N-RN Evaluating a Special Exponential Expression

Task

Three students disagree about what value to assign to the expression 0^0. In each case, critically analyze the student's argument.

a. Juan suggests that $0^0 = 1$:

I know that $2^0 = 1$ and $1^0 = 1$ and $x^0 = 1$ for any non-zero real number x. So 0^0 should also be 1.

b. Briana thinks that $0^0 = 0$:

I know that $0^1 = 0$ and $0^2 = 0$ and $0^x = 0$ for any non-zero real number x. So 0^0 should be 0.

c. Kristin says that $0^0 = -1$:

If I try some negative numbers, I find $\left(-\frac{1}{3}\right)^{-\frac{1}{3}}$ is about -1.44, $\left(-\frac{1}{5}\right)^{-\frac{1}{5}}$ is about -1.38, and $\left(-\frac{1}{1001}\right)^{-\frac{1}{1001}} = -1.007$. As the base and exponent both get closer to 0, it looks like the values are getting closer and closer to -1. So 0^0 should be -1.

IM Commentary

The purpose of this task is to study the rules of exponents in the context of trying to make sense of a very interesting mathematical expression. The first two arguments are relatively common while the third is not. The third argument is a bit off the beaten track and is intended to stimulate conversation. First, it is not common to use negative numbers as a base because many powers will not make sense within the real numbers system, $(-1)^{\frac{1}{2}} = i$ being the most famous example. Second, it is a good exercise for students to make sense of expressions with negative exponents. For example, to calculate $\left(-\frac{1}{3}\right)^{-\frac{1}{3}}$, we have

$$\begin{aligned}\left(-\frac{1}{3}\right)^{-\frac{1}{3}} &= \frac{1}{\left(-\frac{1}{3}\right)^{\frac{1}{3}}}\\ &= \frac{1}{-\left(\frac{1}{3}\right)^{\frac{1}{3}}}\\ &= \frac{1}{-\frac{1}{3^{1/3}}}\\ &= -3^{1/3}.\end{aligned}$$

It is vital going from the first line to the second line that 3 is an odd number and thus $(-1)^3 = -1$, or equivalently, $(-1)^{1/3} = -1$. Students are not likely to be convinced by this third argument and the teacher can take advantage of this to have them offer criticism. Much of this criticism is likely applicable to the ''more standard'' arguments given in (a) and (b).

In algebraic notation, the three arguments correspond to evaluating 0^0 by studying x^0, 0^x, and x^x. In none of these cases is the expression defined if $x = 0$. The goal of each method is to argue that if we study the given expression for values of x close to 0, it looks like the expression should take the value 0, 1, or -1 respectively.

The teacher may wish to encourage students to evaluate 0^0 before presenting the different approaches outlined here. They may come up with other interesting ideas which reinforce the notion that more information is needed to make sense of 0^0. The teacher might also emphasize the fact, which comes up in the solution to part (a), that the difficulty in making sense of 0^0 is closely linked to another expression with 0's, namely $\frac{0}{0}$. The number 0 is full of surprises and paradoxes!

This task provides arguments why 0^0 could be assigned the values $0, 1,$ and -1. Because of this ambiguity, the expression 0^0 is taken to be undefined, just like $\frac{0}{0}$. In a specific context, such as those described by Juan, Briana, and Kristin, we might choose to assign, for convenience, a particular value to the expression 0^0. This is legitimate as long as we make it clear that the value we assign comes from the context and should not be used outside of the context.

Solution

a. Juan's argument is that for any non-zero number x we have $x^0 = 1$ so from here it would seem natural to make $0^0 = 1$. Examining this in a little more detail, we can write

$$\begin{aligned} x^0 &= x^{1-1} \\ &= \frac{x}{x} \\ &= 1. \end{aligned}$$

The expression $\frac{x}{x}$ does not make any sense if we substitute $x = 0$ as we obtain $\frac{0}{0}$. This problem goes away if we rewrite $\frac{x}{x}$ as 1 but this is not legitimate in the special case where $x = 0$ which is what is of interest to Juan. The strength of Juan's argument lies in making a uniform choice for x^0 regardless of the value of x.

b. Briana's argument is very similar to Juan's except that instead of studying what happens when the base varies, she is looking at what happens when the exponent varies. In other words, Juan studies the expression x^0 for different values of x while Briana studies the expression 0^x for different values of x. When x is a positive integer, $0^x = 0$. For positive rational numbers a little more analysis is required. If p, q are positive integers then we know that $0^{\frac{p}{q}}$ satisfies the equation

$$\left(0^{\frac{p}{q}}\right)^q = 0^p.$$

Since $0^p = 0$ we must have $0^{\frac{p}{q}} = 0$ as 0 is the only number which gives 0 when raised to a whole number power.

Having determined that $0^{\frac{p}{q}} = 0$ for every positive rational number $\frac{p}{q}$ it is natural to define 0^x to be 0 when $x = 0$ or when x is a positive irrational number.

As with Juan's argument, the strength of Briana's argument lies in making a uniform choice of values for 0^x for all non-negative real numbers x. The weakness lies in the fact that while a value is being assigned to these expressions, they still have not been given any concrete meaning. Moreover, 0 as an exponent is very different from the rational numbers considered as no multiple of 0 gives a non-zero integer.

c. Kristin's argument is an interesting one. We begin by making a table of values for some expressions of the form Kristin is considering:

x	x^x **(approximate value)**
-1	-1
$-\frac{1}{3}$	-1.442
$-\frac{1}{11}$	-1.244
$-\frac{1}{51}$	-1.080
$-\frac{1}{101}$	-1.047
$-\frac{1}{1001}$	-1.007

According to the table, it appears as if the values of x^x are getting closer and closer to -1 as x gets closer and closer to 0. There are, however, some problems with this argument and also problems when it is juxtaposed with the other students' ideas. First, Kristin has chosen to extract odd roots which is possible for a negative number because a negative number raised to a negative odd power is negative. The method does not work, however, for even roots. So for example, the expression

$$\left(-\frac{1}{2}\right)^{-\frac{1}{2}}$$

is not equal to any real number. When x is a negative irrational number it will also be difficult to make sense of x^x. So Kristin is basing her argument on some very specific and special calculations.

More serious issues arise if we observe that Kristin's table also applies to the positive values $x = \frac{p}{q}$. The only thing that changes is the sign of x^x which is now positive instead of negative. So Kristin's reasoning would at the same time establish that $0^0 = 1$ (as well as $0^0 = -1$).

What we can conclude from these three arguments is that 0^0 can be interpreted in different ways. So depending on the context, it might make sense to assign a value of 0 or 1 (or even -1). If all we have is the expression 0^0 and no context, it is not possible to make a good choice and for this reason the expression 0^0 is taken as being undefined.