Fixed-income securities

Topics

- This class: **Bonds**
- Valuation of annuity contracts: hedging and arbitrage
- Yield-to-maturity of a bond
- Duration and bond portfolio immunization

Terminology

- A **financial instrument** represents the right on a cash stream
- A **security** is a financial instrument that is traded on a well-developed market
- A **fixed income security** represents a fixed (definite) cash stream

  Fixed income securities are important because they define the market for money, and most investors participate in this market. They also define a reference benchmark for other investment opportunities that are not traded in markets, such as a firm's research projects, oil leases and royalty rights.

Bonds

- A bond is a fixed-income security that promises a fixed amount at its maturity, called the **face value**; there are two general ways in which interest is paid
- A bond with a single terminal cash flow issued at a discount is called a **discount bond** or **zero-coupon bond**
- A bond with fixed periodic interest payments (coupons) is called a **coupon bond**; the coupon amount is expressed as a percentage of the face value

Bonds

Cash stream of bonds

Bonds

Bonds issued by the US government

- **Treasury bills**: maturities of 13, 26 and 52 weeks, issued at a discount each week, highly liquid market
- **Treasury notes**: maturities of 1 to 10 years, pay a fixed coupon every 6 months
- **Treasury bonds**: maturities of more than 10 years, pay a fixed coupon every 6 months
- **Treasury strips** are bonds issues in stripped form: each of the coupons is issued separately, as is the principal, generating several zero-coupon bonds

Bonds

Other bond issuers

- **Municipal bonds** are issued by agencies of state and local governments; exempt from federal and local state taxes
  - General obligation bonds are backed by a governing body
  - Revenue bonds are backed by the revenue of the project the bond was issued for or the sponsoring agency
- **Corporate bonds** are issued by corporations to raise money
  - Often an agency rating describes the credit quality of the issuer
  - Most are traded over-the-counter (OTC)
- **Sovereign bonds** are issued by foreign governments

Bonds

Bond indenture (contract of terms)

- A bond is **callable** if the issuer has the right to repurchase the bond at a specified price
- A bond is **putable** if the owner has the right to sell the bond back to the issuer at face value on designates dates
  - A bond with put or call features has **embedded options**
- To protect bond investors, they may be guaranteed that in the event of bankruptcy, payment to them takes priority over payments of other debt–the other debt being **subordinated**

Bonds

Bond rating

- Many bonds are subject to default by the issuer
  - Greece, Orange County, Lehman Brothers
- Private rating agencies such as Moody's, Standard and Poor's and Fitch assign a rating to bond issues, which is supposed to describe the creditworthiness of the issuer
  - Based primarily on accounting information such as various financial ratios, and industry assessments
- The rating is an important pricing factor
- Many investors are subject to restrictions on the class of bonds they can invest in

Bonds

Rating classifications

	Moody's	S & P
Investment Grade	Aaa	AAA
	Aa	AA
	A	A
	Baa	BBB
Speculative Grade	Ba	BB
("junk")	B	B
	Caa	CCC
	Ca	CC
	C	C
		D

Mortgage

- A loan secured by property
- Pre-payment by lender possible
- Fixed-rate and adjustable rate mortgages
- Residential and commercial mortgages
- Mortgage-backed securities are securities issued against a large pool of mortgages (carry pre-payment and default risk)

Annuity

- Pays the holder money periodically according to a fixed schedule over a period of time
  - Pension benefits often taken the form of an annuity
- A perpetual annuity, or perpetuity, pays a fixed sum forever
  - In the UK, these are known as consol bonds

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Valuation

Valuing a perpetuity

- Suppose the perpetuity pays A at the end of each period starting at the end of the first period
- With a compounding cycle equal to that period and corresponding per-period rate $r > 0$, we get the present value

$$\begin{aligned} PV &= \sum_{i=1}^{\infty} \frac{A}{(1+r)^i} = \frac{A}{1+r} + \sum_{i=2}^{\infty} \frac{A}{(1+r)^i} = \frac{A}{1+r} + \sum_{i=1}^{\infty} \frac{A}{(1+r)^{i+1}} \\ &= \frac{A}{1+r} + \frac{1}{1+r} \sum_{i=1}^{\infty} \frac{A}{(1+r)^i} = \frac{A}{1+r} + \frac{1}{1+r} PV \qquad (1) \end{aligned}$$

- Solving equation (1) for PV gives $PV = \frac{A}{r}$
- We have used the PV as a **pricing rule**

Valuation

Valuing an annuity with life of n periods

- Consider a portfolio of two perpetuities paying A
  - Long a perpetuity that starts today (time 0): this position has present value $\frac{A}{r}$ (=cost)
  - Short a perpetuity that starts at time n (it pays first at time $n+1$): this position has present value $\frac{A}{r}\frac{1}{(1+r)^n}$ (=income)
- Since the cash flows to the portfolio are the same as the cash flows to a long position in an annuity paying A with life n, the cost of that annuity must be equal to the cost to buy the portfolio
- It follows that the value of the annuity is

$$\frac{A}{r}\left(1-\frac{1}{(1+r)^n}\right) \tag{2}$$

Letting $n \to \infty$, we get the value of a perpetuity as a special case

Valuation

Valuation

Hedging

- The valuation principle we just applied is one of the most important concepts in finance
- We replicated the finite life annuity using two perpetuities
  - That is, we generated a synthetic annuity of finite life
- This is called **hedging** in finance; the corresponding portfolio is called the **hedging portfolio**

Valuation

Building a hedge

- Suppose we have sold at time 0 a finite life annuity for $\frac{A}{r}(1 - \frac{1}{(1+r)^n})$ so we have to pay A at the end of every period until time n to the buyer
- Here is how we can cover ourselves for this obligation:
  - At time 0, we borrow $\frac{A}{r}\frac{1}{(1+r)^n}$ from the bank, to be returned with interest at time n
  - Also at time 0, using that money together with the proceeds from the sale of the finite life annuity, we buy a perpetuity paying A from 0, costing $\frac{A}{r}$
  - At time n, we sell a perpetuity starting at n, giving us $\frac{A}{r}$
  - We use these proceeds to pay off our debt in the bank at n

Valuation

Arbitrage

- The cost of the hedge (the initial price of the portfolio) offsets the selling price of the annuity and we are hedged perfectly over the life of the annuity (but we also give up any profit)
- Suppose the equality does not hold and the annuity can be sold for more than the value of the portfolio:
  - We can manufacture the annuity for a cost that is less than the price we can sell it for
  - If there is a liquid market for the perpetuities and the annuity, we can make **riskless** profits
  - Such an opportunity is called an **arbitrage**, or “free lunch”
- It is a basic postulate in finance that such opportunities do not exist

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Yield

Yield to maturity for an option-free bond

- The internal rate of return on a bond with fixed maturity is called the **yield to maturity** (YTM)
- Suppose a bond with face value F makes m coupon payments of C/m dollars each year and that there are n coupon periods remaining; if P is the current price of the bond observed in the market, then this is the value of λ such that

$$P = \frac{F}{(1+\frac{\lambda}{m})^n} + \sum_{i=1}^{n} \frac{C/m}{(1+\frac{\lambda}{m})^i} \tag{3}$$

- The YTM λ is the annual interest rate implied by the bond price if interest is compounded m times a year
- Quoting convention

Yield

Yield to maturity for an option-free bond

- Note that the coupon payments C/m from now until time n can be viewed as a finite annuity
- Applying the valuation formula (2) for a finite annuity to the coupons C/m (set $r = \lambda/m$ and $A = C/m = Cr/\lambda$ in (2)), we find that a bond with YTM $\lambda > 0$ and price P satisfies

$$P = \frac{F}{(1+\frac{\lambda}{m})^n} + \frac{C}{\lambda}\left(1 - \frac{1}{(1+\frac{\lambda}{m})^n}\right) \tag{4}$$

Yield

Price vs. yield

- The market price P of a bond with a given coupon is determined by supply and demand on the bond market
  - Level of current interest rates in the economy: if rates change (Fed), then the bond price adjusts so that the yield matches these new rates
  - Creditworthiness of the issuer ("promised YTM")
  - Flight to quality
- Understanding this relation is important for
  - Trading
  - Managing interest rate risk for bond investments

Yield

Price-yield curves for an option-free bond

- The relationship between a bond's price P and its yield is shown in a price-yield curve
  - The bond price is decreasing in the yield; the curve is convex
  - If the yield is 0, then the price is equal to the sum of all payments
  - If the yield matches the coupon rate, then the bond price is equal to the face value and we say the bond is **at par**
- The steepness of the curves increases with maturity: the price of a bond is more sensitive to changes in yield the longer the maturity

Yield

Price-yield curves for an option-free bond

Yield

- If $\lambda = 0$, then the definition (3) gives

$$P = \frac{F}{1^n} + \sum_{i=1}^{n} \frac{C/m}{1^i} = F + \frac{C}{m} n$$

- With $C = cF$ and c the annual coupon rate, for $\lambda = c$ we get from the alternative definition (4)

$$P = \frac{F}{(1 + \frac{\lambda}{m})^n} + \frac{\lambda F}{\lambda} \left(1 - \frac{1}{(1 + \frac{\lambda}{m})^n} \right) = F$$

since we discount at the same rate as the bond accumulates interest

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Duration

First definition

- Maturity itself gives only a crude measure of a bond's interest rate sensitivity
- We consider a better measure of time length called **duration**, which is the PV-weighted average of the times at which payments are received:

$$\frac{1}{PV}\sum_{i=0}^{n} t_i \cdot PV_i$$

where t_i is the time of a payment received, PV_i is the present value of that payment and $PV = \sum_i PV_i$ is the present value of the bond

Duration

Simple properties of duration

- It has units of time
- If the cash flows are all ≥ 0, then the duration is in $[t_0, t_n]$
- The duration of a zero-coupon bond is equal to its maturity
- The duration of a coupon bond is less than its maturity

Duration

Macaulay duration for an option-free bond

- If we use the yield of the bond to discount its payments in the duration formula, we obtain the **Macaulay duration**, denoted D
- For a bond making payments m times a year, with c_i the payment in period i, and n periods remaining, D is given by

$$D = \frac{1}{PV} \sum_{i=1}^{n} \frac{i}{m} \cdot \frac{c_i}{(1 + \frac{\lambda}{m})^i}$$

where λ is the YTM and

$$PV = \sum_{i=1}^{n} \frac{c_i}{(1 + \frac{\lambda}{m})^i}$$

Since the factor i/m is time in years, D is measured in years

Duration

Price sensitivity

- **Theorem.** Consider a bond with price $P(\lambda)$ as a function of yield λ and Macaulay duration D, all based on m compounding cycles per year. We have

$$P'(\lambda) = \frac{dP(\lambda)}{d\lambda} = -D_M P(\lambda)$$

where the variable

$$D_M = \frac{D}{1 + \frac{\lambda}{m}}$$

is called the **modified duration**.

- **Proof.** Exercise in taking derivatives (homework).

Duration

Taylor's theorem

- Let $h > 0$. If $P = P(\lambda)$ is continuous on $[\lambda, \lambda + h]$ and has continuous derivatives of order $n - 1$ and the derivative of order n exists, then

$$P(\lambda + h) - P(\lambda) = hP'(\lambda) + \frac{h^2}{2!}P''(\lambda) + \ldots + \frac{h^n}{n!}P^{(n)}(\lambda + \Theta h)$$

where $\Theta \in (0, 1)$

Duration

Estimating the impact of yield variations

- A first order approximation gives for small changes $\Delta\lambda$ in the yield the corresponding change ΔP in the price as

$$\Delta P \approx -D_M P \Delta\lambda$$

- The change in the bond price ΔP for a 1 basis point change $\Delta\lambda = 0.0001$ in the yield is called the **price value of a basis point** or the **dollar value of a basis point (DV01)**

Duration

A picture

Duration

Example

- Consider a 30yr zero-coupon bond with current yield $\lambda = 10\%$
  - What is the relative price change $\frac{\Delta P}{P}$ if the yield changes to 11%?

Duration

Example

- Consider a 30yr zero-coupon bond with current yield $\lambda = 10\%$
  - The Macaulay duration is $D = 30$ years
  - The modified duration is $D_M = \frac{D}{1+\lambda} = \frac{30}{1+.1} \approx 27$ years
- If the yield changes to 11%, then the relative price change is

$$\frac{\Delta P}{P} \approx -D_M \Delta\lambda \approx -27 \cdot 0.01 = -27\%$$

Duration

A better approximation

- We include a second-order (quadratic term) based on **convexity** C, the relative curvature at a given point of the price-yield curve:

$$C = \frac{1}{P(\lambda)} \frac{d^2 P(\lambda)}{d\lambda^2}$$

- $C \geq 0$ since $P(\lambda)$ is convex
- C has units of time squared
- Taylor's theorem yields the second order approximation

$$\Delta P \approx -D_M P \Delta\lambda + \frac{PC}{2}(\Delta\lambda)^2$$

Duration

Duration of a portfolio

- **Theorem.** Consider n fixed-income securities with prices P_i and Macaulay durations D_i, respectively, all computed at a common yield. The portfolio of these securities has price $P = \sum_{i=1}^{n} P_i$ and duration

$$D = \sum_{i=1}^{n} w_i D_i$$

  where $w_i = \frac{P_i}{P}$ is the weight of security i.

- **Proof.** Homework.

Immunization

Suppose you have a stream $x = (x_1, x_2, \ldots, x_n)$ of future obligations. How do you cover yourself today?

- You buy zero coupon bonds that match the x_i, but
  - Most corporate bonds are coupon bonds
  - Corporate bonds are subject to default risk
  - It is hard to find maturities that exactly match
- You buy a bond portfolio whose value today matches the $PV(x)$ calculated at the current yield (assuming there is one such yield)
  - If a cash flow is due, you sell parts of the portfolio
  - Excess cash generated by the portfolio is used to buy more bonds
  - As long as the yield does not change, the value of the portfolio will continue to match the PV of the remaining obligations

Immunization

Hedging a portfolio against interest rate risk

- If yields change, then the change in the value of your portfolio is likely to be different from the change in $PV(x)$ so you are exposed
- This problem can be solved by constructing a bond portfolio that matches not only the $PV(x)$, but also
  - The duration of x
  - The convexity of x
- In this case, the portfolio value responds to changes in the yield in approximately (to second order) the same way as $PV(x)$, and hence the portfolio will still be adequate to cover your obligations
- Re-balance the immunization portfolio from time to time
- You will work on an explicit example in one of the projects