On The Discovery of Deductive Science

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by Curtis Wilson

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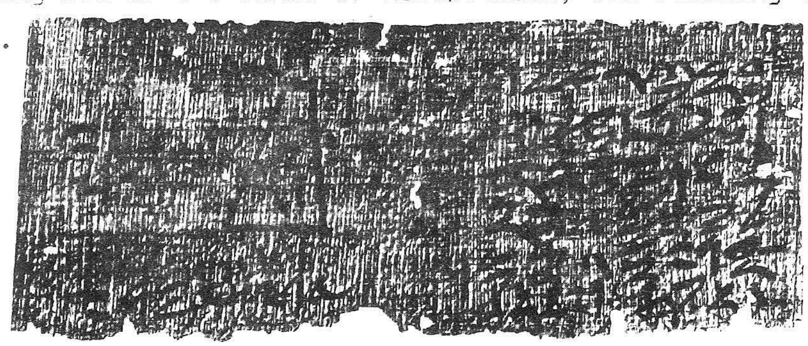
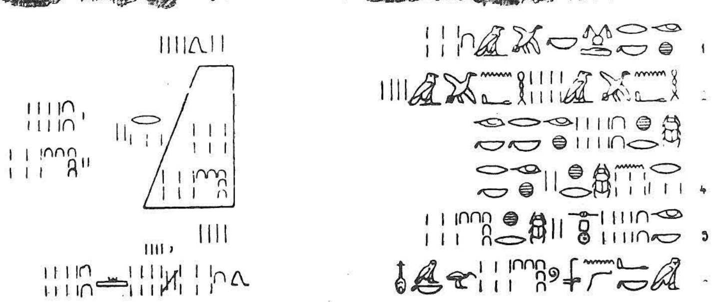
How did the notion of deductive science--science based on definitions, postulates, and axioms, science consisting of a sequence of propositions, each of which is deduced, either from previously deduced propositions, or from the definitions, postulates, and axioms initially set out--how did this notion first come to be thought of, and then realized? For there seems to have been a particular moment in which this idea was first conceived; so far as we can tell, it did not make its appearance at different times and places, independently. Can we learn anything about the original conception? I am going to pursue this question, althougn as you will at once realize, it is not the sort of question that is likely to receive a non-conjectural answer. The ground here has been worked into a deep and slippery mud by the trampling feet of contending scholars; mere non-classicists or not yet classicists like myself are liable to stumble over the mouldering carcasses of defunct theories, not yet decently interred. Certain questions of historical fact that are material to this discussion I am able to answer only conjecturally. At the same time, I wish to affirm that my primary aim is not to establish historical facts, nor yet to hypothesize possible causes for those facts, but rather to locate the meaning of facts that, it seemed to me, come nearest to being reliable.[1]

I want to begin by saying something about pre-Greek mathematics. The oldest mathematical documents known from any place on this earth are Egyptian papyri stemming from the Middle Kingdom, 2000-1800 B.C., and clay tablets dug out of the sands of Mesopotamia, and stemming from about 1800-1600 B.C.





In figure I you see a transcription from a papyrus now in Moscow, showing the computation of the volume of a truncated pyramid with square base and top. The base is four cubits on a side, the top two cubits on a side, and the height or distance between base and top is six cubits. The text says: "Add together this 16 with this 8 and this 4. [16 is the area of the base, 4 the area of the top, and 8 the product of the side of the base by the side of the top.] You get 28. Compute one-third of 6 [the height]; you get 2. Multiply 28 by 2. You get 56. Behold: it is 56. You have found right."

Now the result <u>is</u> right. It is something you might want to know if you were building pyramids; but by the time of the Middle Kingdom the Egyptians had ceased building pyramids, enjoyable though that occupation seems to have been, as we gather from the inscriptions of rival work gangs. It is not clear that there was any immediate practical reason for anyone in the Middle Kingdom to know the rule for computing the volume of a truncated pyramid. But the real puzzle is how this rule was discovered in the first place. It is a complicated rule, and there is no plausible empirical way of arriving at it by, say, weighing certain objects; therefore reasoning was involved. But on the other hand, the Egyptian mathematicians would not fall back on algebraic transformations in the modern manner, since their mathematics dealt explicitly only with particular numbers. There are a number of hypotheses as to how the Egyptians' procedure could have been arrived at, the most plausible, I think, involving a slicing of the pyramid into parts.

Let us take another example.

"A square and a second square whose side is $\overline{2} + \overline{4}$ of the first square, have together an area of 100. Show me how to calculate this."

> [I note that Egyptian fractions, with one exception, are unit fractions, fractions we would write with 1 as numerator. They are written by putting a line above the number we call the denominator. The exception was 2/3, written by putting two of these lines above the numeral 3.
>
> Now for the solution:]

"Take a square of side 1, and take $\overline{2} + \overline{4}$ (3/4) of 1 as the side of the other square.

"Multiply $\overline{2} + \overline{4}$ by itself; this gives $\overline{2} + \overline{16}$.

"Hence, if the side of one of the areas is taken to be 1, and that of the other is $\overline{2} + \overline{4}$, then the addition of the areas gives $1 + \overline{2} + \overline{16}$.

"Take the square root of this; it is $1 + \bar{4}$.
"Take the square root of the given number 100; it is 10.
"How many times is $1 + \bar{4}$ contained in 10? Answer 8."

The two squares then have sides $8 \times 1 = 8$ and $8 \times \frac{3}{4} = 6$, the sum of their squares being 100.

Now Egyptian mathematics has certain general characteristics. First, Egyptian mathematics, whatever it is dealing with--areas, volumes, numbers of bricks or loaves of bread or jugs of beer--is always a matter of numerical calculation. The mathematician is a computer who uses both integers and fractions. Second, there are no explicit proofs whatever, but reasonings <u>have</u> to have been employed in the solution of problems. Finally, while the problems presented in the papyri seldom appear to be actual practical problems, they give the general impression of being the sort of problems that an instructor might think up for his students, in order to prepare them for solving practical problems. Instructors seldom succeed in being strictly practical, but the Egyptian ones appear to have understood their activity as occurring within the horizon of the practical.

Aristotle claimed that the mathematical arts had been founded in Egypt, because there the priestly class was allowed leisure; but this is incorrect. The Egyptian calculative art was the possession not of a priestly class, but of scribes who had practical functions in the state, and among whom there was rivalry. So we find one scribe ridiculing another:

> "You come to me to inquire concerning the rations for the soldiers, and you say 'reckon it out.' You are deserting your office!.... I cause you to be abashed when I bring you a command of your lord, you who are his Royal Scribe. A building ramp is to be constructed, 730 cubits long, 55 cubits wide, 55 cubits high at its summit.... The quantity of bricks needed for it is asked of the generals, and the scribes are all asked together, without one of them knowing anything. They all put their trust in you.... Behold your name is famous... Answer us how many bricks are needed for it?"

It seems likely, then, that the mathematical papyri were textbooks used in the school for scribes.

In Babylonia, the mathematical texts appear to have been produced by a similar class of scribes. The texts give problems with their solutions; proofs are entirely absent; the procedures are always numerical. Are the problems practical problems? Once again, yes and no. Here is an example from the time of Hammurabi, 1700 B.C.:

> "I have multiplied length and width, thus obtaining the area. Then to the area I added the excess of the length over the width. The total result is 183. I have also added the length and width, with the result 27. Required: length, width, and area."

I omit the solution. For us it would involve the solution of a quadratic equation. This Babylonian problem does not strike me as a practical problem, or a near neighbor to one. The adding of a length to an area seems to me decidedly impractical, perhaps even nonsensical. This is mathematics gone a bit haywire: a pedagogue might invent it to bemuse his pupils, always understanding, of course, that calculating is a good thing.

Babylonian mathematics, however, is a good deal more powerful than Egyptian mathematics. When the Babylonian scribe wrote: $\sqrt{2} = 1;\ 24,51,10$ (I am using the Indian numerals in place of the Babylonian), he meant $1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$. This is the Babylonian approximation to the square root of 2, or diagonal of a square of unit side. The Babylonians definitely knew and used the proposition we call the theorem of Pythagoras, which is involved in getting this approximation, but nowhere do any of the clay tablets that have been deciphered give a proof of this or any other theorem. The approximation, which is probably the result of a series of successively closer approximations, is good to one-millionth. Ptolemy will still be using it, having acquired it probably indirectly from the Babylonians, when he computes his table of chords in the second century A.D.

Now if we turn to other civilizations besides the Egyptian and the Babylonian, but still uninfluenced by Greek thought--the civilization of the Yellow River valley, say, or Mayan civilization--I think we shall once again find a computational art, often highly developed, but not explicit deductions. You may on occasion find the contrary asserted. Joseph Needham in his Science and Civilization in China gives a passage from a Chinese mathematical text which perhaps originated as early as the 4th century B.C.; it is accompanied by a diagram which he labels "proof of the Pythagoras Theorem" (Figure 2, p.5).

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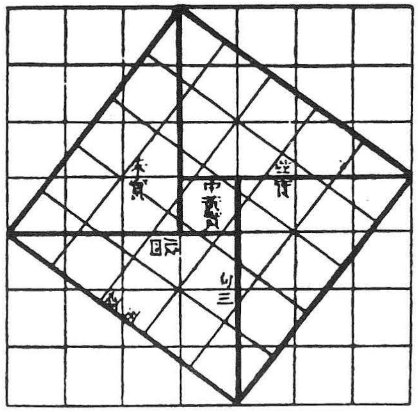


I quote from the text:

"Of old, Chou Kung addressed Shang Kao, saying, "I have heard that the Grand Prefect [that is Shang Kao] is versed in the art of numbering. May I venture to inquire how Fu-Hai anciently established the decrees of the celestial sphere?... I should like to ask you what was the origin of these numbers?

[In the course of his reply Shang Kao says:]

"Let us cut a rectangle diagonally, and make the width 3 units, and the length 4 units. The diagonal between the corners will then be 5 units long. Now after drawing a square on this diagonal, circumscribe it by half rectangles like that which has been left outside, so as to form a square plate. Thus the outer half rectangles of width 3, length 4, and diagonal 5, together make two rectangles [of total area 24]; then the remainder [that is, of the square of area 49] is of area 25. This is called 'piling up the rectangles.'

"The methods used by Yu the Great in governing the world were derived from these numbers... He who understands the earth is a wise man, and he who understands the heavens is a sage. Knowledge is derived from a straight line. The straight line is derived from the right angle. And the combination of the right angle with numbers is what guides and rules the ten thousand things.

"Chou Kung exclaimed, 'Excellent indeed!'"

Nothing here, I would urge, has really been proven, certainly not the theorem of Pythagoras, so-called. Needham has shown in overwhelming detail that between the 5th century B.C. and the 15th century A.D. no people on earth exercised more technical ingenuity than the Chinese. Long in advance of the West, they possessed caste iron, an escapement clock, the navigational compass, gunpowder, printing by movable type, the segmental arch bridge. But as for deductive science, the Chinese would not encounter it until the Jesuits came to China in the late 16th century, bringing the textbooks of their fellow Jesuit, Christopher Clavius. It is a curious fact that, for some centuries thereafter, Chinese students reciting their Euclidean theorem out of Clavius would finish not with our Q.E.D. but with the Chinese word for "nail." Apparently they were citing their authority: "clavus" being the Latin word for "nail."

It is conceivable that some day, in the investigation of early civilizations uninfluenced by Greece, evidence will turn up for the existence of some pieces of deductive mathematics. On the basis of what is known today, the prospects for such a find are dim. Deductive mathematics is a rare bird, which first settled, so far as we know, in Greece.

How did it happen? What did it mean that it happened? Seeking an answer, I turn to a document of late antiquity, a commentary on the first book of Euclid's Elements written by Proclus in the middle of the 5th century A.D. Proclus was a member of the Platonic Academy in Athens during the last century of its 900-year existence. The commentary includes a kind of catalogue of ancient geometers which is based on an earlier history of geometry, now lost, by Eudemus, a disciple of Aristotle writing in the late 4th century B.C. The account begins by saying that geometry was first discovered among the Egyptians, and originated in the remeasuring of their lands necessitated by the annual flooding of the Nile. Proclus then proceeds as follows:

> Thales, having travelled in Egypt, first introduced this theory into Hellas. He discovered many things himself, and pointed the road to the principles of many others, to those who came after him, attacking some questions in a more general way, and others in a way more dependent on sense perception.

After mentioning the names of two other ancient geometers, Proclus continues:

> After these, Pythagoras transformed the philosophy of this (geometry) into a scheme of liberal education. He surveyed its principles from the highest on down, and investigated its theorems separately from matter and intellectually. He it was who discovered the doctrine of irrationals

> and the construction of the cosmic figures.

A little farther on we read:

> Hippocrates of Chios, who invented the method of squaring lunules (crescents formed from arcs of circles) and Theodorus of Cyrene became eminent in geometry. For Hippocrates wrote a book on elements, the first of whom we have any record who did so.

With respect to Hippocrates of Chios, there is no reason to doubt what Proclus says. A fragment of Hippocrates' work on lunules still exists, and it shows a high level of rigor. Thus Hippocrates may very will have written a book on the elements of geometry. Thus, at the time Hippocrates was teaching geometry in Athens, around 430 B.C., the process of turning geometry into a deductive science was in all probability well advanced.

Thales, who was active about a century and a half before Hippocrates of Chios, is a much more shadowy figure, and it is unclear how we should interpret what Proclus says about him. Proclus attributes to Thales the discovery and proof of five propositions:

(1) A circle is bisected by any diameter.

(2) Vertical angles of intersecting straight lines are equal.

(3) The base angles of an isosceles triangle are equal.

(4) Two triangles such that two angles and the included side of one are equal to two angles and the included side of the other, are themselves equal.

(5) The angle at the periphery of a semicircle is right.

Now these are general, theoretical propositions, theorems, propositions to be contemplated rather than mere rules for solution of problems. The enunciation of them may therefore mark a decisive step in the emergence of theoretical science. But how were they proved? The usual guess is that it was by superposition, the visual showing that one figure or part of a figure would coincide with another. If this is right, then it is unlikely that we have here the notion of a logically constructed theory which begins with expressly enunciated premises and advances step by step. Thales need not have enunciated any premises explicitly. He pointed the road to the principles, as Proclus says; the extent to which he laid out principles is totally unclear.

As for Pythagoras, whole books have been devoted in recent times to showing that the ancient accounts of his mathematical exploits are unworthy of trust.[2] These accounts stem from members of the Platonic Academy from the 4th century and later, men who saw in the 6th-century Pythagoras a forerunner of Plato, and who tended to attribute to him discoveries that had been made later on in the Pythagorean tradition. Pythagoras cannot have known all the five cosmic figures, because two of them, the octahedron and the icosahedron, were first discovered by Theaetetus, a contemporary of Plato. Contrary to what Proclus says, there is no good evidence that Pythagoras knew anything about the doctrine of irrational lines. The old verse quoted by Plutarch, according to which Pythagoras, on making a certain geometrical discovery, sacrificed an ox, cannot be true, because it is well attested that Pythagoras was a vegetarian, who believed in transmigration of souls and was opposed to the killing of animals. What we can be fairly sure of, with regard to Pythagoras, aside of course from his having had a golden thigh, is that he had made the flight to the Beyond and had become the leader of a cult, a medicine man, a shaman. He can well have taught that odd numbers are male, even numbers female; that five is the marriage number; that ten is perfect, being the sum of 1, 2, 3, and 4. Somewhat similar beliefs have been found all over the world, in connection with rituals and creation myths, and have not led to deductive mathematics. Pythagoras' thought seems to have been cosmogonic, concerned with the coming-to-be of our world out of something prior and more fundamental. There is no trustworthy evidence that Pythagoras ever carried out an explicit proof.

On the other hand, the transformation in the character of mathematics that Proclus attributes to Pythagoras may well have been brought about by Pythagoreans. The old accounts speak of a split within the Pythagorean tradition; the Mathematikoi, those who wished to discuss and teach openly the mathematical disciplines, separated off from the secret cult, the Akousmatikoi, the hearers of the sacred and secret sayings. Reliable 4th-century sources speak of the arithmetical studies of the 5th-century Pythagoreans. Aristotle says that the so-called Pythagoreans were the first to deal with mathemata, mathematical disciplines. According to the *Epinomis*, a dialogue written either by Plato or a follower of Plato, the first and primary disciplines or mathema of the Pythagoreans was arithmetic. Now it is possible to make a plausible reconstruction of some of this early Pythagorean arithmetic. When this is done, we find ourselves before a piece of deductive science, quite possibly the earliest that ever was; and it is a science in which the principles are explicit, and in which the theorems are, to use Proclus' terms, investigated independently of matter and intellectually.

The reconstruction necessarily starts from Euclid's text, which appears to be, to a certain extent, a compilation from earlier texts which it drove out of circulation, and which are now wholly lost, so that we know of them only from certain references by Aristotle or Plato or other ancient authors. The reconstruction proceeds by a kind of literary archaeology.

Flourishings

Thales	flor. 585 B.C.
Pythagoras	flor. 550 B.C.
Parmenides	flor. 475 B.C.
Hippocrates of Chios	flor. 430 B.C.
Archytas of Tarentum	flor. 400 B.C.
Theaetetus	c. 415-369 B.C.
Plato	c. 428-348 B.C.
Aristotle	384-322 B.C.
Euclid	flor. 300 B.C.

Permit me to give here a set of not very reliable dates. Flourishing was something Greeks did as a rule at age 40, just as they often died at 80, to suit the taste for symmetry of a certain 2nd-century B.C. chronographer named Apollodorus. Euclid wrote about 300 B.C. There are good grounds to believe that a good deal of geometry had been organized as a deductive science by the time of Hippocrates of Chios, about 430 B.C.; and there are plausibilities in assuming that portions of arithmetic had been organized deductively even earlier. In discussing this development, I shall want to refer to Parmenides, who lived in the first half of the 5th century; to Archytas of Tarentum, a Pythagorean and friend of Plato living around the turn of the 5th and 4th centuries; and to Theaetetus, another friend of Plato, who died as a result of battle wounds in 369 B.C., and was one of the great mathematicians of antiquity, being the author, in all probability, of nearly all of books X and XIII of Euclid's Elements.

In 1936, Oskar Becker pointed out a number of peculiar facts concerning Propositions 21-34 of Book IX of Euclid. These theorems are for the most part so obvious that it is hard to imagine why anyone would be so fussy as to want them proved. "If as many even numbers as we please be added together, the whole is even." Certainly. "If from an even number an even number be subtracted, the remainder will be even." Who will doubt it?

The proofs, with one exception, do not depend on any previous theorems in Euclid's Elements: they depend rather on certain definitions given at the start of Book VII, the first of the arithmetical books. The one exception, IX.32, depends on IX.13. But Becker suspects the proof as we now have it to be Euclid's emendation of the original proof; he shows that IX.32 follows quite straightforwardly from IX.31. Thus Propositions IX.21 to IX.34 can be a self-sufficient set of propositions dependent only on certain definitions. Moreover, with one curious exception, nothing else in Euclid's Elements depends on these propositions. The exception is the last proposition of Book IX, which modern editors delete as not being integral to Book X. It is the ancient proof of the incommensurability of the side and diagonal of the square, and what it depends on is the doctrine of the even and the odd, and more specifically, Propositions 32-34 of Book IX.

Becker believed that, originally, before incorporation in Euclid's Elements, the doctrine of the even and the odd had led to another consequence, the traces of which have been left in Euclid. Propositions 21-34 of Book IX are followed by two final propositions, 35 and 36; 35 is used for the proof of 36, and 36 shows how to construct a perfect number--perhaps all perfect numbers, but that I believe is not yet known. Euclid's proofs for these two propositions depend on propositions in Book VII having to do with ratios of numbers. Becker shows that 35 and 36 can be proved on the basis of the immediately preceding propositions of Book IX, independently of any reference to ratios. Thus Becker's conjecture is that, long before Euclid, there existed a treatise on the even and the odd, including Propositions 21-36 of Book IX and the last proposition of Book X; that out of piety Euclid or some ancient editor added this treatise to the Elements, then, in an effort to integrate this addition with the whole, changed some of the proofs, making use of propositions on numerical ratios from Book VII. This hypothesis at least accounts for the peculiarities of Book IX that I have cited.

That such a doctrine of the even and the odd already existed in the 5th century is supported by the fact that Plato defines arithmetic as the doctrine of the even and the odd, and refers to this doctrine as a familiar discipline.

Following Becker, Van der Waerden has argued that most of Book VII of Euclid had also been worked out in the 5th century. One of his arguments is that Archytas of Tarentum, in a work on musical theory written about 400 B.C., depends on propositions found in Book VIII, and these propositions depend in turn on propositions in Book VII. Now since Archytas is punctilious in working out the most trivial syllogisms, it is extremely unlikely that he merely assumed the propositions he needed; he must have known them to be already proved. On the other hand, if the propositions of Book VII existed in any form in Archytas' time, then Van der Waerden concludes that they must have been in almost exactly their present form and thus in apple-pie order; for Book VII is worked out with great care and in such a strictly logical fashion that no step can be removed without the whole collapsing. There are other clues that lead Van der Waerden to believe that most of Book VII was complete before Hippocrates of Chios wrote in lunules.

Two pieces of deductive arithmetic, then, along with Hippocrates' quadrature of lunules, constitute the available presumptive evidence for the character of 5th-century deductive mathematics. Can we learn anything from them, which might throw light on the question of what it meant for them to come to be? I want to take up, first, the demonstrations, then, the premises on which they are based.

Every Euclidean proposition ends with the stereotyped formula, ὅπερ ἔδει δεῖξαι, meaning: the very thing that it was necessary to show. The infinitive here, δεῖξαι, seems to have had the original meaning of showing visually. Thus in Plato's dialogue Cratylus Socrates says:

> "Can I not step up to a man and say to him, 'This is your portrait, and show him perhaps his own likeness or, perhaps, that of a woman? And by 'show' (δεῖξαι) I mean, bring before the sense of sight." (430e)

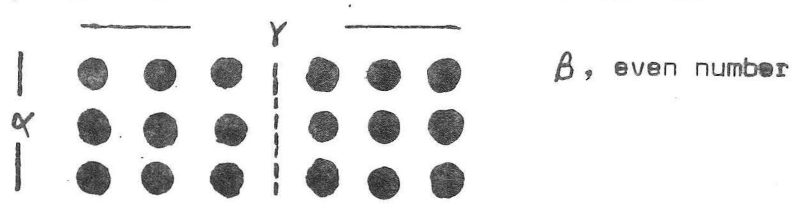
Early geometry must have been primarily a kind of visual showing, the pointing out of a symmetry, or the possibility of the coincidence of two figures, superposition. But in Euclid's text every effort is made to reduce the dependence on superposition to a minimum. Thus we come to suspect that there was present a kind of anti-illustrative, anti-empirical tendency in mathematics, as it was being transformed into deductive science. This same tendency is detectible in arithmetic as well as geometry.

Pythagorean arithmetical doctrines seem to have been originally worked out and taught with the aid of calculating pebbles. There is a fragment of the comic poet Epicharmus, written probably before 500 B.C., that runs as follows:

> "When there is an even number present, or, for all I care, an odd number, and someone wants to add a pebble or to take one away, do you think that the number remains unchanged?"
>
> "Not me!"
>
> "Well, then, look at people: one grows, another one perhaps gets shorter, and they are constantly subject to change. But whatever is changeable in character and does not remain the same, that is certainly different from what is changed. You and I are also different people from what we were yesterday, and we will still be different in the future, so that by the same argument we are never the same."

Presumably the sly rogue goes on to argue that he need not pay the debt he contracted the day before.

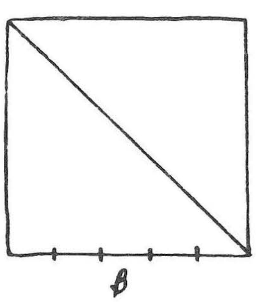
Aristotle, too, speaks of the Pythagorean pebble figures, the triangles, squares, and rectangles formed of pebbles with which the Pythagoreans taught arithmetical truths. We can easily see how they could have satisfied themselves, with their pebble figures, of the propositions concerning the even and the odd. Take Proposition IX.30: if an odd number is the divisor of an even number, then this same odd number is also the divisor of half the even number.



The number will be a rectangular number, with our odd number, the divisor, represented by the pebbles forming one of the sides. But the number as a whole is even, hence divisible in half, as by the vertical line. We see at once, then, that our odd number is a side of the half rectangle, hence a divisor of the half.

The proof of this proposition in Euclid is quite different. The numbers are not represented by points, but rather by lines. We know that Archytas representa numbers in this way, by lines, as a matter of course, and presumably, therefore, this mode of representation had become conventional before his time, that is, already in the 5th century. Now by looking at a line which represents a number, one cannot tell whether the number is even or odd, since any line can be halved: consequently, Euclid's visual representation of the numbers does not help us at all to see <u>why</u> the proposition is true. A pebble configuration could only represent visually a <u>particular</u> number; the new representation has the advantage of generality, but it also has the disadvantage that it forces one to look for an entirely new proof. The new proof that Euclid gives us involves the famous reduction to the absurd, or <u>indirect</u> demonstration. The important step is to show that the odd number α, the divisor, measures the even number β an even number of times, or in other words, that the quotient, γ, is even. Euclid's argument runs as follows: I say that γ is <u>not</u> odd, for <u>if</u> possible, let it be so. Now α multiplying γ makes β, and α was taken at the start to be odd, and an odd number multiplying an odd number yields only an odd number, as Euclid has previously shown. Therefore it would be odd, which is impossible (ἀδύνατον !), because it was taken at the start to be even. Thus the anti-illustrative tendency brings with it the <u>reductio ad absurdum</u> proof.

It is surprising how many <u>reductio</u> proofs occur in the arithmetical treatises that, according to Becker and Van der Waerden, stem from the 5th-century Pythagoreans. In propositions 21-36 of Book IX there are six such proofs, or eight if we accept Becker's reconstructions of 32, 35 and 36. In the first theorems of Book VII there are 15 such proofs. Moreover, the proof of the incommensurability of the side and diagonal of the square is also a <u>reductio</u>, and in this case we have to do with a truth which is altogether non-visualizable. Let me pause to review the strategy of that proof.



α, β relatively prime, therefore not both even.

$\alpha^2 = 2\beta^2$. Therefore α^2 is even.

Therefore α is even.

Let $\alpha = 2\gamma$.

$\alpha^2 = 4\gamma^2 = 2\beta^2$

Therefore $\beta^2 = 2\gamma^2$.

Therefore β^2 = an even number.

Therefore β = an even number.

Ἀδύνατον

Suppose, if possible, that the side and diagonal of a square <u>are</u> commensurable. Then there would be a length that measured both, and also a largest such length. Let this largest such length measure the diagonal α times, and the side β times, where α and β are integers or whole numbers. Now α and β cannot both be even, for otherwise our unit length could have been doubled, and the numbers halved, contrary to the assumption that the unit length was the largest possible; so at least one of the numbers must be odd. The sequence of the proof then shows that both must be even, or as Aristotle says in referring to this proof, that the same number must be both even and odd. The only alternative left is to relinquish the original assumption that α and β exist, or that side and diagonal are commensurable. In this demonstration human reason exhibits a rather astonishing power, the power to discover what eyesight could never in any way disclose. This discovery would encourage the anti-illustrative tendency, and the recourse to indirect proofs. It also implies that geometry cannot be subsumed under arithmetic, and needs therefore to be built up as an independent science in its own right. But the relevant point at this moment is that the emergence of deductive science appears to be connected with this anti-illustrative tendency, and with the closely-connected introduction of <u>reductio</u> proofs.

What about the <u>principles</u> or <u>premises</u> of Pythagorean arithmetic? I have already mentioned that the premises of the doctrine of the even and the odd are to be found among the definitions of Euclid's Book VII, and only there. The same thing goes for the doctrine concerning divisibility and proportionality found in Book VII itself. And fundamentally, all the definitions of Book VII rest on the first two definitions, the definition of number--a number is a multitude composed of units or monads--and then the definition of monad: monad is that according to which each of the things that are, each of the beings, is called one. What these definitions do, above all, is to limit the following discussion to <u>whole</u> numbers. Comparing this Greek arithmetical theory with Egyptian and Babylonian numerical work, we see that the Greek theory is sharply distinguished by its careful avoidance of fractions; and the first definition, whatever else it is doing, is expressing this prohibition against fractions, this insistence on the indivisibility of the one or unit. This insistence was already traditional in Plato's time.

In the Republic Socrates speaks of "the teaching concerning the one" (ἡ περὶ τὸ ἕν μάθησις), and explains what he means by it. I quote:

> ...You are doubtless aware that experts in this study, if anyone attempts to cut up the 'one' in argument, laugh at him and refuse to allow it; but if you mince it up, they multiply, always on guard lest the one should appear to be not one but a multiplicity of parts...Suppose now...someone were to ask them, "My good friends, what numbers are these you are talking about, in which the one is such as you postulate, each unit equal to every other without the slightest difference and admitting no division into parts?" What do you think would be their answer? This, I think---that they are speaking of units which can only be conceived by thought, and which it is not possible to deal with in any other way.

Socrates' explanation tells us why the indivisibility of the one had to be insisted upon; if the one were divisible, then it would be a multiplicity of parts, hence many, not one. In other words, the thought that the one is divisible is self-contradictory. Thus the insistence on the indivisibility of the one, which is Euclidean and also, according to Plato's Socrates in the Republic, pre-Platonic, is the conclusion of an indirect demonstration, a reduction to the absurd.

I have not yet taken up the principles used in early deductive geometry, but let me recapitulate what I have said, and consider what it suggests. The earliest deductive science, as far as we can tell, was the arithmetical theory of the so-called Pythagoreans of the 5th century. Their science differs from all earlier mathematics, first, in exhibiting an anti-empirical tendency, which sought to eliminate mere visual showing, as with the pebble figures; secondly, in making use of indirect demonstration or proof of something by reduction of its opposite to absurdity; thirdly, in insisting upon the indivisibility of the one, on the ground that admission of its divisibility would contradict the very meaning of the word "one." Now these features call to mind certain lines that remain of a poem written early in the 5th century, the poem by Parmenides of Elea; and to no other author of this time can these features be related. try to say some words about the poem of Parmenides.

Only fragments of it remain. Their interpretation is thoroughly controversial. There is widespread assurance that, whatever it was that

Parmenides meant, he was wrong. On the other hand, it will be little contested, I believe, if I say that Parmenides was the founder of Dialectic. Aristotle says that Zeno of Elea, Parmenides' pupil, was the founder of dialectic; but I think that may be because Zeno wrote out arguments in prose, whereas Parmenides wrote a poem in epic verse, while dialectic has essentially nothing to do with verse. I believe there is also rather general agreement that Parmenides, in composing his poem, was responding to, and attacking, earlier cosmogonies, which sought to derive all the variety and diversity of the world out of some underlying stuff, understood to be the real stuff of the world.

The poet begins by describing his journey in a chariot, drawn by mares that know the way, and escorted by the Daughters of the Sun. They arrive, high in the sky, before the gates of Night and Day. The Sun Maidens persuade the Goddess Justice to open the gates, and Parmenides is welcomed by the goddess who takes his hand and assures him that it is right and just that he, a mortal, should have taken this road. He must now learn both the unshaken heart of well-rounded truth, and the unreliable beliefs of mortals. The goddess describes three ways of inquiry: first, "That it is, (ἐστί) and cannot not be; this is the way of Persuasion, for she is the attendant of Truth;" second, "That it is not (οὐκ ἐστί), and must necessarily not be; this I tell you is a way of total ignorance;" third, "That it is, and it is not, the same and not the same; this is the way that ignorant mortals wander, bemused."

An initial difficulty that we face is that, although the pronoun "it" is not expressed in Greek, we can hardly resist the impression that there is something that is being talked about, and we should like to know what it is. The next fragments may be helpful.

> "It is the same thing that can be thought and can be."
> "What can be spoken of and thought must be; for it is possible for it to be, but it is not possible for <u>nothing</u> to be. These things I bid thee ponder."

In a preliminary and superficial way I think I can conclude that the subject of the verb <u>ἐστιν</u> is: that which is <u>intended</u> in thought, what we call the object of thought. The goddess is presenting an argument: that which thought intends <u>can</u> exist; but <u>nothing</u> cannot exist;

therefore that which thought intends cannot be nothing; hence it <u>must</u> exist.

The syllogism holds, I believe, although at that point in time logic had not been invented. But what does it mean? That which in no way is cannot be entertained in thought. Thought always is of something, it is intentional in character. Hence I must accept the Goddess' rejection of the second way, or non-way, of inquiry.

But the Goddess means something more. Some of this "more" emerges as she proceeds to dispose of the third way of inquiry. This is the way whereon, she says, mortals who know nothing wander two-headed; perplexity guides the wandering thought in their breasts; they are borne along, both deaf and blind, bemused, as undiscerning hordes, who have decided to believe that it is, and it is not, the same and not the same, and for whom there is a way of all things that turns back upon itself. "Never," says the goddess, "shall this be proved: that things that are not, are; but do thou hold back thy thought from this way of inquiry, nor let custom that comes of much experience force thee to cast along this way an aimless eye and a noise-cluttered ear and tongue, but judge through logos (through reasoning) the hard-hitting refutation I have uttered."

"It is necessary," adds the Goddess, "to say and to think that Being is."

Now in one way, this is all simple and undeniable. When I entertain an idea, when I use a word to signify some idea, I intend what I am thinking of as a constant, invariable. Never mind that my thought, my intending of what I am thinking about, is a shifting and not very controllable process. What is thought and named is intended as having a certain fixity. Otherwise, as Aristotle puts it, to seek truth would be to follow flying game. We would be reduced to the level of Cratylus, who did not think it right to say anything, and instead only moved his finger, and who criticized Heracleitus for saying that it is impossible to step twice into the same river, for he, Cratylus, said that one could not do it even once. On one level, the words of the Goddess are simply telling us what the prerequisites and necessities are for speech and thought that will be free of contradiction. Parmenides' poem is the earliest document preserved from the past which speaks explicitly of the logical necessities of thought.

Yet the discourse of the Goddess is more strange and frightening, or insane, or as Whitehead might say, important, than I have been making it out to be. The Goddess is not concerned with just anything that might be thought; she is concerned--she says so again and again--with Being. What is all this silly talk about Being? What else is there for Being to do but be? "It is necessary," says the Goddess,"to say and to think that Being is." Is it? Then is it necessary to say and to think that rain rains, that thunder thunders, or that lightening lightenings, and are not these parallel cases? I shall later come back, very briefly, to this question. It is just here that the poem becomes exasperating and impossible, prompting Aristotle to say more than once: the premises are false, and the conclusions do not follow. From the fact that Being just is, the Goddess proceeds to conclude that Being is precisely One, and contains no plurality, no multiplicity or differentiation within it, and no motion. In particular, and to Aristotle's great disgust, the Eleatics claim to have discovered the self-contradictory character of motion. It is Zeno, Parmenides' pupil, who formulates this discovery in the most memorable way. The flying arrow is in every instant exactly where it is, is at rest in the space equal to itself, and since this is true of every moment of its flight, it is always at rest, it does not move. Never mind the mortal wound we think it can inflict; this does not answer the argument, it does not tell us how motion can be consistently thought. The question is not whether Zeno is wrong but how. It is still being debated in the philosophical journals.

In the case of Parmenides, a more insistent question is what he can have meant by his poem. There is a second part to it, called the Way of Seeming or Opinion, of which 40 lines remain, and this speaks of the coming-to-be of the visible things of our ordinary world out of Fire and Night. Did Parmenides intend the Way of Opinion to have any validity at all, or only to present the bemused and erring beliefs of mortals? Plutarch remarks that

> Parmenides has taken away neither fire nor water nor rocks nor precipices, nor yet cities...for he has written very largely of the earth, heaven, sun, moon and stars, and has spoken of the generation of men.

Traditions credit Parmenides with having given laws to the city of Elea, and with having been the first to say that the Earth is round, that the

Moon shines by reflected light, and that the morning star is identical with the evening star--momentous discoveries every one of them. But such actions and discoveries do not seem easily compatible with the teaching about Being that the goddess has set forth, with such emphasis, such imperial absolutism. The heart of well-rounded truth appears to have no place in it for human law, for the earth's rotundity and its conical shadow, for Venus and her irregularities, or for Parmenides or you or me.

The speech of the Goddess is nevertheless fateful. With Parmenides, as I have said, dialectic takes its start. The age of those called sophists begins. One of the earliest of them, Protagoras, is clearly reacting to Parmenides when he makes his famous statement: man, he says, is the measure of all things, of the things that are, that they are, and of the things that are not, that they are not. Who but Parmenides had raised these questions about Being and not-Being? Protagoras has concluded that the Parmenidean standard of truth, that is, freedom from contradiction, is unreachable; thought, he thinks, inevitably involves contradiction. Therefore he turns to sense-experience, and asserts his right to say that the same thing can at one time be, and at another time not be, according as he, Protagoras, holds it to be or not to be. In Protagoras' time and later, there will be other objectors with other formulations, rejecting the speech of the Parmenidean Goddess in other ways. Gorgias, for instance, argues, first, that nothing is; second, that if anything is, it cannot be known; third, that if anything is and can be known, it cannot be expressed in speech.

Among the Parmenidean sequels, I want to suggest, was deductive arithmetic. For according to Aristotle, Parmenides was the first to speak of the One according to logos, according to definition; and arithmetic seems to have become deductive just when the Pythagoreans set out to found the doctrine of the even and the odd on the definition of the One, on its essential indivisibility, and proceeded in Parmenidean style to formulate proofs which relied no longer on visualization but rather on non-contradiction of the logos.

Of course--and this is a crucial qualification--no arithmetician could follow the teaching of the Parmenidean Goddess strictly. When the deductive arithmetician took his start from the indivisibility of the One, he was proceeding in accordance with a Parmenidean necessity of thought. When he went on to multiply the One, in order that arithmetic might be, he was violating the Parmenidean Way of Truth. Parmenidean-wise, how could there be many ones, each exactly the same as every other, and yet each retaining its identity to the extent of remaining separate from the others? The way in which these many ones can be, or are in being, is a question not for arithmetic, but for meta-arithmetic, but apparently the arithmeticians recognized that their discipline depended on the question about being. The Euclidean definition of Monas, One, reads: Monas is that in accordance with which _each of the beings_ is called one. A plurality of beings--what they are remains unclear--is here presupposed.

As for geometry, the violations of the Parmenidean logos that are necessary in order for it to become deductive are more drastic. The definitions of _point_ and _line_ with which Euclid begins Book I were no doubt modelled on the definitions of _One_ and _Number_, but there is a world of difference between the cases. The definitions tell us that a point is without parts, that a line is without breadth, but we cannot go on to derive any geometrical proposition from these rather problematic denials. It was the questionable character of the geometrical things that led Protagoras to reject the possibility of geometry altogether: a wheel, he said, does not touch a straight pole in one point only; therefore geometry is impossible, Q.E.D. But even if the geometrical definitions are granted, they do not provide a sufficient basis for the organization of geometry as a deductive science.

At the beginning of Euclid's _Elements_ three kinds of principles are set out. First, definitions or ὅροι; second, postulates or αἰτήματα; third, common notions or κοιναὶ ἔννοιαι. About a century ago, it was argued that the term κοιναὶ ἔννοιαι had to be of late Stoic origin, and therefore not due to Euclid. Was there an earlier Greek term? It may well have been ἀξιώματα; this is the term that Proclus constantly uses instead of κοιναὶ ἔννοιαι, and it may have been the term in front of him in his Euclidean text. Instead of ὅροι for definitions, Proclus commonly uses ὑποθέσεις; this usage is found earlier in Archimedes, and earlier still in Plato's _Republic_, where the odd and the even, and the various kinds of figures and angles are said to be treated in the sciences that deal with them as ὑποθέσεις. All three of these terms, ὑποθέσεις, αἰτήματα, and ἀξιώματα, were connected at one time with the practice of dialectic.

The term αἰτήματα, postulates, comes from the verb αἰτέω, to require, to ask. "Whenever, " Proclus tells us, "the statement is unknown and nevertheless is taken as true without the student's conceding it, then, Aristotle says, we call it an αἴτημα." The term ἀξιώματα comes from ἀξιόω, which can also mean to require, to ask; it is often so used in the Platonic dialogues. To be sure, Proclus says of the axioms that they are deemed by everybody to be true and no one disputes them. I believe this statement reflects an Aristotelian and post-Aristotelian usage. Aristotle himself refers to the earlier, dialectical usage when he says: ἀξιόω is used of a proposition which the questioner <u>hopes</u> the questioned person will concede.

As for the term ὑποθέσεις, there is perhaps little need to mention its dialectical use. At a certain point in Plato's <u>Republic</u>, Socrates speaks of the principle of non-contradiction, the presumably unshakeable principle according to which it is not possible for the same thing at the same time in the same respect and same relation to suffer, be, or do opposite things. And having enunciated the principle, he says, "Let us proceed on the hypothesis that this is so, with the understanding that, if it ever appear otherwise, everything that results from the assumption shall be invalidated." (437a) And as even the not-so-dialectical Aristotle recognizes, this principle can only be established controversially, that is to say, dialectically, against an adversary who offers to say something.

My general point is a simple one. The first book of the elements of geometry of which we have record was written in the middle of the fifth century, by Hippocrates of Chios. An anti-visual, anti-illustrative tendency that had first emerged, so far as I know, in Parmenidean dialectic, is already present in the geometrical proofs of Hippocrates of Chios that have come down to us, e.g.proofs of inequalities that would be obvious to visual inspection. The fact that at an early stage the terms adopted for the premises of geometry were terms of dialectic, terms referring to assumptions or concessions that do not entirely lose their provisional character but are required in order that a discussion might proceed, reinforces the impression that the transformation of geometry into a deductive science was carried out in a context determined by the practice of dialectic.

It is also important to realize here that the premises of geometry had to be *concessions*: propositions needed to derive what not only geometers but even surveyors and carpenters knew, yet propositions which violated, in the most obvious way, the canons of the Parmenidean logos. Two things equal to the same thing, Euclid tells us, are equal to each other. But what is equality but sameness, and how can three things that are exactly the same be three? How, moreover, are we to perform the absolutely impossible feats that the αἰτήματα require--to draw a straight line from point to point, to extend a line, to describe a circle? Part of the paradox here is described by Socrates in the *Republic*: "The science (of geometry)," he says, "is in direct contradiction to the language spoken by its practitioners. They speak in a ludicrous way, although they cannot help it; for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas the entire discipline is directed towards knowledge." (527[a-b]) It is probably this peculiar mixture that Timaeus is referring to when he speaks of geometry as apprehending what it deals with by a bastard kind of reasoning.

I should like to conclude with a short summary of and comment on what I have been saying, followed by a brief epilogue.

Deductive science appears to have been first discovered by a few Greeks; so far as I know, this discovery remained unique. Knowledge of it fell into oblivion during certain times; at whatever later times the possibility of deductive science has been recognized, the recognition has come through the recovery of Greek deductive science. What did the original discovery involve, what did it mean, for those who made it? That is the question I have sought to examine. From a plausible reconstruction of Pythagorean deductive arithmetic, I am led to conclude that the essential moves were
(1) the turning away from visualization and taking recourse in *logos*;
(2) the application of a negative test, the method of indirect proof or reduction to the absurd. Now these two steps are dialectical steps, they are the steps of the method that Socrates in the *Phaedo* describes as his own: "I was afraid," he says, "that my soul might be blinded altogether if I looked at things with my eyes or tried to apprehend them only by the help of the senses. And I thought I had better have recourse to the *logos*.... This was the method I adopted. I first assumed some principle, which I

judged to be the strongest, and then I affirmed as true whatever seemed to agree with this, and that which disagreed I regarded as untrue."

But in all the features that Socrates mentions, Socratic method is essentially Eleatic, Parmenidean dialectic. The search for the sources of Pythagorean deductive arithmetic thus leads us back to Parmenides, or to someone else, who lived about the same time, and whose utterances had the same effect.

What was so special, so peculiar, about the discourse of the Parmenidean Goddess, that it could precipitate what followed?

"Thinking and the thought that it is," says the Goddess, "are one and the same. For you will not find thought apart from that which is..; for there is and shall be no other besides what is, since Destiny has fettered it so as to be whole and immovable."

"It is necessary to say and to think," the Goddess adds,"that Being is." These words are spoken not by Parmenides but to Parmenides. He is being called upon to say and to think, and the saying and thinking are not separated, although the order in which the Goddess names them is worth noticing, being the opposite of that which we moderns tend to choose. The thinking, we had better remind ourselves, is Greek thinking; the verb is noein, which once meant: to perceive by the eyes, to observe, to notice. It is not to conceive, to analyze, to grasp, to attack in our thinking. And that which Parmenides is asked to say and to notice, what will it do for him to say and to notice it? The sentence, "Being is," does indeed offer nothing to grasp, nothing to conceptualize, nothing to attack in our thinking, nothing to analyze. Except--there is a twoness there. There is the noun and the verb, essentially, of course, the same word. Yet, there is that which is present, and there is its presence. To say and to notice not only what is present but its presence is to be arrested in front of something. It is to be, at least a little bit, astonished. It is to respect what lies before us. It is to think appropriately, as befits the matter. At some point Greek thought ceased asking: Out of what do the many things come to be? and began to ask instead: What is the Being of that which is in front of us? Ti to on is the Greek: what is the being? In this question, there are implicit the so-called laws of logic: A is A, A is not not-A. Deductive science, I am proposing, takes it start here. What seems to have been important, for these beginnings, was not answering the question but pursuing it. Even Aristotle, from whom we have received more answers than questions, nevertheless says:

Both formerly and now and forever it remains something to be sought and something forever darting away: Ti to on ?

Suppose, if you will, that the account I propose is something like the truth. Then deductive science came to be and perhaps still comes to be as a result both of a logos from beyond the gates of Night and Day, and of the fracturing of Being and of the Motion going on in the Realm of Fire and Night. Or can deductive science proceed on its own way, simply leaving behind what triggered its coming-to-be? It has sometimes attempted to do this: to become, for instance, purely formal, with the specification of every element and every rule of operation, and the exclusion of every bit of explicit or implicit ontology, with the intent of insuring logical completeness and consistency. The effort has led to many refinements; but the odd result of modern metamathematical study is that the effort cannot succeed in its original intention. Mathematics does not succeed in being completely in itself and for itself. Its triumph lies not in isolated grandeur, but in coping as best it can with necessities that appear.

Deductive mathematics, not quite a century after coming to be, underwent a crisis with respect to its foundations. The discovery of incommensurability can well have been early in the 5th century. It implies, rather obviously one would think, the falsity of the old Pythagorean doctrine that all is number, whatever that doctrine may have meant. But if the discovery was early, an important consequence of it was somewhat slow in being realized. The teaching concerning ratios of magnitudes was originally conceived in a numerical fashion: four magnitudes are proportional when the first is the same part, parts or multiple of the second that the third is of the fourth. That definition is still being used by Hippocrates of Chios. Archytas, around 400 B.C., is saying that logistic, the doctrine of ratios of numbers, has the highest rank among the arts, and in particular it is superior to geometry, "since it can treat more clearly than the latter whatever it will." Archytas thus fails to notice that the fact of incommensurability sets a new task for mathematics, the formulation of a new definition of proportionality, one which will apply to magnitudes that may be incommensurable.

The problem is solved by the early fourth century, possibly by Theaetetus; at least he is the first we known to have used the new definition, and he did so extensively. The new definition of same ratio or proportionality is not the one embodied in Euclid, the definition due to Eudoxus, but a precursor of the latter, one which we can argue Euclid excised from Book X as it came down to him from Theaetetus. The manner of the new definition is worth noting. At the beginning of Book VII of Euclid, a method is given of determining the greatest common divisor of two numbers; it has come to be called the Euclidean algorithm.

What is the greatest common divisor of 65, 39?

$65 - 39 = 26$

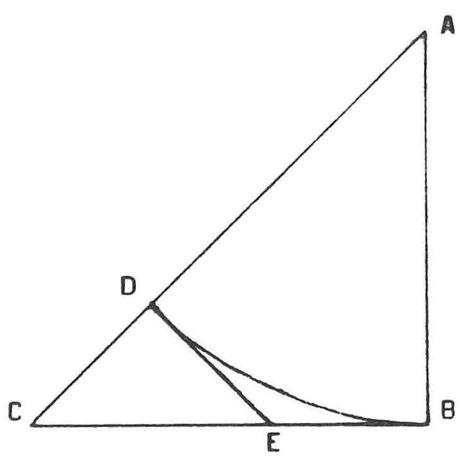
$39 - 26 = 13$

But 13 measures 26.

Ans: 13 is g.c.d. (65,39).

The lesser of the two numbers is subtracted from the greater until a yet smaller number remains. This smaller remainder is subtracted from the preceding subtrahend in the same manner, and so one continues, obtaining a series of decreasing remainders, until one arrives at a remainder that measures the preceding remainder. In the case shown, this number is 13, which is the greatest common divisor of 65 and 39.

This same procedure of successive, in-turn, subtractions--its Greek name was <u>antanairesis</u>---can be applied to magnitudes, in order to determine their common measure. But suppose they are incommensurable; then the subtractions would go on forever, without any remainder being found that measured the preceding remainder. A particular such situation is shown in the following diagram, which shows the side and diagonal of a square:



DE = EB (symmetry)

= CD (isosceles rt. $\triangle$)

CD = AC - AB

CE = AB (or CB) - CD (or EB)

And so on <u>ad infinitum</u>.

first the side is subtracted from the diagonal, leaving CD; CD is then subtracted from the side CB twice, and so on; I will not go into the proof of incommensurability here, which necessarily involves a reduction to the absurd; but one can get a hint from the diagram as to why the process would be infinite. Nevertheless, this infinite process of antanairesis would go on in a determinate way, for a given pair of original magnitudes. The n^{th} remainder, for example, might subtract two times from remainder n-1; and remainder n-1 might subtract three times from remainder n. The two and three, along with the corresponding numbers for all the other subtractions, would characterize and define the antanairesis as a whole. Then same antanairesis could be the definition of same ratio: a first magnitude would have to a second magnitude the same ratio as a third to a fourth if the first and second magnitude had the same antanairesis as the third and the fourth. With this definition, it is possible to prove, for example, that rectangles under the same height are to one another as their bases, because one sees that the antanairesis will go on in the same way with the rectangles as with the bases, even though the antanairesis be infinite.

There are other mathematical exploits of Theaetetus, embodied in books X and XIII, and they are of a kind with the formulation of the definition of proportionality that I have just described. Using theorems about numbers in new ways, Theaetetus succeeds in rendering what was inexpressible expressible. Such achievement, I would suggest, should be put down under the rubric of Pascal's esprit de finesse, rather than under his esprit de geometrie, the geometrical turn of mind, which Pascal so berates for its blindness to the problem of the principles. The Pythagorean mathemata, arithmetic, geometry, and the rest, are not liberal arts merely or primarily in being deductive, in proceeding stepwise in accordance with certain rules. Their liberality, it seems to me, has an essential relation to the awareness not merely of logical necessity, but of that necessity with which they are designed to cope: we are free men when we are aware of that necessity and can begin to cope with it. The liberal arts become fully liberal only as we turn toward the problem of the principles, toward the matrix of necessity in which those principles are embedded, toward the question of being from which those arts take their rise.

Notes

1. (p. 1) This lecture owes everything, or nearly everything, to a number of studies by historians of mathematics, particularly: O. Becker, "Die Lehre vom Geraden und Ungeraden im neunten Buch der euklidischen Elemente," Quellen und Studien zur Geschichte der Mathematik..., Abt. B, Band 3 (1936), 125-145; B. L. van der Waerden, "Die Arithmetik der Pythagoreer," Mathematische Annalen, 120, (1947-1949), pp. 127-153, and Science Awakening (New York: Oxford Univ. Press, 1971): G. Vlastos, "Zeno of Elea" in Enclyclopedia of Philosophy, VIII, 370; O. Neugebauer, The Exact Sciences in Antiquity, 2d ed., 1957; and above all articles by Arpad Szabo: "Zur Geschichte der Dialektek des Denkens," in Acta Antiqua Academiae Scientiarum Hungaricae, II (1954), 17-62, and "The Transformation of Mathematics into Deductive Science and the Beginnings of its Foundation on Definitions and Axioms," in Scripta Mathematica, 27 (1964), 28-48 and 113-139. For the Proclus text I depended on Procli Diadochi in Primum Euclidis Elementorum Commentarii (ed. Friedlein, Teubner, 1873) and the recent translation by the late Glenn R. Morrow, A Commentary on the First Book of Euclid's Elements (Princeton, 1970). In the section on Parmenides there may be recognized a certain inspiration, much diluted, of Martin Heidegger's What is Called Thinking? (tr. Wieck & Graz, New York: Harper & Row, 1954).

2. In particular, see Walter Burkert, Lore and Science in Ancient Pythagoreanism.