A/\gamma_B$.

The first inequality means that the type B investors are unconstrained in at least some states of the economy. The second inequality ensures that there exist states of the economy in which type B investors are constrained. Note that margin constraints with $\bar{w}(s)$ specified by Eq. (6) are endogenous since they are related to the volatility of returns which is endogenously determined in the equilibrium.

E. Consumption and portfolio problem of type A investors

The consumption and portfolio problem of type A investors is quite standard. Indeed, they are competitive, unconstrained, and maximize the standard CRRA utility function over the future consumption stream. Their first order conditions can be rewritten as pricing equations relating the interest rate $r(s_t)$ and the expected excess returns $\mu_Q(s_t)$ to the characteristics of the consumption process $C_t^A$ (e.g., Cochrane, 2005). In terms of the discount factor $\Lambda_t = \exp(-\beta_A t)u'(C_t^A)$, the functions $r(s)$ and $\mu_Q(s)$ are given by the standard formulas:

$$r(s_t) = -\frac{1}{d_t} E_t \left( \frac{d\Lambda_t}{\Lambda_t} \right), \quad \mu_Q(s_t) = -\frac{1}{d_t} E_t \left( \frac{d\Lambda_t}{\Lambda_t} \frac{dS_t}{S_t} \right).$$

(7)

For the CRRA preferences, the discount factor is $\Lambda_t = \exp(-\beta_A t)(C_t^A)^{-\gamma_A}$ and

$$\frac{d\Lambda_t}{\Lambda_t} = -\beta_A dt - \gamma_A \frac{dC_t^A}{C_t^A} + \frac{\gamma_A(\gamma_A + 1)}{2} \left( \frac{dC_t^A}{C_t^A} \right)^2.$$

In terms of the consumption share $s_t$, the consumption of type A investors is $C_t^A = (1-s_t)D_t$. Ito’s lemma applied to this equation yields

$$\frac{dC_t^A}{C_t^A} = \left( \mu_D - \frac{\mu_s(s_t) + \sigma_D \sigma_s(s_t)}{1-s_t} \right) dt + \left( \sigma_D - \frac{\sigma_s(s_t)}{1-s_t} \right) dB_t$$

and, hence,

$$\frac{d\Lambda_t}{\Lambda_t} = -\beta_A dt - \gamma_A \left( \mu_D - \frac{\mu_s(s_t) + \sigma_D \sigma_s(s_t)}{1-s_t} \right) dt - \gamma_A \left( \sigma_D - \frac{\sigma_s(s_t)}{1-s_t} \right)^2 dB_t.$$

Using Eq. (7), it is easy to find the interest rate $r(s_t)$ and the expected excess returns $\mu_Q(s_t)$:

$$r(s_t) = \beta_A + \gamma_A \mu_D - \frac{\gamma_A(\gamma_A + 1)}{2} \sigma_D^2 - \frac{\gamma_A}{1-s_t} (\mu_s(s_t) - \gamma_A \sigma_D \sigma_s(s_t)) - \frac{\gamma_A(\gamma_A + 1) \sigma_s(s_t)^2}{2(1-s_t)^2},$$

(8)
\[
\mu_Q(s_t) = \gamma_A \left( \sigma_D - \frac{\sigma_s(s_t)}{1 - s_t} \right) \sigma_Q(s_t).
\] (9)

The last equation effectively characterizes the market price of risk \( \eta(s_t) = \mu_Q(s_t)/\sigma_Q(s_t) \).

In the equilibrium, the volatility of returns is always positive, so for the unconstrained investors the market is complete. This is the reason why the optimal behavior of type A investors uniquely determines the interest rate and the market price of risk in terms of the drift and the volatility of the state variable. In the extension of the model developed in Section III, jumps in constraints produce unhedgeable jumps in returns and even for unconstrained investors the market becomes incomplete.

F. Consumption and portfolio problem of type B investors

Next, consider the consumption and portfolio problem of type B investors who are restricted by margin constraints. Again, it is assumed that the dynamics of the model can be described by a one-dimensional state variable \( s_t \). The problem of a type B investor is to maximize a CRRA utility function subject to the state-dependent margin constraint (5) and the standard budget constraint

\[
dW^B_{it} = \omega^B_{it} W^B_{it}(\mu_Q(s_t)dt + \sigma_Q(s_t)dB_t) + (r(s_t)W^B_{it} - C^B_{it})dt,
\] (10)

where \( W^B_{it} \) is total wealth of investor \( i \) and \( \omega^B_{it} \) is the share of total wealth invested in the risky asset. The solution to the portfolio problem is given by the following proposition, where time subscripts and individual investor indexes are omitted to simplify notations.

**Proposition 1** The solution to the consumption and portfolio problem of an investor maximizing the CRRA utility function (3) with \( \beta = \beta_B, \gamma = \gamma_B \) subject to the budget constraint (10) and the margin constraint (5) is

\[
C^B = \exp \left( -\frac{H(s)}{\gamma_B} \right) W^B,
\] (11)

\[
\omega^B(s) = \min \left( \bar{\omega}(s), \omega^*(s) \right), \quad \omega^*(s) = \frac{\mu_Q(s)}{\gamma_B \sigma_Q(s)^2} + \frac{\sigma_s(s)}{\gamma_B \sigma_Q(s)} H'(s),
\] (12)

where a twice continuously differentiable function \( H(s) \) solves the following ODE:

\[
\frac{1}{2} \sigma_s(s)^2 (H''(s) + H'(s)^2) + H'(s) \mu_s(s) + r(s)(1 - \gamma_B) - \beta_B + \gamma_B \exp \left( -\frac{H(s)}{\gamma_B} \right) - \frac{\gamma_B(1 - \gamma_B)}{2} \omega^B(s)^2 \sigma_Q(s)^2 + (1 - \gamma_B) \omega^B(s) \left( \mu_Q(s) + H'(s) \sigma_Q(s) \sigma_s(s) \right) = 0,
\] (13)

and the value function is

\[
J(s, W, t) = \frac{1}{1 - \gamma_B} W^{1 - \gamma_B} \exp(H(s)) \exp(-\beta_Bt).
\]

**Proof.** See Appendix A.
Proposition 1 shows that the solution to the portfolio problem has a form of a free boundary problem: to choose the optimal portfolio policy the investor needs to decide not only how much to invest in the risky asset when the constraint does not bind, but also in which states it is optimal to allow the constraint to bind. Both choices are encoded in the function $H(s)$. The point at which the portfolio hits the constraint is determined by the equation $\bar{\omega}(s) = \omega^*(s)$. Thus, the optimal portfolio policy (12) has a two-region structure. If the margin constraint is sufficiently tight ($\bar{\omega}(s)$ is sufficiently low) the investor prefers to invest up to this limit and $\omega^B(s) = \bar{\omega}(s)$. However, if the margin constraint is loose the optimal portfolio is $\omega^B(s) = \omega^*(s)$, where $\omega^*(s)$ is given by Eq. (12).

The unconstrained demand $\omega^*(s)$ has an intuitive structure. The first term is a standard myopic demand. The second term is a hedging demand produced by time-variation in i) the interest rate $r(s)$; ii) the market price of risk $\mu_Q(s)/\sigma_Q(s)$; iii) the portfolio constraint $\bar{\omega}(s)$. The first two hedging motives are standard and were extensively analyzed in the literature (e.g., Merton, 1971; Detemple, Garcia, and Rindisbacher, 2003; Liu, 2007). The third motive is new and specific to this paper. Since the interest rate, market price of risk, and margin constraints are all driven by the same state variable $s$, it is hard to disentangle the hedging demands produced by them individually. Nevertheless, state-dependent constraints should be hedged: even small probability that the margin constraint may bind in future changes the current indirect utility function of investors (see Eq. (13)). As a result, the current optimal portfolio is affected.

G. Equilibrium

The equilibrium in the model is determined by the interaction between type A and type B investors. The definition of the equilibrium is very standard: it implies that (i) all investors solve their maximization problems given market conditions and subject to their budget and margin constraints; (ii) aggregate consumption is equal to the aggregate dividend; (iii) all asset markets clear. For investors, the market conditions are described by the functions $r(s_t)$, $\mu_Q(s_t)$, $\sigma_Q(s_t)$, and $\bar{\omega}(s_t)$, and the dynamics of the state variable $s_t$ are specified by Eq. (4) which implies the knowledge of $\mu_s(s_t)$ and $\sigma_s(s_t)$. All these functions are endogenously determined by equilibrium conditions.

The characterization of the equilibrium is simplified if we note that the equilibrium expected returns and the volatility of returns can be conveniently represented in terms of the price-dividend ratio $S_t/D_t = f(s_t)$ (Ang and Liu, 2007). The exact result is stated in Lemma 1.

**Lemma 1** The expected excess returns on the risky asset $\mu_Q(s)$ and their volatility $\sigma_Q(s)$ are completely determined by the state variable dynamics, the risk-free rate, and the price-dividend ratio of the risky asset:

$$\mu_Q(s) = \mu_D - r(s) + \frac{1}{2}\sigma_s(s)^2\frac{f''(s)}{f(s)} + (\mu_s(s) + \sigma_D\sigma_s(s))\frac{f'(s)}{f(s)} + \frac{1}{f(s)},$$  \hspace{1cm} (14)

$$\sigma_Q(s) = \sigma_D + \frac{f'(s)}{f(s)}\sigma_s(s).$$  \hspace{1cm} (15)
The equilibrium of the model is completely characterized by the functions linked to the dynamics of the risk-free rate and the market price of risk) and the price-dividend ratio $f$. Proposition 2 implies that when the state variable is procyclical ($\sigma_s(s) > 0$) the volatility of returns $\sigma_Q$ exceeds the volatility of dividends $\sigma_D$ if the price-dividend ratio $f(s)$ is also procyclical ($f'(s) > 0$). As demonstrated below, this is exactly the case for the chosen set of model parameters.

The next proposition describes the equilibrium in the model.

**Proposition 2** The equilibrium of the model is completely characterized by the functions $r(s)$, $\mu(s)$, $\sigma_s(s)$, $H(s)$, and $f(s)$ that solve the following system of equations:

\[
\frac{1}{2} \sigma_s(s)^2 f''(s) + \left( \mu_s(s) + (1 - \gamma_A) \sigma_D \sigma_s(s) + \frac{\gamma_A \sigma_s(s)^2}{1-s} \right) f'(s) + \left( \mu_D - r(s) - \gamma_A \sigma_D + \frac{\gamma_A \sigma_D \sigma_s(s)}{1-s} \right) f(s) + 1 = 0, \quad (16)
\]

\[
\frac{\sigma_s(s)^2}{2 \gamma_B} \left( H''(s) + \frac{1}{\gamma_B} H'(s) \right) + \frac{H'(s)}{\gamma_B} \left[ \mu_s(s) + (1 - \gamma_B) \sigma_s(s) \left( \sigma_D + \frac{\sigma_s(s)}{s} \right) \right] - \beta_B + (1 - \gamma_B) \left[ \mu_s(s) + (1 - \gamma_B) \sigma_D \sigma_s(s) \right] + \mu_D - \frac{\gamma_B}{2} \left( \sigma_D^2 + \frac{\sigma_s(s)^2}{s^2} \right) + \exp \left( - \frac{H(s)}{\gamma_B} \right) = 0, \quad (17)
\]

\[
\sigma_D + \frac{\sigma_s(s)}{s} = \min \left[ \tilde{\omega}(\sigma_s(s), s) \left( \sigma_D + \frac{f'(s)}{f(s)} \sigma_s(s) \right) - \sigma_s(s) \frac{H'(s)}{\gamma_B} , \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma_s(s)}{1-s} \right) \right], \quad (18)
\]

\[
\mu_s(s) = \left( \frac{\gamma_A}{1-s} + \frac{\gamma_B}{s} \right)^{-1} \left[ \beta_A - \beta_B + (\gamma_A - \gamma_B) \left( \mu_D + \frac{\sigma_D \sigma_s(s)}{s} \right) + \frac{\gamma_A(1-\gamma_A)}{2} \left( \sigma_D - \frac{\sigma_s(s)}{1-s} \right)^2 - \frac{\gamma_B(1-\gamma_B)}{2} \left( \sigma_D + \frac{\sigma_s(s)}{s} \right)^2 - \frac{\gamma_A \sigma_s(s)^2}{(1-s)^2} + \left( (\gamma_A - \gamma_B) \sigma_D - \sigma_s(s) \left( \frac{\gamma_A}{1-s} + \frac{\gamma_B}{s} \right) \right) \frac{H'(s)}{\gamma_B} \right], \quad (19)
\]

\[
r(s) = \beta_A + \gamma_A \mu_D - \frac{\gamma_A}{2} \sigma_D^2 - \frac{\gamma_A}{1-s} (\mu_s(s) - \gamma_A \sigma_D \sigma_s(s)) - \frac{\gamma_A(\gamma_A + 1) \sigma_s(s)^2}{2(1-s)^2}. \quad (20)
\]

The equilibrium market price of risk is

\[
\eta(s) = \gamma_A \left( \sigma_D - \frac{\sigma_s(s)}{1-s} \right). \quad (21)
\]

**Proof.** See Appendix A.
(which is linked to the expected returns and the volatility of returns). For a general form of margin constraints, the differential equations are intertwined and should be solved simultaneously. However, when the tightness of margin constraints is inversely proportional to the volatility of returns (i.e., they are given by Eq. (6) with $\alpha = 1$), the dynamics of asset prices do not feed back into the dynamics of the state variable $s_t$. In this case, the differential equations for $H(s)$ and $f(s)$ can be disentangled and solved sequentially.

As mentioned in Section I.D, the margin constraint $\bar{\omega}(s)$ is endogenously determined in the equilibrium. Still, to close the model the relation between margins and market conditions should be exogenously specified. Since Eq. (18) is an algebraic equation for $\sigma_s$ in each state, the endogenous dependence of $\bar{\omega}(s)$ on $\sigma_s$ is explicitly indicated in it.

An important implication of Proposition 2 is that it provides an unambiguous comparison of the volatility of the state variable $\sigma_s$ and the market price of risk $\eta_s$ in constrained and unconstrained economies with identical fundamentals. The following Proposition states one of the main results of the paper.

**Proposition 3** In the economy with margin constraints

1. the volatility of the state variable $\sigma^c_s(s)$ is not greater than the corresponding volatility $\sigma^{un}_s(s)$ in the identical, but unconstrained economy: $\sigma^c_s(s) \leq \sigma^{un}_s(s)$ for all $s \in [0, 1]$;

2. the market price of risk $\eta^c_s(s)$ is not smaller than the corresponding market price of risk $\eta^{un}_s(s)$ in the identical, but unconstrained economy: $\eta^c_s(s) \geq \eta^{un}_s(s)$ for all $s \in [0, 1]$.

**Proof.** For $s = 0$ and $s = 1$ the statement is trivial. For any fixed state of the economy $s \in (0, 1)$, $\sigma_s$ is determined by Eq. (18) that can be rewritten as

$$\sigma_D + \frac{\sigma_s}{s} = \min \left[ g(\sigma_s), \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma_s}{1-s} \right) \right],$$

where $g(\sigma_s)$ is a complicated function whose form depends on the form of margin constraints $\bar{\omega}$. Figure 1 represents the solution to this equation as an intersection of graphs representing the right and left hand sides of the equation. For the unconstrained economy, it is an intersection of two straight lines. For the constrained economy, it is an intersection of an upward sloping straight line and a lower envelope of $g(\sigma_s)$ and $(\gamma_A/\gamma_B)(\sigma_D - \sigma_s/(1-s))$ (it is shown as a bold curve). From the graph it is obvious that all points of possible intersections of $\sigma_D + \sigma_s/s$ with the lower envelope are to the left from the unconstrained equilibrium, so $\sigma^c_s \leq \sigma^{un}_s$. Note that the proof is relatively simple because the volatility of the state variable in the unconstrained economy can be found in a closed form and does not depend on the function $H$. The latter results from market completeness in the absence of portfolio constraints.

More formally, assume that there exists a state of the economy $s$ in which $\sigma^c_s > \sigma^{un}_s$. Then, using Eq. (22) we have

$$\sigma_D + \frac{\sigma^c_s}{s} = g(\sigma^c_s) \leq \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma^c_s}{1-s} \right) < \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma^{un}_s}{1-s} \right) = \sigma_D + \frac{\sigma^{un}_s}{s}.$$
This chain of inequalities implies that $\sigma_s^c < \sigma_s^{un}$, and this contradiction completes the proof of the first statement in the proposition. The second statement immediately follows from Eq. (21). □

FIGURE 1 IS HERE

It should be emphasized that Proposition 3 is valid for a very general specification of margin constraints. In particular, they can have a complicated dependence on the volatility of returns or be explicitly state-dependent. As follows from the proof, the form of the constraint determines the function $g(\sigma_s)$, which is irrelevant for the result.

The reduction in the volatility of the state variable produced by portfolio constraints admits an intuitive explanation. The consumption shares of investors vary over time because investors dynamically share risks. This is a direct consequence of their heterogeneity. Portfolio constraints put a limit on how different positions of investors can be and effectively reduce their heterogeneity. As a result, the volatility of consumption shares declines.

Together with Eq. (15), Proposition 3 says that when the dividend-price ratio is procyclical $(f(s)' > 0)$ the volatility of returns in the constrained economy is not higher than in its unconstrained counterpart unless portfolio constraints substantially increase the semi-elasticity of the price-dividend ratio $(\log f(s))'$ with respect to the state variable $s$. As demonstrated below, the impact of portfolio constraints on the price-dividend ratio is quite weak and $(\log f(s))'$ is almost the same in both economies. Thus, the volatility of returns is tightly linked to the volatility of the state variable and decreases in states where margins bind. Again, this is a very general conclusion that does not rely on the form of margin constraints. It means that, in contrast to a widely spread belief, binding margin constraints alone cannot produce an increase in volatility in a general equilibrium in response to a negative shock to fundamentals. To explain such increase, other market imperfections are required.

II. Analysis

A. Equilibrium without margin constraints

As a benchmark case, consider a setup without margin constraints in which type B investors can borrow without limit. This case can be obtained when $\bar{\omega}(s) \to \infty$ for all $s$, i.e. the margin requirements are infinitely loose. Without margin constraints, the model is essentially the same as in Longstaff and Wang (2008) and Bhamra and Uppal (2010), so I only briefly summarize the main properties of the equilibrium to facilitate its comparison with the constrained economy.

In the absence of portfolio constraints, the characterization of the equilibrium substantially simplifies. In particular, the drift and volatility of consumption shares have explicit analytical representations:

$$\mu_s(s) = \left(\frac{\gamma_A}{1-s} + \frac{\gamma_B}{s}\right)^{-1}\left[\beta_A - \beta_B + (\gamma_A - \gamma_B)\left(\mu_D + \frac{1}{2}\sigma_D^2\left(\frac{\gamma_A \gamma_B}{\gamma_A s + \gamma_B (1-s)} - 2\right)\right)\right].$$  \hspace{1cm} (23)
\[ \sigma_s(s) = \sigma_D(\gamma_A - \gamma_B) \left( \frac{\gamma_A}{1 - s} + \frac{\gamma_B}{s} \right)^{-1}. \]  
(24)

Using Eq. (24) together with Eq. (9), the market price of risk can be written as

\[ \eta(s) = \sigma_D \left( \frac{1 - s}{\gamma_A} + \frac{s}{\gamma_B} \right)^{-1}. \]  
(25)

Moreover, the differential equations for the functions \( f(s) \) and \( H(s) \) in Proposition 2 decouple, and only the first one should be solved to find \( \mu_Q(s) \) and \( \sigma_Q(s) \). Longstaff and Wang (2008) provide a closed-form solution in terms of hypergeometric functions for the price-dividend ratio \( f(s) \) when the risk aversion of one group of investors is twice as high as the risk aversion of the other group. Chabakauri (2010) generalizes this result to models with arbitrary risk aversion parameters. Bhamra and Uppal (2010) provide a solution to a similar model in the form of a sum of an infinite series.

Eqs. (23) and (24) imply that the dynamics of the model are generated not only by fluctuations in the dividend, but also by wealth reallocation between agents with different degrees of risk aversion. As follows from Eq. (24), the volatility of the state variable \( \sigma_s \) is determined by the ratio \( \gamma_A/\gamma_B \) which characterizes the degree of heterogeneity between investors. Without the heterogeneity in preferences, the consumption share of each agent is constant and the price of the risky asset is proportional to the current dividend. Eq. (25) shows that the market price of risk is solely determined by the aggregate relative risk aversion (Bhamra and Uppal, 2010).

To demonstrate the main properties of the equilibrium, I use a specific calibration of model parameters. Following other studies (e.g., Kogan, Makarov, and Uppal, 2007; Chabakauri, 2010), I calibrate the process for fundamentals \( D_t \) using the historical mean and volatility of the aggregate consumption growth rate in the US. In particular, I set \( \mu_D = 0.018 \) and \( \sigma_D = 0.032 \). Also, I choose \( \beta_A = \beta_B = 0.01 \). To illustrate the impact of the heterogeneity of investors on equilibrium properties, I consider several combinations of the risk aversion parameters. In particular, I set \( \gamma_A = 10 \) and consider the cases \( \gamma_B = 1, \gamma_B = 2, \) and \( \gamma_B = 5 \).

To visualize the structure of the equilibrium, several statistics of interest are plotted as functions of the state variable \( s \). The results are presented in Figure 2, where different curves are associated with different \( \gamma_B \). Since \( \sigma_s \) is positive (it immediately follows from Eq. (24)), the state variable \( s_t \) is procyclical. The boundary points \( s = 0 \) and \( s = 1 \) correspond to economies dominated by investors of one type: when \( s \to 0 \) (bad states) almost all dividends are consumed by type A whereas when \( s \to 1 \) (good states), type B investors prevail.

**FIGURE 2 IS HERE**

Figure 2 shows that the heterogeneity of preferences increases the volatility of the consumption share, makes the interest rate and the market price of risk countercyclical, and the price-dividend ratio procyclical. As discussed above, the first result follows immediately from Eq. (24). The downward slope of \( r \) and \( \eta \) and the upward slope of \( P/D \) are intuitively explained by dependence of these
characteristics on the aggregate risk aversion which decreases with \( s \) as less risk averse type B investors consume a bigger share of the dividend. In the limits \( s = 0 \) and \( s = 1 \) these characteristics are solely determined by type A and type B investors, respectively.

The middle right panel of Figure 2 presents the volatility of returns \( \sigma_Q \), which has a hump-shaped form. In the extremes \( s = 0 \) and \( s = 1 \) it coincides with the fundamental volatility \( \sigma_D \) since there all changes in the price are linearly related to changes in the dividend \( D_t \). However, in the intermediate states the volatility is amplified by optimal risk sharing between heterogeneous investors, and the magnitude of the effect increases with the degree of heterogeneity.\(^2\) Expected returns \( \mu_Q \) depicted in the upper right panel also demonstrate a hump-shaped form, but in the majority of the states they are countercyclical.

The bottom right panel of Figure 2 shows that the optimal weight of the risky asset in the portfolio of type B investors is always greater than one and tends to one when \( s \rightarrow 1 \). Thus, type B investors borrow from type A investors and invest in the risky asset. Noteworthy, the optimal leverage of type B investors is higher in bad states when investment opportunities are especially attractive from their perspective. It means that, when imposed, margin constraints are likely to bind in bad states of the economy.

B. Equilibrium with margin constraints

To explore the impact of margin constraints on the equilibrium prices and returns, I use again the setting from Section I but now type B investors are subject to margin requirements. Unfortunately, the system of non-linear differential equations from Proposition 2 is rather complex and does not admit a closed-form solution. Thus, I have to rely on numerical techniques. To find a numerical solution, I use the projection method which approximates the functions \( H(s) \) and \( f(s) \) by linear combinations of Chebyshev polynomials (Judd, 1998). One of the advantages of this method is that the obtained solutions are automatically twice continuously differentiable for all \( s \in (0, 1) \). The details on the implementation of the projection method are relegated to Appendix B.

To make the results comparable to those from the previous section, I consider an economy with the same dynamics of fundamentals as before and choose \( \gamma_A = 10 \) and \( \gamma_B = 2 \). On the one hand, due to substantial heterogeneity of agents all effects are quite strong in such an economy. On the other hand, \( \gamma_B > 1 \) and type B investors have a non-trivial hedging demand. To close the model, I assume that margin constraints are specified by Eq. (6). The parameter \( \bar{m} \) determining the tightness of constraints is chosen such that the constraint binds in some (bad) states of the economy, but not in all states. In the main numerical example \( \bar{m} = 1.3 \).

I examine three types of margin constraints corresponding to different values of \( \alpha \). The first one is the simplest case with constant margin requirements (\( \alpha = 0 \)). The second one is the VaR-type

\(^2\)The form of \( \sigma_Q \) as a function of \( s \) may be different for alternative parameter values. In particular, it may take a U-shaped form, meaning that in many states the volatility of returns is lower than the volatility of fundamentals (Longstaff and Wang, 2008). Bhamra and Uppal (2009) identify the exact condition that determines whether the volatility of returns is larger or smaller than the fundamental volatility.
constraint; it corresponds to $\alpha = 1$ and the tightness of margins is inversely related to the volatility of returns. The third one is an “aggressive” constraint with $\alpha = 2$. It is more sensitive to the volatility of returns than the VaR-type constraint. To visualize the effect of constraints, I graph the ratios of various equilibrium characteristics in the constrained and unconstrained economies. The results are presented in Figure 3.

FIGURE 3 IS HERE

The first important observation from Figure 3 is that qualitatively the impacts of various types of constraints are very similar. In particular, the effect of time-varying margins resembles the impact of constant portfolio constraints. The only difference is in magnitudes: since the volatility of returns $\sigma_Q$ is higher than the volatility of fundamentals $\sigma_D$, time-varying margins from Eq. (6) with $\alpha > 0$ are more restrictive on average than constant margins, and their average tightness increases with $\alpha$. This explains why the effect of “aggressive” constraints with $\alpha = 2$ is the most pronounced.

The middle left panel of Figure 3 depicts $\sigma_s$ and illustrates the first statement of Proposition 3: all portfolio constraints reduce the volatility of the state variable. Since the impact of constraints on the sensitivity of the price-dividend ratio to the state variable is very small, a decrease in $\sigma_s$ translates into a decrease in the volatility of returns $\sigma_Q$ presented in the middle left panel of Figure 3 (this follows from Eq. (15)). The graph of $\sigma_Q$ also shows that although the exact point at which the constraint starts binding is determined by the form of the constraint, the volatility decreases only in the states where the constraint actually binds. This is an implication of Eq. (18) and can be inferred from Figure 1.

Next, portfolio constraints tend to increase the expected excess returns $\mu_Q$. This effect has an intuitive explanation: since portfolio holdings of less risk averse investors are constrained in some states, more risk averse investors are forced to hold more risky assets than they would without constraints. To induce them to buy more risky assets, the risk premium should be higher. This intuition is quite general and does not rely on the specific form of margin constraints.

Figure 3 also shows that portfolio constraints tend to increase the market price of risk and reduce the risk-free rate. The former effect illustrates the second statement of Proposition 3 and immediately follows from the decrease in volatility and increase in expected excess returns. The decrease in the risk-free rate produced by margin constraints is also quite intuitive. Without constraints, type B investors borrow from type A investors and build leveraged portfolios. If maximum leverage is restricted, the demand for credit is lower and, as a result, the equilibrium risk-free rate is also lower. Again, this is a quite general effect which does not depend on the exact form of constraints. These two effects are reported by Kogan, Makarov, and Uppal (2007) for an economy with constant borrowing constraints and less risk averse investors having logarithmic preferences. My results indicate that their conclusions are very general: they are valid for other forms of portfolio constraints and can be extended to economies with arbitrary levels of risk aversion.
The middle panel in the bottom row of Figure 3 shows the impact of portfolio constraints on the price-dividend ratio. Although it is widely believed that margin constraints decrease asset valuations, my results indicate that it is not the case in a general equilibrium: when margins bind, the price-dividend ratio increases. Intuitively, this effect can be related to the lower risk-free rate. Even though constraints increase the risk premium, total expected returns decrease since the lower risk-free rate cannot be offset by higher excess expected returns. However, the magnitude of the price-dividend ratio increase is quite limited: the maximum increase produced by the most restrictive “aggressive” constraint is only around 10%.

C. Equilibrium with myopic type B investors

So far it has been assumed that the type B investors have rational CRRA preferences. Since investment opportunities are state-dependent in the equilibrium, the optimal portfolio of type B investors contains both a myopic and a hedging component when \( \gamma_B \neq 1 \), and their consumption-wealth ratio is also state-dependent. In this section, I explore the impact of the hedging demand on the properties of the equilibrium. For this purpose, I compare a fully rational equilibrium discussed in the previous section with an equilibrium in a modified model in which type B investors are assumed to be myopic. To construct the latter equilibrium, it is sufficient to set \( H = \text{const} \) in Eqs. (16), (18), (19), and (20). This eliminates the hedging demand and makes the consumption-wealth ratio constant.

To visualize the effect of hedging, I plot the ratios of various characteristics in the equilibria with rational and myopic type B investors. Again, I consider an unconstrained equilibrium along with constrained economies with margins from Eq. (6). The results are presented in Figure 4.

FIGURE 4 IS HERE

The first interesting observation from Figure 4 is that in the unconstrained economy the hedging demand has no impact on the equilibrium statistics (all ratios are equal to one) although in some states the hedging component accounts for almost 20% of the myopic demand. This is a consequence of market completeness for all investors and homothetic preferences. Note that this property also holds in economies with portfolio constraints, but only in those states where these constraints do not bind.

Next, Figure 4 shows that hedging of time-varying investment opportunities by type B investors affects the properties of the equilibrium when portfolio constraints bind, and the effect is quite strong. Since in the unconstrained region investment opportunities are worse in good states, the hedging demand is positive and increases the leverage desired by type B investors. As a result, the presence of

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3 Under certain conditions, equilibrium prices in economies with portfolio constraints may even contain rational bubbles (Hugonnier, 2010). In the case \( \alpha = 1 \), Prieto (2011) shows that the stock price is free of bubbles if more risk averse investors are unconstrained and less risk averse investors are unconstrained in some states of the economy.

4 The same combination of properties made it possible to find explicit formulas for \( \mu_s \), \( \sigma_s \), and \( \eta \) in Section II.A.
hedging demand expands the set of states in which type B investors hit the margin constraint. These are the states where the impact of hedging is particularly strong. However, even in those states where type B investors are constrained both with and without hedging demand, the equilibrium statistics are affected by the hedging component.

As before, the effects of constant portfolio constraints and time-varying constraints are qualitatively similar. Also, comparing Figures 3 and 4 we can conclude that portfolio constraints change equilibrium statistics in the same direction in economies with and without hedging demand, but hedging demand amplifies the impact of constraints. In particular, the presence of the hedging component substantially reduces the volatility of the state variable as well as its drift when the constraint binds. Intuitively, it can be explained by better consumption smoothing by type B investors when the consumption-wealth ratio is allowed to be state dependent.

Overall, my analysis reveals that in the presence of portfolio constraints the hedging motive of potentially constrained investors may strongly influence the properties of the equilibrium amplifying the effects of portfolio constraints.

III. Extension: random jumps in margin requirements

The analysis in the previous sections assumes that margin constraints are either constant or deterministically determined by continuously evolving market conditions. However, in practice there is always some uncertainty about margin requirements in the next moment, and changes in margin requirements tend to be discrete. In this section, I consider a model with jumps in margin requirements and explore the robustness of my previous results.

A. Setup

The main setup is the same as before: this is a continuous-time exchange economy with heterogeneous investors. However, now margin requirements are not linked to market conditions but can randomly jump. To keep the model as simple as possible, it is assumed that type B investors have logarithmic preferences, and margin requirements can take only two values $\bar{m}^+$ and $\bar{m}^-$, where $\bar{m}^- < \bar{m}^+$. The margin constraint of type B investors is $\omega^B \leq \bar{m}(q)$, where $\bar{m}(q) = qm^- + (1 - q)m^+$ and $q = \{0, 1\}$ is a new binary state variable evolving as a continuous Markov chain. The dynamics of $q_t$ can be represented as

$$dq_t = (1 - q_t) dq^+_t + q_t dq^-_t,$$

where $q^+$ and $q^-$ are counting processes with $E[dq^+_t] = \lambda dt$ and $E[dq^-_t] = (1 - \lambda) dt$. For simplicity, the transition probability $\lambda$ is assumed to be constant. Thus, the state $q = 0$ is associated with loose margin requirements $\bar{m}^+$, whereas in the state $q = 1$ the constraints are more restrictive. The transition between states is random and is uncorrelated with the dividend process. The structure of the transition matrix is such that unconditionally margin constraints are tight with probability $\lambda$. It is still assumed that there are only two assets in the economy: a risky asset representing a claim on
the exogenous dividend and a risk-free asset in a zero supply, and there is no derivative asset allowing agents to hedge the jumps in margin requirements.

Since margin constraints affect a large number of investors, in the equilibrium a jump in the level of constraints produces a jump in the asset price. Hence, the jump risk is systematic, and the processes for returns and consumption share should contain jump components. In general, these processes can be written in the following form:

\[
dQ_t = \mu_Q(s_t, q_t)dt + \sigma_Q(s_t, q_t)dB_t + \kappa_Q^+(s_t)(1 - q_t)(dq_t^+ - \lambda dt) + \kappa_Q^-(s_t)q_t(dq_t^- - (1 - \lambda)dt),
\]

\[
ds_t = \mu_s(s_t, q_t)dt + \sigma_s(s_t, q_t)dB_t + \kappa_s^+(s_t)(1 - q_t) dq_t^+ + \kappa_s^-(s_t)q_t dq_t^-.
\]

As before, \(\mu_Q\) and \(\sigma_Q\) are expected returns and the volatility of returns, respectively. Note that \(\mu_Q\) measures the compensation for both the diffusion risk and the jump risk. The coefficients \(\kappa_Q\) and \(\kappa_s\) measure the jumps in returns and the state variable \(s_t\) when margin requirements switch the state.

B. Consumption and portfolio problem of type A investors

As in Section I, the solution to the optimization problem of type A investors yields the equilibrium risk-free rate and the market price of diffusion risk in terms of the functions determining the dynamics of the state variable \(s\) and the ratios \(\kappa_Q^+/\sigma_Q, \kappa_Q^-/\sigma_Q\). However, now the market is incomplete and the jump risk cannot be hedged by investors. As a consequence, the market prices of diffusion and jump risks cannot be inferred individually from the optimal behavior of type A investors. The following Proposition states the result.

**Proposition 4** The optimality of consumption and portfolio strategies of type A investors implies the following representation for the risk-free rate and the market price of diffusion risk:

\[
r(s_t, q_t) = \beta_A + \gamma_A \mu_D - \frac{\gamma_A(\gamma_A + 1)}{2} \left( \frac{\sigma_D - \sigma_s(s_t, q_t)}{1 - s_t} \right)^2 - \frac{\gamma_A(\mu_s(s_t, q_t) + \sigma_D \sigma_s(s_t, q_t))}{1 - s_t} - \lambda (1 - q_t) \left[ \left( 1 - \frac{\kappa_s^+(s_t)}{1 - s_t} \right)^{-\gamma_A} - 1 \right] - (1 - \lambda) q_t \left[ \left( 1 - \frac{\kappa_s^-(s_t)}{1 - s_t} \right)^{-\gamma_A} - 1 \right],
\]

\[
\frac{\mu_Q(s_t, q_t) - \lambda (1 - q_t) \kappa_Q^+(s_t) - (1 - \lambda) q_t \kappa_Q^-(s_t)}{\sigma_Q(s_t, q_t)} = \gamma_A \left( \frac{\sigma_D - \sigma_s(s_t, q_t)}{1 - s_t} \right) - \lambda (1 - q_t) \frac{\kappa_Q^+(s_t)}{\sigma_Q(s_t, q_t)} \left( 1 - \frac{\kappa_s^+(s_t)}{1 - s_t} \right)^{-\gamma_A} - (1 - \lambda) q_t \frac{\kappa_Q^-(s_t)}{\sigma_Q(s_t, q_t)} \left( 1 - \frac{\kappa_s^-(s_t)}{1 - s_t} \right)^{-\gamma_A}.
\]

**Proof.** See Appendix A.

Eqs. (28) and (29) are analogs of Eqs. (8) and (9), respectively, and reduce to them in the absence of jumps in margin constraints (e.g., when \(\lambda = 0\) and \(q = 0\)). Later it will be shown that
\[ \kappa_s^+ > 0, \kappa_s^- < 0. \] It means that for admissible \( \kappa_s^+ \) and \( \kappa_s^- \) we have \( (1 - \kappa_s^+/(1 - s))^{-\gamma_A} > 1 \) and \( (1 - \kappa_s^-/(1 - s))^{-\gamma_A} < 1 \). Hence, keeping the drift and the volatility of the state variable fixed, the possibility of jumps reduces the interest rate in the states with loose margins (when \( q = 0 \)) and increases the interest rate in the states with tight margins (when \( q = 1 \)). This is intuitive, since as demonstrated in the previous section portfolio constraints tend to decrease the risk-free rate.

C. Consumption and portfolio problem of type B investors

Due to logarithmic preferences of type B investors, their optimal consumption-wealth ratio is constant and the optimal portfolio does not contain a component resulting from hedging diffusion risk. As a result, there is no need to solve a differential equation to find optimal portfolio weights. The exact solution to the utility maximization problem of type B investors is stated in Proposition 5.

**Proposition 5** The optimal consumption of a logarithmic investor with the time preference parameter \( \beta = \beta_B \) subject to the standard budget constraint and the margin constraint \( \omega_B \leq \bar{m}(q) \) is \( C_B = \beta_B W_B \) whereas the optimal portfolio of the investor is

\[
\omega^B(s, q) = \min (\bar{m}(q), \omega^*(s, q)),
\]

where \( \omega^*(s, q) \) solves the following algebraic equation

\[
\omega^*(s, q) = \frac{\mu_Q(s, q)}{\sigma_Q(s, q)^2} - \frac{\kappa_Q(s)\sigma_Q(s, q)^2\omega^*(s, q)}{1 + \omega^*(s, q)\kappa_Q^+(s)} - (1 - \lambda)q\left(\frac{\kappa_Q^-(s)\sigma_Q(s, q)^2\omega^*(s, q)}{1 + \omega^*(s, q)\kappa_Q^-(s)}\right). \tag{31}
\]

**Proof.** See Appendix A.

As in the case without jumps, the optimal portfolio strategy has two regimes: in the constrained regime the portfolio is on the boundary and \( \omega^B(s, q) = \bar{m}(q) \); in the unconstrained regime the portfolio \( \omega^B(s, q) = \omega^* \) coincides with the optimal portfolio in a setting without constraints but with jumps in returns. Thus, due to logarithmic preferences and the absence of hedging, the solution in the unconstrained regime is unaffected by the presence of constraints (c.f. Tepla, 2000). This contrasts with the optimal portfolio policy in Proposition 1 whose unconstrained part accounts for the possibility of hitting the constraint.

Although there is no standard hedging demand, the optimal portfolio is not myopic in the presence of jumps in returns (Liu, Longstaff, and Pan, 2003; Wu, 2003). Because the additional terms in Eq. (31) are unambiguously negative, for the given expected returns and the volatility of returns the optimal weight of the risky asset is lower in both states of margin constraints. Also note that Eq. (31) can be reduced to a quadratic equation for \( \omega^*\sigma_Q \) and solved analytically in terms of \( \mu_Q/\sigma_Q, \kappa_Q^+/\sigma_Q, \) and \( \kappa_Q^-/\sigma_Q \). However, Eq. (31) is more convenient for finding the equilibrium.
D. Equilibrium

As in Section I.G, I describe the equilibrium in terms of the process for the state variable $s$ and the price-dividend ratio. However, now all characteristics of the equilibrium also depend on the current state of margin requirements which represents an additional discrete state variable. The equilibrium is described in Proposition 6.

**Proposition 6** The equilibrium of the model is completely characterized by the functions $r(s, q)$, $\mu_s(s, q)$, $\sigma_s(s, q)$, $\kappa^+_s(s, q)$, $\kappa^-_s(s, q)$, and $f(s, q)$ that solve the following system of equations:

\[
\frac{1}{2} \sigma_s(s, q)^2 f''(s, q) + (\mu_s(s, q) + \sigma_D \sigma_s(s, q)) f'(s, q) + (\mu_D - r(s, q)) f(s, q) + 1 \\
- \frac{\sigma_D f(s, q) + \sigma_s(s, q) f'(s, q)}{s \sigma_D + \sigma_s(s, q)} (\mu_s(s, q) + \sigma_D \sigma_s(s, q) + s(\mu_D + \beta_B - r(s, q))) = 0, \tag{32}
\]

\[
\sigma_D + \frac{\sigma_s(s, q)}{s} = \min \left[ m(q) \left( \sigma_D + \frac{f'(s, q)}{f(s, q)} \sigma_s(s, q) \right), \right.
\gamma_A \left( \sigma_D - \frac{\sigma_s(s, q)}{1 - s} \right) - \lambda(1 - q) \kappa^+_s(s) \left( 1 - \frac{\kappa^+_s(s)}{1 - s} \right)^{-\gamma_A} - \left( 1 + \frac{\kappa^+_s(s)}{s} \right)^{-1} \left( \sigma_D + \frac{\sigma_s(s, q)}{s} \right)^{-1}
\]

\[
- (1 - \lambda) q \left( 1 - \frac{\kappa^-_s(s)}{1 - s} \right)^{-\gamma_A} - \left( 1 + \frac{\kappa^-_s(s)}{s} \right)^{-1} \left( \sigma_D + \frac{\sigma_s(s, q)}{s} \right)^{-1} \right], \tag{33}
\]

\[
\mu_s(s, q) = \left( \frac{\gamma_A}{1 - s} + \frac{1}{s} \right)^{-1} \left[ \beta_A - \beta_B + (\gamma_A - 1) \left( \mu_D + \frac{\sigma_D \sigma_s(s, q)}{s} \right) - \frac{\gamma_A \sigma_s(s, q)^2}{s(1 - s)^2} \right.
\]

\[
+ \frac{\gamma_A(1 - \gamma_A)}{2} \left( \sigma_D - \frac{\sigma_s(s, q)}{1 - s} \right)^2 - \lambda(1 - q) \left( 1 - \frac{\kappa^+_s(s)}{1 - s} \right)^{-\gamma_A} \left( 1 + \frac{\kappa^+_s(s)}{s} \right) - 1
\]

\[
- (1 - \lambda) q \left( 1 - \frac{\kappa^-_s(s)}{1 - s} \right)^{-\gamma_A} \left( 1 + \frac{\kappa^-_s(s)}{s} \right) - 1 \right], \tag{34}
\]

\[
\kappa^+_s(s) = \frac{(s \sigma_D + \sigma_s)(f(s + \kappa^+_s(s), 1) - f(s, 0))}{\sigma_D f(s, 0) + \sigma_s(s, 0) f'(s, 0)}, \quad \kappa^-_s(s) = \frac{(s \sigma_D + \sigma_s)(f(s + \kappa^-_s(s), 0) - f(s, 1))}{\sigma_D f(s, 1) + \sigma_s(s, 1) f'(s, 1)}, \tag{35}
\]

\[
r(s, q) = \beta_A + \gamma_A \mu_D - \frac{\gamma_A(\gamma_A + 1)}{2} \left( \sigma_D - \frac{\sigma_s(s, q)}{1 - s} \right)^2 - \gamma_A (\mu_s(s, q) + \sigma_D \sigma_s(s, q))
\]

\[
- \lambda(1 - q) \left( 1 - \frac{\kappa^+_s(s)}{1 - s} \right)^{-\gamma_A} - 1 - (1 - \lambda) q \left( 1 - \frac{\kappa^-_s(s)}{1 - s} \right)^{-\gamma_A} - 1 \right]. \tag{36}
\]

**Proof.** See Appendix A.

Proposition 6 is an analog of Proposition 2 and generalizes it for the case with jumps in margin requirements. However, the equilibrium with jumps is constructed for an economy with less risk averse investors having logarithmic preferences. As a consequence, only differential equations for the price-dividend ratio should be solved. This substantially reduces the computational burden and facilitates the analysis.
E. Analysis

For consistency with the analysis of economies without jumps, the calibration of almost all model parameters is the same as in Section II. I set the margin levels $m^+$ and $m^-$ such that in the state with loose margin requirements $q = 0$ they never bind, but in the other state $q = 1$ they are relatively restrictive. In particular, I choose $m^- = 1.1$ and $m^+ = 10$. I set $\lambda = 0.977$, so the tightness of margins averaged over time is the same as in the benchmark economy with constant margins. The equilibrium statistics in both states of margin constraints are presented in Figure 5. For comparison, Figure 5 also depicts equilibria in the economies with constant constraints fixed at the levels $\bar{\omega} = 1.1$ and $\bar{\omega} = 10$.

FIGURE 5 IS HERE

The right and left bottom panels of Figure 5 plot $\kappa_s$ and $\kappa_Q$, which show how the state variable and returns jump in response to the switch in the margins regime. Since portfolio constraints tend to increase the value of the risky asset, returns jump upward when constraints become tight ($\kappa_Q^+ > 0$) and jump downward when they become loose ($\kappa_Q^- < 0$). The increase in the price of the risky asset increases wealth of type B investors. Since they optimally maintain a constant consumption-wealth ratio, their consumption also increases explaining the upward jump in their consumption share ($\kappa_s^+ > 0$). The opposite effect works when constraints become loose ($\kappa_s^- < 0$).

However, since the effect of constraints on the price-dividend ratio is quite small (see Section II.B), the magnitude of the jump in returns measured by $\kappa_Q$ is also quite small. As a result, investors of both types have only weak incentives to hedge such jumps leaving their optimal portfolios almost unchanged. This explains the main observation from Figure 5: almost all equilibrium characteristics in states $q = 0$ and $q = 1$ are practically indistinguishable from their counterparts in the benchmark economies with $\bar{\omega} = 10$ and $\bar{\omega} = 1.1$, respectively. The only exception is $\mu_s$ which differs from its benchmark in the state $q = 0$. This is explained by the definition of $\mu_s$: it does not include the compensated variation of jumps in the state variable $s$. The irrelevance of possible jumps is particularly remarkable for $q = 0$: given that $\lambda = 0.977$, the probability of switching to the state with tight margins is high but this does not change substantially the equilibrium statistics.

Overall, we can conclude that the presence of unhedgeable jumps in portfolio constraints does not invalidate the main results obtained in Section II and the assumption that the level of constraints is deterministically related to market conditions is innocuous. Even if the level of constraints can jump, the properties of the equilibrium are predominantly determined by a direct impact of constraints on the portfolio of less risk averse investors, but not by an anticipation of discrete changes in constraints.

IV. Conclusion

To summarize, this paper presents a general equilibrium model with heterogeneous investors and endogenously determined state-dependent margin requirements. In contrast to many existing studies,
investors may have arbitrary levels of risk aversion and the margin requirements are assumed to have a quite general form allowing the levels of constraints to be determined endogenously in the equilibrium. The analysis of this model delivers several insights: 1) binding margin constraints reduce the risk-free rate and the volatility of returns but increase the expected returns, the market price of risk, and the price of the risky asset; 2) qualitatively, all documented effects are very robust and not sensitive to the form of margin constraints which may have a complex dependence on market conditions; 3) since the volatility of returns unambiguously decreases and the price-dividend ratio increases when margin constraints bind, margins themselves (even time varying) cannot be responsible for increase in volatility and decrease in the asset value when fundamentals deteriorate and some investors are hit by constraints; 4) hedging of time-varying investment opportunities by constrained investors may strongly affect the properties of the equilibrium; 5) since the presence of margin constraints has only a small effect on the value of the risky asset, the hedging of discrete changes in margin requirements is very weak and the possibility of such changes does not significantly affect the properties of the equilibrium.

In this paper, I assume that there is only one risky asset in the model. However, it may be interesting to generalize the model and consider a setup with multiple risky assets. Although such a generalization is relatively straightforward when constrained investors have logarithmic preferences (Garleanu and Pedersen, 2011; Prieto, 2011), the equilibrium is much more complicated when the constrained investors have a hedging demand. The main reason is that in the latter case the process for the consumption share also depends on the shares of each asset in the aggregate consumption which become new state variables hedged in the equilibrium (Martin, 2009). The analysis of equilibrium models with multiple assets and constrained investors is an interesting direction for future research.

Appendix A.

Proof of Proposition 1

Because in this proof I consider type B investors only, I omit the subscript B to simplify notations. All functions depend on \( s_t \) only, and this argument is also omitted. The value function of an investor maximizing the CRRA utility function satisfies the following Bellman equation:

\[
\max_{\{C, \omega \leq \omega\}} \left[ e^{-\beta t} u(C) + DJ \right] = 0, \tag{A1}
\]

where

\[
DJ = J_W (\omega W \mu_Q + r W - C) + \frac{1}{2} J_{WW} W^2 \omega^2 \sigma_Q^2 + J_s \mu_s + \frac{1}{2} J_{ss} \sigma_s^2 + J_{Ws} W \omega \sigma_Q \sigma_s + J_t.
\]

Due to the homotheticity of preferences, it is natural to look for the value function in the following standard form:

\[
J(s, W, t) = \frac{1}{1 - \gamma} W^{1-\gamma} \exp(H) \exp(-\beta t), \tag{A2}
\]

where the function \( H(s) \) is assumed to be twice continuously differentiable. The maximization in Eq. (A1) over \( C \) yields

\[
e^{-\beta t} u'(C) - J_W = 0 \quad \text{or} \quad C = W \exp \left( -\frac{H}{\gamma} \right), \tag{A3}
\]
The constrained maximization over $\omega$ reduces to

$$
\omega = \arg \max_{\omega \leq \bar{\omega}} \left[ (J_W \mu_Q + J_W \sigma_Q \sigma_s) \omega + \frac{1}{2} J_W W \sigma_Q^2 \omega^2 \right].
$$

Given that $J_W W < 0$, the solution to the maximization problem is either an internal point satisfying the first order condition or a point on the boundary $\omega = \bar{\omega}$. Thus, solving the first order condition and using Eq. (A2) we get

$$
\omega = \min \left( \bar{\omega}, \frac{\mu_Q}{\gamma \sigma_Q^2} + \frac{\sigma_s}{\gamma \sigma_Q} H' \right).
$$  \hspace{1cm} (A4)

Substituting the optimal consumption (A3) and the optimal portfolio policy (A4) along with the value function (A2) back into Eq. (A1), we get Eq. (13). ■

**Proof of Lemma 1**

By definition, the price of the asset is $S_t = D_t f(s_t)$. Applying Ito’s lemma we get

$$
\frac{dS_t}{S_t} = \frac{dD_t}{D_t} + \frac{df(s_t)}{f(s_t)} + \frac{dD_t}{D_t} \frac{df(s_t)}{f(s_t)},
$$

where

$$
df(s_t) = f'(s_t)(\mu_s(s_t)dt + \sigma_s(s_t)dB_t) + \frac{1}{2} f''(s_t)\sigma_s(s_t)^2 dt.
$$

Using the structure of the dividend process from Eq. (1) and performing simple algebra, the process for returns can be written as

$$
\frac{dS_t + D_t dt}{S_t} - r_t dt = \left( \mu_D - r(s_t) + \frac{\sigma_s(s_t)^2}{2} f''(s_t) + (\mu_s(s_t) + \sigma_D \sigma_s(s_t)) \frac{f'(s_t)}{f(s_t)} + \frac{1}{2} \left( \sigma_D + \frac{f'(s_t)}{f(s_t)} \sigma_s(s_t) \right)^2 dt
\right)
$$

Comparing this equation with Eq. (2) we get the statement of the lemma. ■

**Proof of Proposition 2**

The equation for the price-dividend ratio $f(s)$ is derived by combining Eqs. (14) and (15) with Eq. (9). Next, the definition of $s_t$ implies that $C_t^B = s_t D_t$, and Ito’s lemma yields

$$
\frac{dC_t^B}{C_t^B} = \left( \frac{\mu_s(s_t)}{s_t} + \mu_D + \frac{\sigma_D \sigma_s(s_t)}{s_t} \right) dt + \left( \sigma_D + \frac{\sigma_s(s_t)}{s_t} \right) dB_t.
$$  \hspace{1cm} (A5)

From Eq. (11) the optimal consumption-wealth ratio of type B investors is $C_t^B / W_t^B = \exp(-H(s_t) / \gamma_B)$. Denoting it by $h(s)$, the wealth process of type B investors is

$$
\frac{dW_t^B}{W_t^B} = (r(s_t) - h(s_t)) dt + W^B(s_t)(\mu_Q(s_t)dt + \sigma_Q(s_t)dB_t).
$$  \hspace{1cm} (A6)

Applying Ito’s lemma to the consumption-wealth ratio and using Eq. (A6) we get

$$
\frac{dC_t^B}{C_t^B} = \left[ \frac{h''(s_t)}{2h(s_t)} \sigma_s(s_t)^2 + \frac{h'(s_t)}{h(s_t)} (\mu_s(s_t) + \sigma_s(s_t) \omega^B(s_t) \sigma_Q(s_t)) + \omega^B(s_t) \mu_Q(s_t) + r(s_t) - h(s_t) \right] dt
$$

$$
+ \left[ \frac{h'(s_t)}{h(s_t)} \sigma_s(s_t) + \omega^B(s_t) \sigma_Q(s_t) \right] dB_t.
$$  \hspace{1cm} (A7)
The optimality of consumption implies that the processes in Eqs. (A5) and (A7) are identical. Hence, their drifts and diffusions should coincide and omitting the argument $s_t$ we get

$$\mu_s + \sigma_D \sigma_s = r - \mu_D - h + \omega_B \mu_Q + \frac{h''}{2h} \sigma_Q^2 + \frac{h'}{h} (\mu_s + \sigma_s \omega_B \sigma_Q), \tag{A8}$$

$$\sigma_D + \frac{\sigma_s}{s} = \frac{h'}{h} \sigma_s + \omega_B \sigma_Q. \tag{A9}$$

Using the definition of $h$, these equations can be rewritten as

$$\omega_B \mu_Q = (\mu_s + \sigma_D \sigma_s) \left( \frac{1}{s} + \frac{h'}{\gamma_B} \right) - r + \mu_D + \exp \left( -\frac{H}{\gamma_B} \right) + \frac{\sigma_s^2}{2\gamma_B} \left( H'' + \frac{1}{\gamma_B} (H')^2 \right) + \frac{H' \sigma_Q^2}{\gamma_B s}, \tag{A10}$$

$$\omega_B \sigma_Q = \sigma_D + \frac{\sigma_s}{s} + \frac{h'}{\gamma_B} \sigma_s. \tag{A11}$$

Substituting Eqs. (A10) and (A11) into Eq. (13) and performing some algebra, we arrive at

$$\frac{1}{2\gamma_B} \left( H'' + \frac{1}{\gamma_B} (H')^2 \right) \sigma_s^2 + \mu_s \left( \frac{1}{s} + \frac{h'}{\gamma_B} \right) - r + \mu_D + \exp \left( -\frac{H}{\gamma_B} \right) + \frac{\sigma_s^2}{2\gamma_B} \left( H'' + \frac{1}{\gamma_B} (H')^2 \right) + \frac{H' \sigma_Q^2}{\gamma_B s} = \frac{\mu_Q}{\sigma_Q} \left( \sigma_D + \frac{\sigma_s}{s} + \frac{h'}{\gamma_B} \sigma_s \right). \tag{A12}$$

Substituting the market price of risk from Eq. (9) and the risk-free rate from Eq. (8) and doing some algebra we get another equation for $H$:

$$\frac{1}{2\gamma_B} \left( H'' + \frac{1}{\gamma_B} (H')^2 \right) \sigma_s^2 + \mu_s \left( \frac{1}{s} + \frac{h'}{\gamma_B} + \frac{\gamma_A}{1-s} \right) + (1 - \gamma_B) \sigma_s \left( \sigma_D + \frac{\sigma_s}{s} \right) \frac{H'}{\gamma_B}$$

$$- \frac{\gamma_A (1 - \gamma_A)}{2} \left( \sigma_D - \frac{\sigma_s}{1-s} \right)^2 + (1 - \gamma_A) \left( \mu_D + \frac{\gamma_A \sigma_A^2}{s} \right) - \beta_A + \exp \left( -\frac{H(s)}{\gamma_B} \right) + \frac{\gamma_A \sigma_A^2}{s(1-s)} \frac{H'}{\gamma_B} = 0. \tag{A13}$$

Subtracting Eq. (A13) from Eq. (A12), it is easy to obtain an explicit representation for $\mu_s$:

$$\mu_s = \left( \frac{\gamma_A}{1-s} + \frac{\gamma_B}{s} \right)^{-1} \left[ \beta_A - \beta_B + (\gamma_A - \gamma_B) \sigma_s \left( \sigma_D + \frac{\sigma_s}{s} \right) \frac{H'}{\gamma_B} + (\gamma_A - \gamma_B) \left( \mu_D + \frac{\gamma_A \sigma_A^2}{s} \right) \right]$$

$$+ \frac{\gamma_A (1 - \gamma_A)}{2} \left( \sigma_D - \frac{\sigma_s}{1-s} \right)^2 - \frac{\gamma_B (1 - \gamma_B)}{2} \left( \sigma_D + \frac{\sigma_s}{s} \right)^2 - \frac{\gamma_A \sigma_A^2}{s(1-s)} \frac{1}{1-s} \frac{H'}{\gamma_B}. \tag{A14}$$

Alternatively, it can be rewritten as

$$\mu_s = \left( \frac{\gamma_A}{1-s} + \frac{\gamma_B}{s} \right)^{-1} \left[ \beta_A - \beta_B + (\gamma_A - \gamma_B) \left( \mu_D + \frac{\gamma_A \sigma_A^2}{s} \right) + \frac{\gamma_A (1 - \gamma_A)}{2} \left( \sigma_D - \frac{\sigma_s}{1-s} \right)^2 \right]$$

$$- \frac{\gamma_B (1 - \gamma_B)}{2} \left( \sigma_D + \frac{\sigma_s}{s} \right)^2 - \frac{\gamma_A \sigma_A^2}{s(1-s)} \frac{1}{1-s} + \left( (\gamma_A - \gamma_B) \sigma_D - \sigma_s \left( \frac{\gamma_A}{1-s} + \frac{\gamma_B}{s} \right) \right) \sigma_s \frac{H'}{\gamma_B}. \tag{A15}$$

This is Eq. (19). To obtain Eq. (18), it is sufficient to combine Eq. (12) with Eq. (A11) taking into account the form of the margin constraints from Eq. (5) and the market price of risk from Eq. (9). Eq. (20) mimics Eq. (8) and Eq. (21) mimics Eq. (9).
Proof of Proposition 4

As in the previous proofs, the index A and obvious arguments of functions are omitted to make formulas less cumbersome. The portfolio problem of type A investors is standard except that the processes for returns (26) and the state variable (27) contain a jump component. The Bellman equation for the utility maximization problem is

$$\max_{\{C, \omega\}} \left[ e^{-\beta t} u(C) + DJ \right] = 0, \quad (A14)$$

where

$$DJ = J_W(\omega W(\mu_Q - \lambda(1 - q)\kappa_Q^+ - (1 - \lambda)q\kappa_Q^-) + rW - C) + J_t + \frac{1}{2} J_{WW} W^2 \omega^2 \sigma_Q^2$$

$$+ J_s \mu_s + \frac{1}{2} J_{s^2} + J_{W}s W \omega \sigma_Q \sigma_s + \lambda(1 - q)(J(W(1 + \omega\kappa_Q^+), s + \kappa_s^+, 1 - q, t) - J(W, s, q, t))$$

$$+ (1 - \lambda)q(J(W(1 + \omega\kappa_Q^-), s + \kappa_s^-, 1 - q, t) - J(W, s, q, t)). \quad (A15)$$

As before, we look for the value function in the following standard form:

$$J(W, s, q, t) = \frac{1}{1 - \gamma} W^{1-\gamma} \exp(H(s, q)) \exp(-\beta t), \quad (A16)$$

where the function $H(s, q)$ is assumed to be twice continuously differentiable in $s$. The maximization in Eq. (A14) over $C$ yields

$$e^{-\beta t} u'(C) - J_W = 0 \quad \text{or} \quad C = W \exp \left( -\frac{H(s, q)}{\gamma} \right). \quad (A17)$$

Using the value function (A16), the maximization over $\omega$ reduces to

$$\omega = \arg \max_\omega \left[ \omega(1 - \gamma)(\mu_Q - \lambda(1 - q)\kappa_Q^+ - (1 - \lambda)q\kappa_Q^-) - \frac{1}{2} \gamma(1 - \gamma)\sigma_Q^2 \omega^2 + (1 - \gamma)H'(s, q)\sigma_Q \sigma_s \omega$$

$$+ \lambda(1 - q) \left( (1 + \omega\kappa_Q^+)^{1-\gamma} \exp(H(s + \kappa_s^+, 1 - q) - H(s, q)) - 1 \right)$$

$$+ (1 - \lambda)q \left( (1 + \omega\kappa_Q^-)^{1-\gamma} \exp(H(s + \kappa_s^-, 1 - q) - H(s, q)) - 1 \right) \right]. \quad (A18)$$

The first order condition yields

$$\omega = \frac{\mu_Q + H' \sigma_Q \sigma_s}{\gamma \sigma_Q^2} + \frac{\lambda(1 - q) \kappa_Q^+}{\sigma_Q^2} \left( (1 + \omega\kappa_Q^+)^{-\gamma} \exp(H(s + \kappa_s^+, 1 - q) - H(s, q)) - 1 \right)$$

$$+ \frac{(1 - \lambda)q \kappa_Q^-}{\gamma \sigma_Q^2} \left( (1 + \omega\kappa_Q^-)^{-\gamma} \exp(H(s + \kappa_s^-, 1 - q) - H(s, q)) - 1 \right). \quad (A19)$$

In contrast to the case without jumps in margin constraints, the optimal portfolio policy cannot be found in a closed form because Eq. (A18) in general does not have analytical solutions. Substituting the optimal consumption (A17) along with the value function (A16) back into Eq. (A14), we get an equation for $H$:

$$\frac{1}{2} \sigma_s^2 (H'' + (H')^2) + H' \mu_s + r(1 - \gamma) - \beta + \gamma \exp \left( -\frac{H}{\gamma} \right) - \frac{\gamma(1 - \gamma)}{2} \omega^2 \sigma_Q^2$$

$$+ (1 - \gamma)\omega \left( \mu_Q + H' \sigma_Q \sigma_s - \lambda(1 - q)\kappa_Q^+ - (1 - \lambda)q\kappa_Q^- \right)$$

$$+ \lambda(1 - q) \left( (1 + \omega\kappa_Q^+)^{1-\gamma} \exp(H(s + \kappa_s^+, 1 - q) - H(s, q)) - 1 \right)$$

$$+ (1 - \lambda)q \left( (1 + \omega\kappa_Q^-)^{-\gamma} \exp(H(s + \kappa_s^-, 1 - q) - H(s, q)) - 1 \right) = 0. \quad (A19)$$

The definition of $s$ implies that $C = (1 - s)D$. Applying Ito’s lemma, we get

$$\frac{dC_t}{C_t} = \left( \mu_D - \frac{\mu_s + \sigma_D \sigma_s}{1 - s} \right) dt + \left( \sigma_D - \frac{\sigma_s}{1 - s} \right) dB_t - \frac{\kappa_s^+}{1 - s} (1 - q) dq_t^+ - \frac{\kappa_s^-}{1 - s} q dq_t^- \quad (A20)$$
Using the optimal consumption from Eq. (A17) and applying Ito’s lemma again, we obtain another representation for the consumption process:

\[
\frac{dC_t}{C_t} = \left[ \frac{1}{2\gamma} \left( -H'' + \frac{(H')^2}{\gamma} \right) \sigma_s^2 + \omega(\mu_Q - \lambda(1-q)\kappa_Q^+ - (1-\lambda)q\kappa_Q^-) + r - \exp\left( -\frac{H}{\gamma} \right) \right]
\]

\[
- \frac{H'}{\gamma}(\mu_s + \sigma_s \omega \sigma_Q) + \omega(\mu_Q - \lambda(1-q)\kappa_Q^+ - (1-\lambda)q\kappa_Q^-) + r - \exp\left( -\frac{H}{\gamma} \right) \right] dt + \left[ - \frac{H'}{\gamma} \sigma_s + \omega \sigma_Q \right] dB_t
\]

\[
+ (1-q) \left( (1 + \omega \kappa_Q^+) \exp \left( -\frac{1}{\gamma} (H(s + \kappa_s^+, 1 - q) - H(s, q)) \right) - 1 \right) dq_t^+
\]

\[
+ q \left( (1 + \omega \kappa_Q^-) \exp \left( -\frac{1}{\gamma} (H(s + \kappa_s^-, 1 - q) - H(s, q)) \right) - 1 \right) dq_t^-.
\]

(A21)

Comparing Eqs. (A20) and (A21) we get

\[
\mu_D - \frac{\mu_s + \sigma_D \sigma_s}{1 - s} = \frac{1}{2\gamma} \left( -H'' + \frac{(H')^2}{\gamma} \right) \sigma_s^2
\]

\[
+ \omega(\mu_Q - \lambda(1-q)\kappa_Q^+ - (1-\lambda)q\kappa_Q^-) + r - \exp\left( -\frac{H}{\gamma} \right) \right] dt + \left[ - \frac{H'}{\gamma} \sigma_s + \omega \sigma_Q \right] dB_t
\]

(A22)

\[
\omega \sigma_Q = \sigma_D - \frac{\sigma_s}{1 - s} + \frac{H'}{\gamma} \sigma_s,
\]

(A23)

\[
- \frac{\kappa_s^+}{1 - s} = (1 + \omega \kappa_Q^+) \exp \left( -\frac{1}{\gamma} (H(s + \kappa_s^+, 1) - H(s, 0)) \right) - 1,
\]

(A24)

\[
- \frac{\kappa_s^-}{1 - s} = (1 + \omega \kappa_Q^-) \exp \left( -\frac{1}{\gamma} (H(s + \kappa_s^-, 0) - H(s, 1)) \right) - 1.
\]

(A25)

The last two equations imply that

\[
(1 + \omega \kappa_Q^+)^{-\gamma} \exp (H(s + \kappa_s^+, 1) - H(s, 0)) = \left( 1 - \frac{\kappa_s^+}{1 - s} \right)^{-\gamma},
\]

(A26)

\[
(1 + \omega \kappa_Q^-)^{-\gamma} \exp (H(s + \kappa_s^-, 0) - H(s, 1)) = \left( 1 - \frac{\kappa_s^-}{1 - s} \right)^{-\gamma}.
\]

(A27)

The combination of Eqs. (A18), (A23), (A26), and (A27) yields

\[
\frac{\mu_Q}{\sigma_Q} = \gamma_A \left( \sigma_D - \frac{\sigma_s}{1 - s} \right) - \lambda(1-q) \frac{\kappa_Q^+}{\sigma_Q} \left[ \left( 1 - \frac{\kappa_s^+}{1 - s} \right)^{-\gamma} - 1 \right] - (1-\lambda)q \frac{\kappa_Q^-}{\sigma_Q} \left[ \left( 1 - \frac{\kappa_s^-}{1 - s} \right)^{-\gamma} - 1 \right].
\]

This is Eq. (29). Combining Eqs. (A19) and (A22), using Eq. (A23) for $H'$ and doing some algebra, the derivatives of $H$ can be eliminated:

\[
\omega(\mu_Q - \lambda(1-q)\kappa_Q^+ - (1-\lambda)q\kappa_Q^-) = \gamma \left( \sigma_D - \frac{\sigma_s}{1 - s} \right) \omega \sigma_Q - r + \beta + \gamma \mu_D - \frac{\gamma(\gamma + 1)}{2} \left( \sigma_D - \frac{\sigma_s}{1 - s} \right)^2
\]

\[
- \gamma(\mu_s + \sigma_D \sigma_s) \left( 1 - q \right) \left( (1 + \omega \kappa_Q^+) \exp (H(s + \kappa_s^+, 1 - q) - H(s, q)) - 1 \right)
\]

\[
- (1-\lambda)q \left( (1 + \omega \kappa_Q^-) \exp (H(s + \kappa_s^-, 1 - q) - H(s, q)) - 1 \right).
\]

(A28)

Substituting $\mu_Q$ from Eq. (29) and using again Eqs. (A26) and (A27) we arrive at Eq. (28).
Proof of Proposition 5

The proof of this proposition is very similar to the proofs of Propositions 1 and 4, so only its sketch is presented. Again, the index B as well as arguments of functions are omitted. The Bellman equation for the utility optimization problem is

$$\max_{\{C, \omega \leq \bar{m}\}} [e^{-\beta t} \log(C) + DJ] = 0,$$

where $DJ$ is given by Eq. (A15). The value function is assumed to have the standard form:

$$J(W, s, q, t) = (\beta \log W + H(s, q)) \exp(-\beta t).$$

(A30)

The maximization over $C$ immediately yields $C = \beta W$. The maximization over $\omega$ reduces to

$$\omega = \arg \max_{\omega \leq \bar{m}} \left[ \omega(\mu_Q - \lambda(1 - q)\kappa^+_Q - (1 - \lambda)q\kappa^-_Q) - \frac{1}{2}\sigma^2 Q \omega^2 \right.\left. + \lambda(1 - q) \log(1 + \omega\kappa^+_Q) + (1 - \lambda)q \log(1 + \omega\kappa^-_Q) \right].$$

Denoting the function to be maximized by $g(\omega)$, it is easy to show that $g''(\omega) < 0$. Hence, the maximization problem has a unique solution which is either an internal point $\omega^*$ satisfying the first order condition $g'(\omega^*) = 0$ or a point on the boundary $\omega = \bar{m}$ if $\omega^* > \bar{m}$. Hence, $\omega = \min(\bar{m}, \omega^*)$. The first order condition yields Eq. (31). ■

Proof of Proposition 6

Using the definition of the state variable $C^B_t = s_tD_t$ and applying Ito’s lemma we get

$$\frac{dC^B_t}{C^B_t} = \left( \frac{\mu_s(s_t, q_t)}{s_t} + \mu_D + \frac{\sigma_D \sigma_s(s_t, q_t)}{s_t} \right) dt + \left( \sigma_D + \frac{\sigma_s(s_t, q_t)}{s_t} \right) dB_t + \frac{\kappa^+_B(s_t)}{s_t} (1 - q_t) dq_t^+ + \frac{\kappa^-_B(s_t)}{s_t} q_t dq_t^-.$$

(A31)

Since the optimal consumption-wealth ratio of type B investors is constant, $dC^B_t/C^B_t = dW^B_t/W^B_t$ where

$$\frac{dW^B_t}{W^B_t} = (r(s_t, q_t) - \beta_B) dt + \omega^B(s_t, q_t)(\mu_Q(s_t, q_t) dt + \sigma_Q(s_t, q_t) dB_t$$

$$+ \kappa^+_Q(s_t)(1 - q_t)(dq_t^+ - \lambda dt) + \kappa^-_Q(s_t)q_t(dq_t^- - (1 - \lambda) dt)).$$

(A32)

Hence, omitting the arguments and the subscript $t$

$$\omega^B \mu_Q = \frac{\mu_s + \sigma_D \sigma_s + \kappa^+ B(1 - q)\lambda + \kappa^- B q(1 - \lambda)}{s} + \mu_D + \beta_B - r,$$

(A33)

$$\omega^B \sigma_Q = \sigma_D + \frac{\sigma_s}{s},$$

(A34)

$$\omega^B \kappa^+_B = \frac{\kappa^+_Q}{s}, \quad \omega^B \kappa^-_B = \frac{\kappa^-_Q}{s}.$$  

(A35)

The elimination of $\omega_B$ yields

$$\frac{\mu_Q}{\sigma_Q} = \frac{\mu_s + s(\mu_D + \beta_B - r) + \sigma_D \sigma_s + \kappa^+_Q(1 - q)\lambda + \kappa^-_Q q(1 - \lambda)}{s \sigma_D + \sigma_s},$$

(A36)

$$\frac{\kappa^+_Q}{\sigma_Q} = \frac{\kappa^+_s}{s \sigma_D + \sigma_s}, \quad \frac{\kappa^-_Q}{\sigma_Q} = \frac{\kappa^-_s}{s \sigma_D + \sigma_s}. $$

(A37)
Comparing Eqs. (29) and (A36) and using Eq. (A37) we get

\[
\frac{\mu_s + \sigma_D \sigma_s}{s} + \mu_D + \beta_B - r = \gamma_A \left( \sigma_D + \frac{\sigma_s}{s} \right) \left( \sigma_D - \frac{\sigma_s}{1-s} \right) - \lambda(1-q)\frac{\kappa_s^+}{s} \left( 1 - \frac{\kappa_s^+}{1-s} \right)^{-\gamma_A} - (1-\lambda)\frac{\kappa_s^-}{s} \left( 1 - \frac{\kappa_s^-}{1-s} \right)^{-\gamma_A}.
\]  

(A38)

Using Eq. (28), Eq. (A38) can be resolved for \( \mu_s \) and we obtain Eq. (34).

The derivation of Eq. (32) follows the proof of Lemma 1. Applying Ito’s lemma to \( S_t = D_tf(s_t,q_t) \) and using the structure of the dividend process, we get

\[
dQ_t = \left( \mu_D - \frac{\sigma_s^2}{2} f''(s_t,q_t) + (\mu_s + \sigma_D \sigma_s) f'(s_t,q_t) + \frac{1}{f(s_t,q_t)} \right) dt + \left( \sigma_D + \frac{f'(s_t,q_t)}{f(s_t,q_t)} \right) dB_t + f(s_t + \kappa_s^+,1-q_t) - f(s_t,q_t)(1-q_t) dq^+ + f(s_t + \kappa_s^+,1-q_t) - f(s_t,q_t)(1-q_t) dq^-.
\]

This process should coincide with the process from Eq. (26). Hence,

\[
\sigma_Q = \sigma_D + \frac{f'(s_t,q_t)}{f(s_t,q_t)} \sigma_s,
\]

(A39)

\[
\kappa_Q^+ = \frac{f(s_t + \kappa_s^+,1) - f(s_t,0)}{f(s_t,0)}, \quad \kappa_Q^- = \frac{f(s_t + \kappa_s^+,1) - f(s_t,1)}{f(s_t,1)}.
\]

(A40)

Combining these equations with Eq. (A37) we arrive at Eqs. (35).

Next, the formulas representing \( \mu_Q \) and \( \sigma_Q \) in terms of \( f(s,q) \) yield

\[
\frac{1}{2} \sigma_s^2 f''(s,q) + (\mu_s + \sigma_D \sigma_s) f'(s,q) + (\mu_D - r) f(s,q) + 1 - \frac{\mu_Q}{\sigma_Q} (\sigma_D f(s,q) + \sigma_s f'(s,q))
\]

\[
+ (f(s + \kappa_s^+,1) - f(s,0))(1-\lambda) + (f(s + \kappa_s^-,0) - f(s,1))q(1-\lambda) = 0.
\]

(A41)

Substituting \( \mu_Q/\sigma_Q \) from Eq. (A36) into Eq. (32) and eliminating non-local terms using Eqs. (35), we obtain a system of differential equations for \( f(s,0) \) and \( f(s,1) \). Lastly, rewriting Eqs. (30) and (31) as

\[
\omega B \sigma_Q = \min \left( \tilde{m} \sigma_Q, \frac{m_Q}{\sigma_Q} - \lambda(1-q)\frac{(\kappa_Q^+/\sigma_Q)^2 \omega B \sigma_Q}{1 + \omega B \kappa_Q^+} - (1-\lambda)q \frac{(\kappa_Q^-/\sigma_Q)^2 \omega B \sigma_Q}{1 + \omega B \kappa_Q^-} \right)
\]

and using Eqs. (A36), (A34), and (A35) we get

\[
\sigma_D + \frac{\sigma_s}{s} = \min \left( \tilde{m} \sigma_Q, \frac{m_s + \sigma_D \sigma_s}{s} + \mu_D + \beta_B - r + \frac{(1-q)\kappa_s^+}{s + \kappa_s^+} + \frac{q(1-\lambda)\kappa_s^-}{s + \kappa_s^-} \right).
\]

Substituting \( \sigma_Q \) from Eq. (A39) and eliminating \( \mu_s \) and \( r \) using Eq. (A38), we arrive at Eq. (33). This completes the proof. ■

Appendix B.

This Appendix reports some details on numerical methods used to solve non-linear differential equations from Sections I and III. The key idea is to use the projection method, which is essentially equivalent to looking for an approximate solution in the form of a linear combination of orthogonal polynomials (Judd, 1998). As an orthogonal basis Chebyshev polynomials of the first kind has been used. Since Chebyshev polynomials form an orthogonal basis in \( L^2([-1,1]) \), the state variable \( s \) defined on the interval \([0,1]\) is rescaled as

\[
z = 2s - 1,
\]

(A42)
and the range of $z$ is $(-1, 1)$. Clearly, this transformation monotonically and smoothly maps the interval $[0, 1]$ into an interval $[-1, 1]$: the boundary $s = 0$ corresponds to $z = -1$ and $s = 1$ corresponds to $z = 1$. The inverse map to (A42) is

$$s = \frac{1 + z}{2}.$$  

The change of variables (A42) also affects derivatives appearing in the differential equations:

$$\frac{\partial}{\partial s} = \frac{1}{2} \frac{\partial}{\partial z}, \quad \frac{\partial^2}{\partial s^2} = \frac{1}{4} \frac{\partial^2}{\partial z^2}.$$

More specifically, denote the solution to the system of differential equations as $\{Y^i(z)\}$, $i = 1, ..., L$ ($\{Y^1(z), Y^2(z)\} ≡ \{H(z), f(z)\}$ in Section I, $\{Y^1(z), Y^2(z)\} ≡ \{f(z, 0), f(z, 1)\}$ in Section III). Then, the projection method prescribes to look for a solution in the following form

$$Y^i(z) = \sum_{j=0}^{N} a^i_j T_j(z),$$

where $\{T_j(z), j = 0, ..., N\}$ are Chebyshev polynomials of the first kind and $N$ denotes the highest order. Thus, the problem is to find the coefficients $a^i_j$ such that $Y^i(z)$ minimizes the deviation of the left hand side of differential equations from zero and satisfies the boundary conditions. Since $T_j(-1) = (-1)^j$ and $T_j(1) = 1$ the boundary conditions at $z = -1$ and $z = 1$ put linear restrictions on the unknown coefficients $a^i_j$

$$Y^i(-1) = \sum_{j=0}^{N} a^i_j (-1)^j, \quad Y^i(1) = \sum_{j=0}^{N} a^i_j.$$

To set the objective function, I use the overidentified collocation method, which prescribes to minimize the sum of squared errors computed at points $\{z_m, m = 0, ..., M\}$ where $M > N$. For better approximation, instead of a uniform grid I use a Chebyshev array, which is a set of points where $T_M(z_m) = 0$ for all $\{z_m, m = 0, ..., M\}$. Since the optimization problem is overidentified, it is possible to ensure the existence of the solution if the value of the objective function at the optimal point is close to zero. In the practical implementation of the projection method I use $M = 100$ and $N = 30$. For these parameters, the error of approximation is typically of order $10^{-8}$. Moreover, the results are stable with respect to variation in the degree of Chebyshev polynomials and the number of points, and this stability provides additional evidence that the numerical approximation indeed converges to the exact solution.

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Figure 1. Proof of Proposition 3. This figure illustrates the idea of the proof of Proposition 3. The points $\sigma_s^c$ and $\sigma_s^{un}$ correspond to the equilibria in the constrained and unconstrained economies, respectively. The bold curve depicts the right hand side of Eq. (18) in the constrained economy.
Figure 2. The equilibrium without portfolio constraints. This figure presents the drift $\mu_s$ and the volatility $\sigma_s$ of the state variable $s$, the risk-free rate $r$, the market price of the risk $\eta$, the drift $\mu_Q$ and the volatility $\sigma_Q$ of excess returns on the risky asset, the price-dividend ratio $P/D$ and the portfolio weight $\omega^B$ of the risky asset held by type B investors. The function $H$ determines the indirect utility function of type $B$ investors. All statistics are functions of the consumption share $s$ of type B investors. The model parameters are as follows: $\mu_D = 0.018$, $\sigma_D = 0.032$, $\beta_A = \beta_B = 0.01$, $\gamma_A = 10$. The solid line corresponds to $\gamma_B = 1$, the dashed line corresponds to $\gamma_B = 2$, and the dashed-dotted line corresponds to $\gamma_B = 5$. 
Figure 3. The equilibrium with various types of margin constraints. This figure presents the ratios of various statistics in the constrained and unconstrained equilibria. $\mu_s$ is the drift and $\sigma_s$ is the volatility of the state variable $s$, $r$ is the risk-free rate, $\eta$ is the market price of risk, $\mu_Q$ is the drift and $\sigma_Q$ is the volatility of excess returns on the risky asset, $P/D$ is the price-dividend ratio and $\omega^B$ is the portfolio weight of the risky asset held by type B investors. The function $H$ determines the indirect utility function of type B investors. All statistics are functions of the consumption share $s$ of type B investors. The model parameters are as follows: $\mu_D = 0.018$, $\sigma_D = 0.032$, $\beta_A = \beta_B = 0.01$, $\gamma_A = 10$, $\gamma_B = 2$. The margin constraints are specified by Eq. (6) with $\bar{m} = 1.3$. The solid line corresponds to a constant portfolio constraint ($\alpha = 0$), the dashed line corresponds to the VaR-type constraint ($\alpha = 1$), and the dashed-dotted line corresponds to the “aggressive” constraint ($\alpha = 2$).
Figure 4. The impact of hedging demand of type B investors on the equilibrium in the presence of portfolio constraints. This figure presents the ratios of various statistics in the equilibria with fully rational and myopic preferences of type B investors. $\mu_s$ is the drift and $\sigma_s$ is the volatility of the state variable $s$, $r$ is the risk-free rate, $\eta$ is the market price of risk, $\mu_Q$ is the drift and $\sigma_Q$ is the volatility of excess returns on the risky asset, $P/D$ is the price-dividend ratio and $\omega^B$ is the portfolio weight of the risky asset held by type B investors. All statistics are functions of the consumption share $s$ of type B investors. The model parameters are as follows: $\mu_D = 0.018$, $\sigma_D = 0.032$, $\beta_A = \beta_B = 0.01$, $\gamma_A = 10$, $\gamma_B = 2$. The margin constraints are specified by Eq. (6) with $\bar{m} = 1.3$. The solid line corresponds to an unconstrained equilibrium, the dashed line corresponds to an equilibrium with a constant portfolio constraint ($\alpha = 0$), and the dashed-dotted line corresponds to an equilibrium with the VaR-type constraint ($\alpha = 1$).
Figure 5. The equilibrium with jumps in margin requirements. This figure presents various statistics in the equilibrium with switching margin requirements. \( \mu_s \) is the drift and \( \sigma_s \) is the volatility of the state variable \( s \), \( \kappa_s \) is the size of the jump in the state variable \( s \) when the margin constraint switches the state \( q \), \( r \) is the risk-free rate, \( \eta \) is the market price of diffusion risk, \( \mu_Q \) is the drift and \( \sigma_Q \) is the volatility of excess returns on the risky asset, \( \kappa_Q \) is the size of the jump in returns when the margin constraint switches the state \( q \), \( P/D \) is the price-dividend ratio. All statistics are functions of the consumption share \( s \) of type B investors. The model parameters are as follows: \( \mu_D = 0.018, \sigma_D = 0.032, \beta_A = \beta_B = 0.01, \gamma_A = 10, \gamma_B = 1, m^+ = 1.1, m^- = 10, \lambda = 0.977 \). The solid lines represent equilibria in the benchmark economies with constraints fixed at the levels \( \bar{\omega} = 1.1 \) and \( \bar{\omega} = 10 \). The dashed line corresponds to the state with loose margin requirements \( (q = 0) \), and the dashed-dotted line corresponds to the state with tight margin requirements \( (q = 1) \).