N-CN, A-REI Completing the square

Task

Renee reasons as follows to solve the equation $x^2 + x + 1 = 0$.

First I will rewrite this as a square plus some number.

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Now I can subtract $\frac{3}{4}$ from both sides of the equation

$$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}.$$

But I can't take the square root of a negative number so I can't solve this equation.

a. Show how Renee might have continued to find the complex solutions of $x^2 + x + 1 = 0$.

b. Apply Renee's reasoning to find the solutions to $x^2 + 4x + 6 = 0$.

IM Commentary

The goal of this task is to solve quadratic equations with complex roots by completing the square. Students could of course directly use the quadratic formula, but going through the process of completing the square helps reinforce the mathematics behind the quadratic formula. The teacher may wish to have students graph the solutions so that they can see that the imaginary solutions are reflections of one another about the real axis. In the case of the equation $x^2 + x + 1 = 0$ these two solutions also lie on the unit circle and they are third roots of unity, that is, the roots of this equation equal 1 when they are raised to the third power.

If desired, the teacher might add a third part to this question: can you apply Renee's reasoning to solve $ax^2 + bx + c = 0$ for real numbers a, b, c? The method used to solve parts (a) and (b) extends naturally. First, as long as $a \neq 0$, that is as long as the polynomial is not linear, we can mulitply by $\frac{1}{a}$: the solutions to $ax^2 + bx + c = 0$ are the same as the solutions to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Next, for $\frac{b}{a}$ to be the linear part of a perfect square we have

$$x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}.$$

So we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}.$$

The equation $ax^2 + bx + c = 0$ holds when

$$\begin{aligned}\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a}\\ &= \frac{b^2 - 4ac}{4a^2}.\end{aligned}$$

Extracting a square root on both sides and then subtracting $\frac{b}{2a}$ from both sides gives the quadratic formula for the roots of the equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Whether or not to do this third part of the question depends on how comfortable the students are with algebraic manipulations of symbols and numbers. An alternative would be to give the students several quadratic equations with different numbers for coefficients.

Solution

a. Renee is right that any real number squared is non-negative and so the square root of a negative number can not be equal to a real number. In the complex numbers, however, we have the number i which satisfies $i^2 = -1$ and this allows us to take the square root of a negative number. Looking at the equation

$$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4},$$

The square root of $\frac{3}{4}$ is $\frac{\sqrt{3}}{2}$ and so

$$\begin{aligned}\left(\frac{\sqrt{3}}{2}i\right)^2 &= \left(\frac{\sqrt{3}}{2}\right)^2 i^2 \\ &= \frac{3}{4} \times (-1) \\ &= -\frac{3}{4}.\end{aligned}$$

So we have

$$x + \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i$$

with the plus or minus sign coming from the fact we have squared the quantity $x + \frac{1}{2}$. Thus the two solutions to the equation $x^2 + x + 1 = 0$ are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

b. To solve $x^2 + 4x + 6 = 0$ using Renee's technique we first look to write $x^2 + 4x$ as part of a perfect square. Since $(x + 2)^2 = x^2 + 4x + 4$ we have

$$x^2 + 4x + 6 = (x + 2)^2 + 2.$$

So we need to find the solutions to $(x + 2)^2 + 2 = 0$. Subtracting 2 from both sides and taking the square root we find

$$x = -2 \pm \sqrt{2}i$$

For any number a we have $(x + a)^2 = x^2 + 2ax + a^2$. So when we want to write $x^2 + 4x + 6$ as a sum of a square plus a real number we need $x^2 + 2ax = x^2 + 4x$

which means $2a = 4$ or $a = 2$. So $(x + 2)^2$ is the *only* perfect square whose quadratic and linear terms are x^2 and $4x$.