Illustrative Mathematics

A2.4 Polynomials and Rational Functions

- **Add, subtract, and multiply polynomials and express them in standard form using the properties of operations (A-APR.A.1).**
- **Prove and make use of polynomial identities (A-APR.C.4).**
- **Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior (A-APR.B.3★, F-IF.C.7c★).**
- **Use the remainder theorem to find factors of polynomials (A-APR.B.2, A-APR.D6).**
- **Use various strategies including graphing and factoring to solve problems in contexts that can be modeled by polynomials in one variable.**
- **Build a rational function that describes a relationship between two quantities (F-BF.A.1).**
- **Graph rational functions, interpret features of the graph in terms of a context, and use the graphs to solve problems (A-SSE.A.1a, A-REI.D.11★, F-IF.B.4, F-IF.C.7d(+)).**
- **Express rational functions in different forms to see different aspects of the situation they model (A-APR.D.6).**
- **Solve simple rational equations and understand why extraneous roots can arise (A-REI.A.2).**

Coming into this unit, students understand that one can do arithmetic on quadratic expressions, and have generalized that understanding to polynomial expressions. They have solved quadratic equations with complex solutions using the methods of factoring, completing the square, and the quadratic formula. They have graphed quadratic functions and understand the relationship between zeros and factors of quadratics. They are familiar with modeling situations from which quadratic relations arise.

Students have modeled contexts with simple rational functions. They have graphed these functions and interpreted features of the graphs in context, including vertical asymptotes and end behavior. They can relate the domain of a function to its graph.

In this unit, students extend their previous work with quadratics and polynomials (in unit A4) to achieve a more general understanding of polynomials. They work with polynomials as a system where you can add, subtract, and multiply, analogous to the integers. They graph polynomial functions and they understand and use the relationship between factors, zeros, and intercepts on the graph. They model with polynomial functions.

Students study the graphs of simple rational functions. They consider contexts which can be modeled with rational functions and interpret vertical and horizontal asymptotes in terms of the context. They rewrite simple rational expressions in different forms to see different aspects of the context, and they find approximate solutions using graphical methods to rational equations that arise from the context. They also solve simple rational equations algebraically and study how and why extraneous solutions may arise.

After this unit, students going into STEM fields will see more examples, including polynomial approximations to other functions. Power series expansions (like Taylor series) show up in calculus. The characteristic polynomial of a matrix is a useful tool in university level linear algebra. Students going into higher mathematics will encounter polynomials as objects that help form abstract algebraic structures such as rings and fields.

Students may encounter simple rational relationships in real life. If they take a calculus course with a rigorous algebra component, they should do more work with more complicated rational expressions and equations beforehand.

A2.4.0 Pre-unit diagnostic assessment

Assess students' ability to
• graph a quadratic by factoring to find zeros, and correctly interpreting end behavior (A-APR.B.3, F-IR.C.7c★);
• solve a quadratic equation with a method suitable to the given form of the equation (A-REI.B.4b);
• write a simple rational equation that models a situation A-CED.A.1★, and use the equation to solve problems (A-REI.A.2).

A2.4.1 What is a polynomial?

• Add, subtract, and multiply polynomials and express them in standard form using the properties of operations (A-APR.A.1).
• Prove and make use of polynomial identities (A-APR.C.4).

In this section students become familiar with the arithmetic of polynomials. They add, subtract, and multiply them, and they use the properties of operations, particularly the distributive law, to express them as a sum of powers with coefficients. They recognize that every polynomial can be put in this form. They use polynomials to express and verify numerical patterns.

The emphasis in this section should not be on formal definitions or formal proofs of closure properties. Rather the emphasis is on preparing students for the manipulations they will be using the coming sections, when they start to study polynomial functions.

Tasks
A-APR Powers of 11
A-APR Trina's Triangles
A-APR Non-Negative Polynomials

A2.4.2 Graphing polynomials

• Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior (A-APR.B.3, F-IF.C.7c ⋆).
• Use the remainder theorem to find factors of polynomials (A-APR.B.2, A-APR.D.6).

The last activity in the previous section prepared students to start viewing polynomials in one variable as defining functions. In this section they study the graphs of polynomial functions. They see that the long run behavior of a polynomial is determined by its highest degree term. The use the relationship between factors and zeros to sketch the graph of a polynomial or to choose an appropriate viewing window for a graph produced by technology. The learn The Remainder Theorem and use it to find factors of polynomials.

Tasks
F-IF Graphs of Power Functions
F-IF Running Time

A-APR Graphing from Factors I
A-APR Graphing from Factors II
A-APR Graphing from Factors III
A-APR Zeroes and factorization of a quadratic polynomial I

A2.4.3 Modeling with polynomials

Use various strategies including graphing and factoring to solve problems in contexts that can be modeled by polynomials in one variable.

The main focus of modeling in this course is situations that can be modeled by linear, exponential, and quadratic functions. However, some contexts naturally give rise to polynomial models. In this section students make use of all that they have learned about polynomial functions to solve problems in such contexts.

Tasks
A-CED, A-REI Introduction to Polynomials - College Fund

A2.4.4 Rational Functions

- **Build a rational function that describes a relationship between two quantities (F-BF.A.1*).**
- **Graph rational functions (A-SSE.A.1a*, F-IF.C.7d*).**
- **Interpret the graph of a rational function in terms of a context (F-IF.B.4*).**

In this section students study simple rational functions. The emphasis is on rational functions that arise naturally out of a real-world context, and on interpreting features of their graphs in terms of that context. Students experiment with graphs using technology to learn the relationship between features of the graph and the structure of the expression defining the function.

Tasks
F-BF,IF The Canoe Trip, Variation 1
F-IF Average Cost
F-IF Graphing Rational Functions

A2.4.5 Modeling with rational functions

• Graph rational functions and use the graphs to solve problems (A-REI.D.11 ⋆).
• Express rational functions in different forms to see different aspects of the situation they model (A-APR.D.6).
• Solve simple rational equations and understand why extraneous roots can arise (A-REI.A.2).

In the previous section students encountered simple rational functions. Here, they extend that work. They consider contexts which can be modeled with rational functions. They rewrite simple rational expressions in different forms to see different aspects of the context, and they find approximate solutions using graphical methods to rational equations that arise from the context. They also solve simple rational equations algebraically and study how and why extraneous solutions may arise.

Tasks
A-REI Ideal Gas Law
A.APR Combined Fuel Efficiency
A-REI An Extraneous Solution

A2.4.6 End Assessment

Assess students' ability to
• graph a polynomial by factoring to find zeros, and correctly interpreting end behavior (A-APR.B.3, F-IF.C.7c⋆);
• apply the remainder theorem to solve a mathematical problem (A-APR.B.2);
• model with a polynomial (A-CED.A.1⋆), and use the model to solve a problem (A-REI.D.11⋆);
• rewrite a rational expression in a different form to solve a problem (A-REI.D.11⋆);
• write a rational equation that models a situation (A-CED.A.1⋆), and use the equation to solve problems (A-REI.A.2);
• demonstrate understanding that when solving a rational equation, extraneous roots may emerge as a result of the solution process (A-REI.A.2).