An Oblivious Ellipsoid Algorithm for Solving a System of (In)Feasible Linear Inequalities

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This talk is based on our paper ("almost finished!"):

*An Oblivious Ellipsoid Algorithm for Solving a System of (In)Feasible Linear Inequalities*
1. Problem(s) of interest

2. Preliminaries and motivation

3. Schematic of Oblivious Ellipsoid Algorithm (OEA)

4. Condition measures

5. Main results
A critique of the ellipsoid algorithm

Given data \((A, u) \in \mathbb{R}^{n \times m} \times \mathbb{R}^m\), the ellipsoid algorithm computes a solution to

\[
(P) : \quad A^T x \leq u
\]

when the set of solutions

\[
\mathcal{P} := \{ x \in \mathbb{R}^n : A^T x \leq u \}
\]

has positive volume

When \((P)\) is infeasible:

- Existing versions of the ellipsoid algorithm can fail to correctly decide so (in the real number model of computation)
- Existing versions do not produce any suitable certificate of infeasibility

In this talk: a certificate of infeasibility is (informally) a mathematical object that yields a proof that \((P)\) is infeasible
Type-1 certificate of infeasibility

In this talk: a certificate of infeasibility is (informally) a mathematical object that yields a proof that \((P)\) is infeasible

Example: when and only when \((P)\) is infeasible, there exists a solution \(\lambda \in \mathbb{R}^m\) to

\[
(Alt) : \begin{cases}
  A\lambda = 0 \\
  \lambda \geq 0 \\
  u^\top \lambda < 0
\end{cases}
\]

- a solution to \((Alt)\) is a certificate of infeasibility
- we call such a solution a type-1 certificate of infeasibility to distinguish it from other certificates of infeasibility developed herein

We view a type-1 certificate of infeasibility as special because

- like a solution to \((P)\), it is a solution to a system of linear inequalities
- it is easy to store, and it is easy to verify \((Alt)\)
A question of Todd [2018]

Both the simplex method and interior point methods will return solutions to \((P)\) or \((Alt)\)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Solution to ((P))</th>
<th>Solution to ((Alt))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplex method</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Interior Point Method</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Ellipsoid Algorithm</td>
<td>✓</td>
<td>?</td>
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</tbody>
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Todd [2018] asked if an ellipsoid algorithm that yields solutions to \((P)\) and \((Alt)\) can be developed?

Related:

- Burrell and Todd [1985] develop an ellipsoid algorithm that “generates dual variables”
- but their algorithm is not guaranteed to return a solution to \((Alt)\) when \((P)\) is infeasible, even asymptotically
This motivates the following two problems:

**Problem 1: (Oblivious Certification)**
Develop a version of the ellipsoid algorithm that computes a solution to \((P)\) when \((P)\) is feasible, and computes a solution to \((Alt)\) when \((P)\) is infeasible.

**Problem 2: (Partial Certification/Decidability)**
Develop a version of the ellipsoid algorithm that computes a solution to \((P)\) when \((P)\) is feasible, and otherwise correctly decides if \((P)\) is infeasible by producing an appropriate certificate of infeasibility.

Note: a solution to Problem 1 is also a solution to Problem 2
Why not just run the ellipsoid algorithm in parallel?

Wait a second . . . , can’t we just run the standard ellipsoid algorithm in parallel on \((P)\) and \((Alt)\)?

- There is an aesthetic interest in developing an ellipsoid algorithm that is “on par” with the simplex method and interior-point methods.
- There also ends up being complexity advantages when \(m \gg n\); for Problem 2, the total complexity of our oblivious ellipsoid algorithm (OEA) is

\[
\tilde{O}(nm^3)
\]

while the total complexity of the parallel scheme is

\[
\tilde{O}(m^4)
\]
Outline

1. Problem(s) of interest

2. Preliminaries and motivation

3. Schematic of Oblivious Ellipsoid Algorithm (OEA)

4. Condition measures

5. Main results
Assumptions

\[ (P) : \quad A^\top x \leq u \quad \text{and} \quad \mathcal{P} := \{ x \in \mathbb{R}^n : A^\top x \leq u \} \]

We assume:

- The conic hull of the columns \( a_1, \ldots, a_m \) of \( A \) is \( \mathbb{R}^n \)
- Each column of \( A \) has unit Euclidean norm

Note that:

- The first assumption ensures that the set of solutions \( \mathcal{P} \) is bounded if \( (P) \) is feasible, and also implies that \( m > n \)
- The second assumption is without loss of generality because \( \mathcal{P} \) does not change under positive rescaling of the constraints of \( (P) \).
Bounding $\mathcal{P}$

\[
(P) : \quad A^T x \leq u \\
\mathcal{P} := \{ x \in \mathbb{R}^n : A^T x \leq u \}
\]

We suppose that we know how to bound $\mathcal{P}$ in the sense that for all $i \in \{1, \ldots, m\}$, we have lower bounds $\ell_i \in \mathbb{R}$ that satisfy

\[
x \in \mathcal{P} \implies a_i^T x \geq \ell_i
\]
Certified lower bounds and their certificates, cont.

\[(P) : \quad A^\top x \leq u \quad \quad \quad \mathcal{P} := \{x \in \mathbb{R}^n : A^\top x \leq u\}\]

We suppose that for all \(i \in \{1, \ldots, m\}\), we have lower bounds \(\ell_i \in \mathbb{R}\) that satisfy

\[x \in \mathcal{P} \implies a_i^\top x \geq \ell_i\]

This being the case, we suppose that we have a certificate \(\lambda_i\) for each \(\ell_i\) such that

\[(LB_i) : \quad \left\{ \begin{array}{l} A\lambda_i = -a_i \\ \lambda_i \geq 0 \\ -\lambda_i^\top u \geq \ell_i \end{array} \right.\]

Observe that if \(\mathcal{P} \neq \emptyset\), then for all \(x \in \mathcal{P}\) it holds that

\[a_i^\top x = -\lambda_i^\top A^\top x \geq -\lambda_i^\top u \geq \ell_i\]

and hence \(\lambda_i\) certifies that \(\ell_i\) is a lower bound on \(a_i^\top x\) for all \(x \in \mathcal{P}\).

What if \(\mathcal{P}\) is empty?
Certified lower bounds and their certificates, cont.

What if $\mathcal{P} = \emptyset$?

We will still suppose that we have a certificate $\lambda_i$ for each $\ell_i$ such that

$$(LB_i) : \begin{cases} A\lambda_i = -a_i \\ \lambda_i \geq 0 \\ -\lambda_i^T u \geq \ell_i \end{cases}$$

So regardless, we say that $\ell_i \in \mathbb{R}$ is a certified lower bound for the $i$-th inequality of $(P)$ with certificate $\lambda_i \in \mathbb{R}^m$ if $\ell_i$ and $\lambda_i$ satisfy $(LB_i)$.
Let us collect our lower bounds into \( \ell = (\ell_1, \ldots, \ell_m)^\top \in \mathbb{R}^m \) and their certificates into \( \Lambda = [\lambda_1 \mid \ldots \mid \lambda_m] \in \mathbb{R}^{m \times m} \).

We say \( \ell \) is a certified lower bound for \((P)\) with certificate \( \Lambda \in \mathbb{R}^{m \times m} \) if \( \ell \) and \( \Lambda \) satisfy the system

\[
\begin{align*}
A\Lambda &= -A \\
\Lambda &\geq 0 \\
-\Lambda^\top u &\geq \ell
\end{align*}
\]

Like before, if \((P)\) is feasible, then for any \(x\) satisfying \((P)\):

\[
A^\top x = -\Lambda^\top A^\top x \geq -\Lambda^\top u \geq \ell
\]

Our Oblivious Ellipsoid Algorithm – OEA – is premised on having an initial certified lower bound \( \ell \) along with its certificate \( \Lambda \).
Our OEA is premised on having an initial certified lower bound $\ell$ along with its certificate $\Lambda$

In general, it is not clear how to construct such a lower bound and certificate – short of solving linear inequality systems of size at least as large as $A^T x \leq u$

However, when $A^T x \leq u$ contains box constraints (as is often the case), for example:

\[
(P_B): \quad \begin{cases}
\hat{A}^T x \leq \hat{u} \\
x \leq \bar{b} \\
x \geq b
\end{cases}
\]

such lower bounds and certificates are simple to describe and construct

Let us briefly see how this is done
Initial certified lower bounds and their certificates, cont.

Data $(\hat{A}, \hat{u}, \underline{b}, \overline{b}) \in \mathbb{R}^{n \times \hat{m}} \times \mathbb{R}^{\hat{m}} \times \mathbb{R}^n \times \mathbb{R}^n$

For $i \in 1, \ldots, \hat{m}$, define $\hat{\ell}_i \in \mathbb{R}$ to be

$$\hat{\ell}_i := \min_{\underline{b} \leq x \leq \overline{b}} \hat{a}_i^T x = (-(a_i)^-)^T \overline{b} + ((a_i)^+)^T \underline{b}$$

For $i = 1, \ldots, \hat{m}$, define

$$\lambda_i = [0_{\hat{m}} / \hat{a}_i^- / \hat{a}_i^+]$$

for $i = \hat{m} + 1, \ldots, \hat{m} + n$, define

$$\lambda_i = [0_{\hat{m}} / 0_n / e_i]$$

and for $i = \hat{m} + n + 1, \ldots, \hat{m} + 2n$, define

$$\lambda_i = [0_{\hat{m}} / e_i / 0_n]$$
Containing ellipsoids

\[(P) : \quad A^\top x \leq u \quad \mathcal{P} := \{x \in \mathbb{R}^n : A^\top x \leq u\}\]

For a certified lower bound \(\ell\) together with \(d \in \mathbb{R}^m\) satisfying \(d > 0\), construct

\[E(d, \ell) := \{x \in \mathbb{R}^n : (A^\top x - u)^\top D(A^\top x - \ell) \leq 0\}\]

where \(D := \text{diag}(d)\) is the diagonal matrix with diagonal \(d\)

Observe that \(\mathcal{P} \subseteq E(d, \ell)\) when \((P)\) is feasible

We can do some algebraic manipulation and obtain

\[E(d, \ell) = \{x \in \mathbb{R}^n : (x - y(d, \ell))^\top ADA^\top (x - y(d, \ell)) \leq f(d, \ell)\}\]

where

\[y(d, \ell) := \frac{1}{2} B(d)^{-1} AD(u + \ell)\]

\[f(d, \ell) := \frac{1}{4}(u + \ell)^\top DA^\top (ADA^\top)^{-1} AD(u + \ell) - \ell^\top Du\]

Note that \(y(d, \ell)\) is the center of \(E(d, \ell)\)
Updating parameters

\[ (P) : \quad A^\top x \leq u \quad \quad \mathcal{P} := \{ x \in \mathbb{R}^n : A^\top x \leq u \} \]

\[ E(d, \ell) : = \{ x \in \mathbb{R}^n : (A^\top x - u)^\top D (A^\top x - \ell) \leq 0 \} \]
\[ = \{ x \in \mathbb{R}^n : (x - y(d, \ell))^\top ADA^\top (x - y(d, \ell)) \leq f(d, \ell) \} \]

We update \( E(d, \ell) \) by updating \( d \) and \( \ell \)

\( \ell \) is a parameter that we update (and we know how to initialize \( \ell \))

We also update \( \Lambda \):

- We update \( \Lambda \) such that updates of \( \ell \) remain certified by the updates of \( \Lambda \)

If eventually \( \ell_j \) satisfies \( \ell_j > u_j \) for some \( j \in \{1, \ldots, m\} \), then \( \bar{\lambda}_j := \lambda_j + e_j \) is a type-1 certificate of infeasibility 😊

We can also use \( \ell \) and \( \Lambda \) to construct other types of certificates of infeasibility. Let us now show two other ways this can be done . . .
Let $d \in \mathbb{R}^m$ such that $d > 0$, and let $\ell$ be a certified lower bound for $(P)$ with certificate matrix $\Lambda$

If $A^\top y(d, \ell) \not\leq u$ and $f(d, \ell) \leq 0$, then it must be true that $(P)$ is infeasible:

If $(P)$ were feasible, then $\mathcal{P} \subseteq E(d, \ell)$

**Case 1.** $f(d, \ell) < 0$, which clearly implies $(P)$ is infeasible

**Case 2.** $f(d, \ell) = 0$, then $\mathcal{P} = \emptyset$ or $\mathcal{P} = \{y(d, \ell)\}$, but the latter is not possible because $A^\top y(d, \ell) \not\leq u$, and thus $(P)$ is infeasible

So $d, \ell$, and $\Lambda$ that satisfy the above comprise a certificate of infeasibility 😊
Type-2 certificate of infeasibility, continued

\[(P) : \quad A^T x \leq u \quad \mathcal{P} := \{x \in \mathbb{R}^n : A^T x \leq u\}\]

\[E(d, \ell) = \{x \in \mathbb{R}^n : (x - y(d, \ell))^T ADA^T (x - y(d, \ell)) \leq f(d, \ell)\}\]

d \in \mathbb{R}^m, \ell \in \mathbb{R}^m, \text{ and } \Lambda \in \mathbb{R}^{m \times m} \text{ that satisfy}

\begin{align*}
    d &> 0 \\
    A\Lambda = -A \\
    \Lambda &\geq 0 \\
    -\Lambda^T u &\geq \ell \\
    A^T y(d, \ell) &\not\leq u \\
    f(d, \ell) &\leq 0
\end{align*}

comprise a certificate of infeasibility that we will call a type-2 certificate of infeasibility.

(In the paper we also show how to construct a type-1 certificate from a type-2 certificate of infeasibility)
Type-3 certificate of infeasibility

\[(P) : \ A^\top x \leq u \quad \mathcal{P} := \{x \in \mathbb{R}^n : A^\top x \leq u\}\]

\[E(d, \ell) = \{x \in \mathbb{R}^n : (x - y(d, \ell))^\top ADA^\top (x - y(d, \ell)) \leq f(d, \ell)\}\]

Let \(d \in \mathbb{R}^m\) such that \(d > 0\), and let \(\ell\) be a certified lower bound for \((P)\) with certificate matrix \(\Lambda\).

Suppose there exists \(j \in \{1, \ldots, m\}\) satisfying

\[u_j < \min_{x \in E(d, \ell)} a_j^\top x = a_j^\top y(d, \ell) - \sqrt{f(d, \ell)(a_j^\top (ADA^\top)^{-1}a_j)}\]

Then \((P)\) must be infeasible:
Type-3 certificate of infeasibility, continued

\((P)\): \quad A^\top x \leq u \quad \mathcal{P} := \{x \in \mathbb{R}^n : A^\top x \leq u\}

\[ E(d, \ell) = \{x \in \mathbb{R}^n : (x - y(d, \ell))^\top ADA^\top (x - y(d, \ell)) \leq f(d, \ell)\} \]

Thus, \(d \in \mathbb{R}^m, \ell \in \mathbb{R}^m, \Lambda \in \mathbb{R}^{m \times m}, \text{ and } j \in \{1, \ldots, m\} \) that satisfy

\[
\begin{align*}
d &> 0 \\
\Lambda A & = -A \\
\Lambda &\geq 0 \\
-\Lambda^\top u &\geq \ell
\end{align*}
\]

\[ u_j < a_j^\top y(d, \ell) - \sqrt{f(d, \ell)(a_j^\top (ADA^\top)^{-1}a_j)} \]

comprise a certificate of infeasibility that we will call a type-3 certificate of infeasibility

(Note: Burrell and Todd [1985] show how to construct a type-1 certificate from a type-3 certificate)
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3. Schematic of Oblivious Ellipsoid Algorithm (OEA)

4. Condition measures

5. Main results
Algorithm 1 Schematic of Oblivious Ellipsoid Algorithm (OEA)

Input: data $A$ and $u$, certified lower bound $\ell$ with certificate matrix $\Lambda$, and $d > 0$.

1: Compute $y(d, \ell)$. If $A^T y(d, \ell) \leq u$, then return $y(d, \ell)$ as a solution of $(P)$ and Stop.
2: Compute $f(d, \ell)$. If $f(d, \ell) \leq 0$, then construct and Return a solution of $(Alt)$ and Stop.
3: Compute the most violated constraint: $j \leftarrow \arg\max_{i \in \{1, \ldots, m\}} a_i^T y(d, \ell) - u_i$.
4: (Possibly) update certificate $\lambda_j$ if its best lower bound can be improved.
5: If $\min_{x \in E(d, \ell)} a_j^T x_j > u_j$, then construct and Return a solution of $(Alt)$ and Stop.
6: Update ellipsoid $E(d, \ell)$ by updating $(d, \ell) \rightarrow (\tilde{d}, \tilde{\ell})$.
7: Re-set $(d, \ell) \leftarrow (\tilde{d}, \tilde{\ell})$ and Goto Step 1.
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Condition Measure $\tau(A, u)$

We introduce the following condition measure which measures the extent of feasibility or infeasibility of $(P)$:

$$\tau(A, u) := |z^*| \quad \text{where} \quad z^* := \max_{x \in \mathbb{R}^n} \min_{i \in \{1, \ldots, m\}} (u_i - a_i^\top x)$$

$\tau(A, u)$ is constructed in a similar spirit to the condition measure $C(A)$ for homogeneous linear inequalities developed by Cheung and Cucker [2001]

Properties:

- If $(P)$ is feasible, then $\tau(A, u) = z^* \geq 0$ and $\tau(A, u)$ is the radius of the largest $\ell_2$ ball contained in $\mathcal{P}$
- If $(P)$ is infeasible, then $\tau(A, u) = -z^* > 0$
- When $(P)$ is infeasible, it holds for every $x \in \mathbb{R}^n$ that there exists $i \in \{1, \ldots, m\}$ for which $a_i^\top x - u_i \geq \tau(A, u)$
Condition Measure $\rho(A)$

We assumed at the outset that the conic hull of the columns $a_1, \ldots, a_m$ of $A$ is $\mathbb{R}^n$.

When $(P)$ is feasible, this assumption ensures that $P$ contains no ray, that is $A^T v \leq 0$ has no non-trivial solution.

In this regard, we also introduce the following condition number (in the spirit of Renegar) which measures the “extent to which $P$ is bounded” if $(P)$ is feasible:

$$\rho(A) := \min_{\Delta A \in \mathbb{R}^{n \times m}} \{ \| \Delta A \|_{1,2} : \text{there exists } v \neq 0 \text{ satisfying } [A + \Delta A]^T v \leq 0 \}$$

$\rho(A)$ is the smallest perturbation of the matrix $A$ that will cause the perturbed feasible region to be unbounded.
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Main results in the case when \((P)\) is infeasible

When \((P)\) is infeasible, OEA will terminate with a solution to \((Alt)\) in at most

\[
2m(m + 1) \ln \left( \frac{m + 1}{2m} \frac{\|u - \ell\|}{\tau(A, u)} \right)
\]

iterations, where

- \(\ell\) is the initial certified lower bound
- \(\tau(A, u)\) captures the extent of infeasibility of \((P)\)

When we have box constraints, OEA terminates with a solution to \((Alt)\) in at most

\[
2m(m + 1) \ln \left( \frac{(m + 1)(\sqrt{\hat{m} + 2})}{2m} \frac{\|\bar{b} - b\|}{\tau(A, u)} \right)
\]

iterations.
Main results in the case when \((P)\) is feasible

When \((P)\) is feasible, OEA will terminate with a solution to \((P)\) in

\[
\left\lfloor 2n(m + 1) \ln \left( \frac{\|u - \ell\|}{2\rho(A) \tau(A, u)} \right) \right\rfloor
\]

iterations, where

- \(\ell\) is the initial certified lower bound
- \(\tau(A, u)\) is the radius of the largest inscribed ball in \((P)\)
- \(\rho(A)\) is a condition measure that captures the extent of boundedness of \(P\)

When we have box constraints, OEA will terminate with a solution to \((P_B)\) in at most

\[
\left\lfloor 2n(m + 1) \ln \left( \frac{\sqrt{m} + 2\|\bar{b} - b\|}{2\tau(A, u)} \right) \right\rfloor
\]

iterations
Recall Problem 1 and Problem 2

Recall the two problems:

Problem 1: (Oblivious Certification)
 Develop a version of the ellipsoid algorithm that computes a solution to \((P)\) when \((P)\) is feasible, and computes a solution to \((Alt)\) when \((P)\) is infeasible.

Problem 2: (Partial Certification/Decidability)
 Develop a version of the ellipsoid algorithm that computes a solution to \((P)\) when \((P)\) is feasible, and otherwise correctly decides if \((P)\) is infeasible by producing an appropriate certificate of infeasibility.
Comparison of methods in the case when \((P)\) is Infeasible

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</thead>
<tbody>
<tr>
<td>Standard ellipsoid algorithm applied to solve ((Alt))</td>
<td>(\tilde{O}((m - n)^2))</td>
<td>(\tilde{O}((m - n)m))</td>
<td>(\tilde{O}((m - n)^3 m))</td>
</tr>
<tr>
<td>Oblivious Ellipsoid Algorithm for Problem 1</td>
<td>(\tilde{O}(m^2))</td>
<td>(\tilde{O}(m^2))</td>
<td>(\tilde{O}(m^4))</td>
</tr>
<tr>
<td>Oblivious Ellipsoid Algorithm for Problem 2</td>
<td>(\tilde{O}(m^2))</td>
<td>(\tilde{O}(mn))</td>
<td>(\tilde{O}(m^3 n))</td>
</tr>
</tbody>
</table>

Suppose we wish to address Problem 2 (compute a solution of \((P)\) if \(P\) is feasible, otherwise produce a proof that \((P)\) is infeasible). When \((P)\) is infeasible, total operations complexity of OEA is:

\[
\tilde{O}(m^3 n)
\]

while the total operations complexity of the standard ellipsoid parallel scheme is

\[
\tilde{O}(m^4)
\]
Comparison of methods in the case when \((P)\) is Feasible

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</tr>
<tr>
<td>Oblivious Ellipsoid Algorithm for Problem 1</td>
<td>(\tilde{O}(mn))</td>
<td>(\tilde{O}(m^2))</td>
<td>(\tilde{O}(nm^3))</td>
</tr>
<tr>
<td>Oblivious Ellipsoid Algorithm for Problem 2</td>
<td>(\tilde{O}(mn))</td>
<td>(\tilde{O}(mn))</td>
<td>(\tilde{O}(n^2m^2))</td>
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Suppose we wish to address Problem 2 (compute a solution of \((P)\) if \(P\) is feasible, otherwise produce a proof that \((P)\) is infeasible). When \((P)\) is feasible, total operations complexity of OEA is:

\[
\tilde{O}(n^2m^2)
\]

while the total operations complexity of the standard ellipsoid parallel scheme is

\[
\tilde{O}(n^3m)
\]
Quick Remarks

• We were rather surprised that OEA attains improved complexity over the standard ellipsoid algorithm in the infeasible case.

• The mechanics of the updates of $d$ and $\ell$ are simple to compute, but their description is very complicated, and we hope others will find a nice simplification of our formulae.

• Unlike the standard ellipsoid method, the OEA algorithm updated ellipsoids are not the minimum volume ellipsoids that contain the current half-ellipsoids. But they would be if $m$ were replaced by $n$ in the formulae.
Thank you! Questions?
Some useful objects

\[ r(\ell) := \frac{1}{2}(u + \ell) , \]
\[ v(\ell) := \frac{1}{2}(u - \ell) , \]
\[ B(d) := ADA^\top , \]
\[ y(d, \ell) := B(d)^{-1}ADr(\ell) , \]
\[ t(d, \ell) := A^\top y(d, \ell) - r(\ell) , \]
\[ f(d, \ell) := v(\ell)^\top Dv(\ell) - t(d, \ell)^\top Dt(d, \ell) , \]
before introducing the update procedures, let us introduce one important definition

for \( i \in \{1, \ldots, m\} \), and for \( d > 0 \) and \( \ell \in \mathbb{R}^m \), define the ellipsoid slab radius

\[
\gamma_i(d, \ell) = \max_{x} a_i^T (x - y(d, \ell)) = \sqrt{(a_i^T (ADA^T)^{-1} a_i)(f(d, \ell))}
\]

s.t. \( x \in E(d, \ell) \)
Let $d > 0$, let $\ell$ be a certified lower bound for ($P$) with certificate matrix $\Lambda$, and suppose that $E(d, \ell)$ is strictly feasible. Therefore $L_i := a_i^T y(d, \ell) - \gamma_i(d, \ell)$ is a valid lower bound on constraint $i$ of ($P$). Can one construct a certificate $\tilde{\lambda}_i$ for the lower bound $L_i$? Burrell and Todd [1985] answer this question in the affirmative.
Updating a Certificate $\lambda_i$, cont.

$L_i := a_i^T y(d, \ell) - \gamma_i(d, \ell)$ is a valid lower bound on constraint $i$ of $(P)$

Can one construct a certificate $\tilde{\lambda}_i$ for the lower bound $L_i$? Burrell and Todd [1985] answer this question in the affirmative

**Proposition**

Let $d > 0$, and $\ell$ be a certified lower bound for $P$ with certificate matrix $\Lambda$, let $E(d, \ell)$ be strictly feasible, and suppose that $d$ has been rescaled so that $f(d, \ell) = 1$. Let $i \in [m]$ be given, and define $L_i := a_i^T y(d, \ell) - \gamma_i(d, \ell)$, and

$$\hat{\lambda}_i := \gamma_i(d, \ell)Dt(d, \ell) - DA^T B(d)^{-1}a_i,$$

$$\tilde{\lambda}_i := \Lambda\hat{\lambda}_i^- + \hat{\lambda}_i^+.$$  

Then $L_i$ is a certified lower bound on constraint $i$ of $P$ with certificate $\tilde{\lambda}_i$. In particular, it holds that $-\tilde{\lambda}_i^T u \geq L_i$. 
Proposition

Let $d > 0$, and $\ell$ be a certified lower bound for $P$ with certificate matrix $\Lambda$, let $E(d, \ell)$ be strictly feasible, and suppose that $d$ has been rescaled so that $f(d, \ell) = 1$. Let $i \in [m]$ be given, and define $L_i := a_i^T y(d, \ell) - \gamma_i(d, \ell)$, and

\[
\hat{\lambda}_i := \gamma_i(d, \ell) D t(d, \ell) - D A^T B(d)^{-1} a_i ,
\]
\[
\tilde{\lambda}_i := \Lambda \hat{\lambda}_i^- + \hat{\lambda}_i^+ .
\]

Then $L_i$ is a certified lower bound on constraint $i$ of $P$ with certificate $\tilde{\lambda}_i$. In particular, it holds that $-\tilde{\lambda}_i^T u \geq L_i$.

We can construct a type-1 certificate from a type-3 certificate:

Corollary

Under the set-up of the Proposition, if $a_i^T y(d, \ell) - \gamma_i(d, \ell) > u_i$, then $\bar{\lambda}_i := \tilde{\lambda}_i + e_i$ is feasible for $(Alt)$. 

If all the slab radii are sufficiently small

$$\gamma_i(d, \ell) < \tau(A, u)$$

for all $i \in \{1, ..., m\}$

then we can construct a type-1 certificate of infeasibility:

**Corollary**

Under the set-up of the Proposition, let $i := \arg\max_{j \in [m]} (a_j^\top y(d, \ell) - u_j)$, and let $L_i$ and $\tilde{\lambda}_i$ be as defined therein. If $P = \emptyset$ and $\gamma_i(d, \ell) < \tau(A, u)$, then $\bar{\lambda}_i := \tilde{\lambda}_i + e_i$ is feasible for $(\text{Alt})$.

We leverage this idea to establish guarantees for our algorithm when $(P)$ is infeasible.
From Type-2 to Type-1

All of the previous results are premised on $E(d,\ell)$ being strictly feasible, i.e. $f(d,\ell) > 0$

We develop a Procedure that constructs a solution to $(Alt)$ when $f(d,\ell) \leq 0$ and $A^T y(d,\ell) \not\leq u$

In other words, the Procedure constructs a type-1 certificate from a type-2 certificate
From Type-2 to Type-1, cont.

**Procedure 2** Procedure for constructing a certificate of infeasibility when $f(d, \ell) \leq 0$ and $A^T y(d, \ell) \not\leq u$.

**Input:** $d > 0$, certified lower bounds $\ell$ with certificate matrix $\Lambda$, satisfying $f(d, \ell) \leq 0$ and $A^T y(d, \ell) \not\leq u$

1. If $\ell \not\leq u$, select an index $j$ for which $\ell_j > u_j$ and Return $\bar{\lambda}_j := \lambda_j + e_j$ and Exit.
2. Select any index $i \in [m]$, and compute $\beta \geq 0$ such that $f(d, \ell - \beta e_i) = 0$.
3. $\ell \leftarrow \ell - \beta e_i$.
4. Compute an index $j \in [m]$ for which $a_j^T y(d, \ell) \leq u_j$.
5. Compute an index $k \in [m]$ for which $a_k^T y(d, \ell) > u_k$.
6. Compute $\varepsilon > 0$ such that $f(d, \ell - \varepsilon e_j) > 0$ and $a_k^T y(d, \ell - \varepsilon e_j) - \gamma_k(d, \ell - \varepsilon e_j) > u_k$.
7. $\ell \leftarrow \ell - \varepsilon e_j$.
8. $d \leftarrow \frac{1}{f(d, \ell)} d$.
9. $\hat{\lambda}_k \leftarrow \gamma_k(d, \ell)Dt(d, \ell) - DA^T B(d)^{-1} a_k$.
10. $\tilde{\lambda}_k \leftarrow \Lambda \hat{\lambda}_k^- + \hat{\lambda}_k^+$.
11. Return $\bar{\lambda}_k := \tilde{\lambda}_k + e_k$ and Exit.
Procedure 3 Procedure for updating ellipsoid \( E(d, \ell) \) by updating its parameters \( d \) and \( \ell \).

**Input:** \( d > 0 \) and certified lower bound \( \ell \) with certificate matrix \( \Lambda \) satisfying \( f(d, \ell) = 1 \), and \( j \in \{1, \ldots, m\} \).

1: \( \ell \leftarrow \ell - \frac{2(t_j(d,\ell) - v_j(\ell))}{d_j \gamma_j(d,\ell)^2} e_j \)
2: Compute \( y(d, \ell) \). If \( A^\top y(d, \ell) \leq u \), then Return \( y(d, \ell) \) as a solution to (Alt) and Stop.
3: Compute \( f(d, \ell) \). If \( f(d, \ell) \leq 0 \), then call Procedure 2 and Stop.
4: \( d \leftarrow \frac{1}{f(d,\ell)} d \)
5: \( \ell \leftarrow \ell + \frac{2(2v_j(d,\ell) - \gamma_j(d,\ell))}{(m-1) d_j \gamma_j(d,\ell)^2 + 2} e_j \)
6: \( d \leftarrow \frac{1}{f(d,\ell)} d \)
7: \( d \leftarrow d + \frac{2}{m-1} \frac{1}{\gamma_j(d,\ell)^2} e_j \)
8: Return \( d \) and \( \ell \), and Stop.
Denote the five updates in the update procedure by

\[ \ell^{(1)} = \ell - \frac{2(t_j(d, \ell) - v_j(\ell))}{d_j\gamma_j(d, \ell)^2} e_j \]  
\[ d^{(1)} = \frac{1}{f(d, \ell^{(1)})} d \]  
\[ \ell^{(2)} = \ell^{(1)} + \frac{2(2v_j(\ell^{(1)}) - \gamma_j(d^{(1)}, \ell^{(1)}))}{(m-1)d_j^{(1)}\gamma_j(d^{(1)}, \ell^{(1)})^2 + 2} e_j \]  
\[ d^{(2)} = \frac{1}{f(d^{(1)}, \ell^{(2)})} d^{(1)} \]  
\[ d^{(3)} = d^{(2)} + \frac{2}{m-1} \frac{1}{\gamma_j(d^{(2)}, \ell^{(2)})^2} e_j \]

We use \( m \) instead of \( n \) in the updates (3) and (5) in order to establish a guarantee when \((P)\) is infeasible... this will become clear.
Updating Ellipsoid, cont.

Theorem

Let $d > 0$ and $\ell$ be certified lower bounds for $P$ with certificate matrix $\Lambda$, and suppose that $f(d, \ell) = 1$. Let $\ell^{(1)}$ be defined as in (1), and suppose that $f(d, \ell^{(1)}) > 0$. Let $d^{(1)}$, $\ell^{(2)}$, $d^{(2)}$, and $d^{(3)}$ be defined as in (2), (3), (4), and (5) respectively. Then,

$$\frac{1}{f(d^{(3)}, \ell^{(2)})} d^{(3)} = \alpha(d, \ell) \left( d + \frac{2}{m-1} \frac{1}{\gamma_j(d, \ell)^2} e_j \right),$$

where $\alpha(d, \ell) > \frac{m^2-1}{m^2}$.

If we used $m$ instead of $n$ in the updates, then $n$ would appear above instead of $m$ like in a standard implementation of the ellipsoid algorithm.
Oblivious Ellipsoid Algorithm

Algorithm 4 Oblivious ellipsoid algorithm

**Input:** data \( A \) and \( u \), certified lower bounds \( \ell \) for \((P)\) with certificate matrix \( \Lambda \), and \( d > 0 \).

1: Compute \( y(d, \ell) \). If \( A^\top y(d, \ell) \leq u \), then Return \( y(d, \ell) \) as a solution of \((P)\) and Stop.
2: Compute \( f(d, \ell) \). If \( f(d, \ell) \leq 0 \), then call Procedure 2 and Stop.
3: \( d \leftarrow \frac{1}{f(d, \ell)} d \)
4: Compute most violated constraint: \( j \leftarrow \arg\max_{i \in [m]} a_i^\top y(d, \ell) - u_i \).
5: If \( \ell_j < L_j := a_j^\top y(d, \ell) - \gamma_j(d, \ell) \), then update the certificate for constraint \( j \):
6: \( \hat{\lambda}_j := \gamma_j(d, \ell) D t(d, \ell) - DA^\top B(d)^{-1} a_j \)
7: \( \tilde{\lambda}_j := \Lambda \hat{\lambda}_j^− + \hat{\lambda}_j^+ \)
8: \( \lambda_j \leftarrow \tilde{\lambda}_j \)
9: If \( L_j > u_j \), then Return type-1 certificate of infeasibility \( \bar{\lambda}_j := \lambda_j + e_j \) and Stop.
10: Update \( E(d, \ell) \) by updating ellipsoid parameters \( d \) and \( \ell \):
11: Call Procedure 3 with input \( d, \ell, \Lambda \) and output \( \tilde{d}, \tilde{\ell} \)
12: Re-set \( (d, \ell) \leftarrow (\tilde{d}, \tilde{\ell}) \) and Goto Step 1.
Infeasible Analysis

Proposition

Let \( d > 0 \) and \( \ell \in \mathbb{R}^m \) such that \( f(d, \ell) > 0 \). For all \( i \in \{1, \ldots, m\} \) it holds then

\[
\gamma_i(d, \ell) \leq \left( \frac{d_i}{f(d, \ell)} \right)^{-\frac{1}{2}}.
\]

Recall if \( \gamma_i(d, \ell) < \tau(A, u) \) for all \( i \in \{1, \ldots, m\} \), then we can construct a solution to \((Alt)\)

So we can construct a solution to \((Alt)\) if the entries of the normalized iterate \( \frac{1}{f(d, \ell)} d \) eventually become large enough so that

\[
\left( \frac{1}{f(d, \ell)} d_i \right)^{-\frac{1}{2}} < \tau(A, u) \quad \text{for all } i \in \{1, \ldots, m\}
\]
Infeasible Analysis, cont.

So we can construct a solution to (Alt) if the entries of the normalized iterate \( \frac{1}{f(d, \ell)} d \) eventually become large enough so that

\[
\left( \frac{1}{f(d, \ell)} d_i \right)^{-\frac{1}{2}} < \tau(A, u) \quad \text{for all } i \in \{1, \ldots, m\}
\]

To this end, we define and consider the potential function

\[
\phi(d, \ell) := \prod_{i=1}^{m} \max \left\{ \left( \frac{1}{f(d, \ell)} d_i \right)^{-\frac{1}{2}}, \frac{m}{m+1} \tau(A, u) \right\}
\]
Infeasible Analysis, cont.

\[
\phi(d, \ell) := \prod_{i=1}^{m} \max \left\{ \left( \frac{1}{f(d, \ell) d_i} \right)^{-\frac{1}{2}}, \frac{m}{m+1} \tau(A, u) \right\}
\]

The potential function decreases by a multiplicative factor at each iteration of OEA:

**Lemma**

Let \(d > 0\) and \(\ell \in \mathbb{R}^m\) satisfy \(f(d, \ell) > 0\), and similarly let \(\tilde{d} > 0\) and \(\tilde{\ell} \in \mathbb{R}^m\) satisfy \(f(\tilde{d}, \tilde{\ell}) > 0\). Let \(j \in [m]\), and suppose that \(d, \ell, \tilde{d}, \tilde{\ell}\) satisfy:

\[
\frac{1}{f(\tilde{d}, \tilde{\ell})} \tilde{d} = \alpha \left( \frac{1}{f(d, \ell)} d + \frac{2}{m-1} \frac{1}{\gamma_j(d, \ell)^2} e_j \right),
\]

for a scalar \(\alpha \geq \frac{m^2 - 1}{m^2}\). If \(\left( \frac{d_j}{f(d, \ell)} \right)^{-\frac{1}{2}} \geq \tau(A, u)\), then

\[
\phi(\tilde{d}, \tilde{\ell}) \leq e^{-\frac{1}{2(m+1)}} \phi(d, \ell).
\]
Theorem

Let \( \ell \in \mathbb{R}^m \) be certified lower bounds for \( P \) with certificate matrix \( \Lambda \). Let \( d := e \in \mathbb{R}^m \). If \( P \) is infeasible, OEA with input \( A, u, \ell, \Lambda, \) and \( d \) will stop and return a certificate of infeasibility for \( P \) in at most

\[
2m(m + 1) \ln \left( \frac{m + 1}{2m} \frac{\|u - \ell\|}{\tau(A, u)} \right)
\]

iterations.
Corollary

Consider the linear inequality system with box constraints \((P_B)\), and let \(A, u, \ell, \) and \(\Lambda\) be defined as in previously stated. Let \(d := e \in \mathbb{R}^m\). If \((P_B)\) is infeasible, OEA with input \(A, u, \ell, \Lambda, \) and \(d\) will stop and return a certificate of infeasibility for \((P_B)\) in at most

\[
2m(m + 1) \ln \left( \frac{(m + 1)(\sqrt{m} + 2)}{2m} \frac{\|\bar{b} - b\|}{\tau(A, u)} \right)
\]

iterations.
For $d > 0$ and $\ell \in \mathbb{R}^m$ satisfying $f(d, \ell) > 0$, the (relative) volume of $E(d, \ell)$ is:

$$\text{vol } E(d, \ell) := \frac{(f(d, \ell))^n}{\sqrt{\det ADA^\top}}$$

The volume decreases by a multiplicative factor at each iteration of OEA:

**Lemma**

Let $d > 0$ and $\ell \in \mathbb{R}^m$ satisfy $f(d, \ell) > 0$, and similarly let $\tilde{d} > 0$ and $\tilde{\ell} \in \mathbb{R}^m$ satisfy $f(\tilde{d}, \tilde{\ell}) > 0$. Let $j \in [m]$, and suppose that $d, \ell, \tilde{d}, \tilde{\ell}$ satisfy:

$$\frac{1}{f(\tilde{d}, \tilde{\ell})} \tilde{d} = \alpha \left( \frac{1}{f(d, \ell)} d + \frac{2}{m - 1} \frac{1}{\gamma_j(d, \ell)^2} e_j \right),$$

for a scalar $\alpha \geq \frac{m^2 - 1}{m^2}$. Then

$$\text{vol } E(\tilde{d}, \tilde{\ell}) \leq e^{-\frac{1}{2(m+1)}} \text{vol } E(d, \ell).$$
Theorem

Let $\ell \in \mathbb{R}^m$ be a certified lower bound for $(P)$ with certificate matrix $\Lambda$. Let $d := e \in \mathbb{R}^m$. If $(P)$ is feasible, OEA with input $A, u, \ell, \Lambda,$ and $d$ will stop and return a feasible solution of $(P)$ in at most

$$\left\lfloor 2n(m + 1) \ln \left( \frac{\|u - \ell\|}{2\rho(A)\tau(A, u)} \right) \right\rfloor$$

iterations.
Corollary

Consider the linear inequality system with box constraints \((P_B)\), and let \(A\), \(u\), \(\ell\), and \(\Lambda\) be defined as previously stated. Let \(d := e \in \mathbb{R}^m\). If \((P_B)\) is feasible, OEA with input \(A\), \(u\), \(\ell\), \(\Lambda\), and \(d\) will stop and return a feasible solution of \((P_B)\) in at most

\[
2n(m + 1) \ln \left( \frac{\sqrt{\hat{m} + 2\|\bar{b} - b\|}}{2\rho(A)\tau(A, u)} \right)
\]

iterations.