Illustrative Mathematics

N-CN, A-REI Complex Square Roots

Task

Working with complex numbers allows us to solve equations like $z^2 = -1$ which cannot be solved with real numbers. Here we will investigate complex numbers which arise as square roots of certain complex numbers.

a. Find all complex square roots of -1, that is, find all numbers $z = a + bi$ which satisfy $z^2 = -1$.

b. Find all complex square roots of 1.

c. Which complex numbers satisfy $z^2 = i$?

IM Commentary

This task is intended as an introduction to the algebra of the complex numbers, and also builds student's comfort and intuition with these numbers. For example, we frequently stress that the "plus or minus rule", that if, for example, $x^2 = 1$, then $x = \pm 1$. This follows because the only two real numbers whose square is 1 are the numbers 1 and -1. With the introduction of a new symbol i satisfying previously unsolvable equations like $x^2 = -1$, it is natural to wonder whether any of these rules need modification. To this end, the task has students deduce algebraically that the rule "$x^2 = 1 \rightarrow x = \pm 1$ is still valid, as even amongst complex numbers, there are only those two solutions. By generalizing the technique, in part (c) students also learn how to start thinking about square roots of complex numbers. This may come as a surprise, that while we need to introduce a new symbol to find a square root of -1, no further symbols are needed to take square roots of i (or anything else!)

The task could serve as a segue to a number of further topics, most notably the complex quadratic formula (which could be used to solve this task rather trivially), and the geometric version of complex multiplication in terms of polar coordinates. For a more advanced version of this task for which these techniques are more appropriate, see N-CN, Complex Cube and Fourth Roots of 1.

Solution

a. We will try to solve the equation $(a+bi)^2 = -1$. Expanding the left hand side we find

$$a^2 + 2abi + bi^2 = -1.$$

Since $i^2 = -1$ this simplifies to

$$a^2 - b^2 + 2abi = -1.$$

The right hand side of this equation is the real number -1 which could also be rewritten as -1 + 0i. In particular, equating the real and complex parts of the two sides we find

$$a^2 - b^2 = -1$$
$$2ab = 0.$$

From the second equation, we conclude that $a = 0$ or $b = 0$ (or both are 0). From the first equation, if $a = 0$ then $-b^2 = -1$ which is possible if $b = \pm 1$. If $b = 0$ then the first equation tells us that $a^2 = -1$ which is impossible since a is real number. So $z^2 = -1$ has exactly two solutions: $a = 0$ and $b = \pm 1$ which correspond to $z = \pm i$.

b. We repeat the same reasoning as in part (a) except that this time we are trying to solve $(a+bi)^2 = 1$. This leads to the system of equations

$$a^2 - b^2 = 1$$
$$2ab = 0.$$

If $a = 0$ this is not possible because $-b^2$ can not be 1. So $b = 0$ and then the first equation tells us that $a^2 = 1$ or $a = \pm 1$. So $z^2 = 1$ has the two expected solutions and no others: $z = \pm 1$.

c. This time we want to solve $(a+bi)^2 = i$ which is equivalent to $a^2 + 2abi + bi^2 = i$. Equating real and complex parts of the two sides as above we find

$$\begin{aligned} a^2 - b^2 &= 0 \\ 2ab &= 1. \end{aligned}$$

The first equation tells us that $a^2 = b^2$. This means that $a = \pm b$. If $a = b$ then the second equation gives us $2b^2 = 1$ or $b^2 = \frac{1}{2}$. This means that $b = \pm\frac{1}{\sqrt{2}}$ which can also be written as

$$b = \pm\frac{\sqrt{2}}{2}.$$

Since $a = b$ this leads to the solutions $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$. If $a = -b$ the second equation leads to $-2b^2 = 1$ or $b^2 = -\frac{1}{2}$. This is not possible, however, because b is real number. So the equation $z^2 = i$ has exactly two solutions.