Illustrative Mathematics

M2.2 Quadratic Functions

• Construct quadratic functions and quadratic sequences.
• Represent quadratic functions using recursive formulas, expressions, tables, and graphs.
• Express quadratic functions in equivalent forms for different purposes; understand the relation between vertex form and the shape of the graph.
• Find the average rate of change of a quadratic function over a unit interval and compare rates for successive intervals.
• Describe properties that distinguish linear, exponential, and quadratic functions.
• Model with quadratic functions.

In unit A1 students developed fluency with linear functions and in unit A2 they learned about simple exponential functions. Students can use graphing technology to plot a function, find intercepts and intersection points, and find a linear regression. They know exponent rules and the application of the distributive property to combining like terms or factoring out a common factor. In this unit students build and interpret quadratic functions. They work with contexts that can be modeled by quadratic functions and compare them with contexts that can be modeled linear and exponential functions. They express quadratic functions using recursive equations (e.g. $f(n+1) = f(n) + 2n + 1$), equations in two variables (e.g. $y = x^2 + 2x + 3$), or function notation (e.g. $f(x) = x^2 + 2x + 3$). They understand the purpose of different forms for the quadratic expression on the right hand side in the last two cases. They graph quadratic functions expressed in different forms, and construct functions expressed in factored or vertex form for a given learn how to put a function in vertex form, and see what information can be most easily obtained from it.

In unit A4 students solve quadratic equations in one variable approximately and exactly using various methods. They continue to develop facility in algebraic manipulation of quadratic expressions and equations. In unit A5, they explore complex numbers and revisit quadratic equations to solve for complex roots.

M2.2.0 Pre-unit diagnostic assessment

• distinguish linear from exponential functions and identify if a function is neither;
• evaluate quadratic expressions and solve simple quadratic equations by inspection. • solve simple quadratic equations by inspection

M2.2.1 It's Neither Linear Nor Exponential

• Model a context with a quadratic function and interpret values of the function in context.
• Graph a quadratic function and interpret the graph.
• Find the average rate of change over a unit interval and compare rates for successive intervals.

This hook lesson touches on topics that arise throughout the unit: modeling with quadratic functions, interpreting their graphs in terms of a context, the way quadratic functions grow, and solving quadratic equations. The specific quadratic function used is of the simplest type and students do not have to carry out extensive manipulations. Rather, the context provides a motivation for learning those manipulations.

M2.2.2 Visual Patterns and Quadratic Sequences

• Look for structure in number sequences arising from visual patterns.
• Model the patterns with quadratic functions given by recursive descriptions, expressions, or equations.

In the previous section students saw a quadratic function where the change over successive intervals of a fixed length grows linearly. It is not until calculus that students will be able to describe precisely the growth law for quadratic functions (namely that their derivatives are linear). However, they can see a discrete version of this law by restricting to quadratic functions whose domains are the whole numbers, that is, quadratic sequences. This allows students to focus on some basic features of quadratic functions without getting bogged down in analysis of real world data and contexts. In this and subsequent sections these sequences often arise from sequences of visual patterns. Each visual pattern lends itself to a geometric interpretation for how it grows, in such a way that the number of additional squares (or other objects) in each successive pattern grows linearly. It is then natural to model the number with a quadratic sequence given what students learned in the previous section. Students conjecture the number of squares at a given stage, create a table, and express the number of squares algebraically. Since the prompts are visual patterns, there are multiple points of entry and students have a context with which to check their conjectures. A note on equations, expressions, and functions: students might initially write an expression, for example n^2, to represent the number of squares in the pattern. This is a natural thing to do. At some later point, in order to help them see that a sequence is a function, it might be helpful to introduce another variable S for the number of squares in the pattern and write an equation $S = n^2$, or to use function notation $f(n) = n^2$. All are acceptable, but it is important to use terminology correctly and not refer to expressions as equations or vice versa.

M2.2.3 Compare Equivalent Expressions for Quadratic Functions

• Understand that different ways of seeing a pattern give rise to different but equivalent expressions for the function arising from the pattern.
• Reason through the equivalence of expressions.

This section explores equivalence of quadratic expressions. Visual patterns that are more complicated than in the previous section lead to different ways of expressing a quadratic function and prompt a discussion about the meaning of equivalence (equivalent expressions define the same function) and what purpose each of the equivalent forms might be useful for. Various equivalences can be explored in this section. Students draw on their previous knowledge of the distributive property to multiply binomials (the principle of multiplying “each by each”), including the special case of expanding the square in an expression given in vertex form. They also begin to think about how these processes might be reversed (factoring and completing the square).

Tasks

A-SSE Seeing Dots

A-SSE Equivalent Expressions

M2.2.4 Compare Quadratic Functions with Linear and Exponential Functions

- **Distinguish between tables representing linear, exponential, and quadratic functions.**
- **Distinguish between graphs of linear, exponential, and quadratic functions.**
- **Use precise language to describe properties that distinguish linear, exponential, and quadratic functions.**

Now that students have been introduced to quadratic functions they compare them with the two other families of functions they have studied, linear and exponential functions. They compare tables and graphs and describe the differences. They also compare the growth of increasing quadratic functions with the growth of increasing linear functions or increasing exponential functions.

Tasks

F-LE Identifying Functions

M2.2.5 Model Simple Contexts with Quadratic Functions

- **Construct a simple quadratic model.**
- **Use the model to solve problems and make predictions.**

Now that students have constructed quadratic functions and compared them with linear and exponential functions, they start to explore contexts that can be modeled by quadratic functions. Different contexts naturally lead to different forms, and students use the skills developed in Section 3 to convert between forms. They also consider what shape graph is suggested by the context and use that information to narrow down the possibilities for expressing the function. This section could revisit the context from Section 1 in a deeper way, or bring in completely new contexts as listed here.

Tasks

F-BF Skeleton Tower

M2.2.6 Mid-Unit Assessment

• write, use and interpret quadratic function that models a given context;
• express quadratic functions in equivalent forms and choose an appropriate form for a given purpose;
• differentiate between graphs and tables of values for linear, exponential, and quadratic functions;
• label features of the graph of a quadratic function with key vocabulary words (axis of symmetry, vertex, maximum, minimum, x-intercept, y-intercept).

M2.2.7 Express Quadratic Functions in Vertex Form

• Understand how the structure of vertex form is related to the maximum or minimum value of the function and to the vertex of its graph.
• Use vertex form to write a possible quadratic function given the maximum or minimum of the function or the vertex of its graph.

The standard form of a quadratic is not always the most useful form for a given situation. When modeling the path of a projectile, for example, it may be useful to express a function in vertex form in order to find the maximum height. In this section students interpret and construct functions expressed in vertex form. It is possible, but not necessary, that students begin work with converting a function from standard form to vertex form by completing the square. They are not expected to gain fluency in this operation until the next unit.

Tasks

F-BF Building a quadratic function from $f(x) = x^2$

M2.2.8 Work with the Three Basic Forms

• Understand how the structure of factored form is related to the zeros of a function and the x-intercepts of its graph.
• Select the best form for expressing a quadratic function to illuminate specific features of graphs.

In this section, students write quadratic functions in different forms to illuminate different features of the function. Through the use of graphing technology or by hand, they explore ways to sketch quadratic functions given key features of the graph. Factored form, which has not been explicitly discussed before now, comes into play in these activities and students explore its equivalence to other forms through graphing. They explore expressions in vertex, factored, or standard form and interpret those expressions in terms of a model. By the end of this section students should be fluent with converting from factored or vertex form to standard form. They are also in the process of learning how to construct factored and vertex form from standard form, and become fluent with this in the next unit.

Tasks

A-SSE Graphs of Quadratic Functions
A-SSE Profit of a company
F-IF Which Function?

M2.2.9 Model Richer Contexts with Quadratic Functions

• Fit functions to verbal descriptions and graphs using key features.
• Solve modeling problems.

Students are now ready to start applying their knowledge of quadratic functions. They draw on their ability to construct expressions for functions to meet a given purpose. They identify intercepts and vertices in order to write functions that match a given situation. The work in this section prepares students for the work on solving quadratic equations in one variable but does not require students to find exact solutions to them.

Tasks

F-BF Medieval Archer
F-LE Choosing an appropriate growth model

M2.2.10 Choosing the Most Convenient Form (Optional)

• Select find a function that matches a given graph.

This section can be used at the end of the unit or intermittently throughout the unit to reinforce different aspects of quadratic functions (and other functions) using technology. The Daily Desmos challenges here mostly contain quadratics, but a few have either linear or exponential components making them a perfect opportunity to think through other types of functions they learned more about in previous units or grade levels and compare them with their new knowledge of quadratics.

M2.2.11 Summative assessment

• generate functions given graphs and graphs given functions;
• interpret expressions for quadratic functions in terms of a context it represents;
• express quadratic functions in different forms for different purposes;
• solve modeling problems using quadratic functions.