A2.3 Logarithms

• Understand the definition of a logarithm as the solution to an exponential equation (F-LE.A.4*).
• Practice evaluating logarithmic expressions and converting between the exponential form of an equation and the logarithmic form (F-LE.A.4*).
• Solve exponential equations using logarithms (F-LE.A.4*).
• Understand the natural logarithm as a special case (F-LE.A.4*).
• Graph exponential and logarithmic functions, both by hand and using technology (F-IF.C.7e*, F-BF.5(+)).
• (Optional) Understand and explain the properties of logarithms.
• (Optional) Use properties of logarithms to solve problems.
• (Optional) Solve problems using the properties of logarithms.

Before this unit, students can construct exponential functions given a graph, table, or description, and interpret exponential functions expressed in different forms in terms of a context. They can graph exponential functions and note key features such as end behavior, asymptotes, and intercepts. They recognize real-world contexts that can be modeled by exponential functions, such as savings accounts and population growth, and they construct functions to model them. For example, they might describe the amount in a savings account with a $1000 initial balance that earns 10% a year compounded monthly using the function $A(t) = 1000(1.0083)^{12t}$. They can solve problems involving exponential equations like $20,000 = 1000(1.0083)^{12t}$ graphically, but not algebraically.

In this unit, students understand the logarithm defined operationally as the inverse of exponentiation, and have opportunities to practice interpreting logarithm notation and evaluating logarithms. They use logarithms to solve for an unknown exponent in situations modeled by exponential functions. These functions include ones expressed with base e, necessitating the introduction of the natural logarithm. Students graph logarithmic functions along with the exponential functions that are their inverses, developing an understanding of logarithmic functions as the inverses of exponential functions.

The Common Core State Standards do not explicitly require students to know and use the properties of logarithms. Student intending to pursue STEM careers in college should go deeper and learn these topics. Two optional sections at the end of the unit develop and apply the properties of logarithms.

After this unit, logarithms become a natural part of the toolkit in working with situations modeled by exponential functions. Applications abound in calculus, engineering, and the sciences. There are many sets of data that reveal their structure when plotted on logarithmic scales.

A2.3.0 Pre-unit diagnostic assessment

Assess students' ability to
• use reasoning and exponent properties to solve for an unknown exponents (A-REI.A.1);
• graph a relatively simple exponential function, showing correct end behavior and intercepts (F-IF.C.7e*);
• solve an exponential function at a given value graphically (A-REI.D.11*);
• summarize an exponential relation by writing an equation, given a table or several points (F-LE.A.2*).

A2.3.1 Motivate the need to undo exponentiation

Generate a need to find an unknown exponent which is not easy to guess and check.

The goal of this section is to help students see why a logarithm might be useful in their mathematical toolkit. Students are presented with a situation where they develop an exponential model and need to find an unknown exponent that yields a specified value. Teachers can revisit and assess solving such a problem by graphing with technology or by guess and check. They can then point out that students can solve all other kinds of equations that they know about by rewriting them in a helpful, equivalent form (for example, $x^3 = 1000$ can be rewritten as $x = 1000^{1/3}$) and suggest that there should be a way to do that to find an unknown exponent (as in, for example, $3^x = 1000$). Teachers can either introduce the logarithm at this point, or leave students in suspense until the next section.

A2.3.2 Understand the definition of a logarithm

• Understand the definition of a logarithm as the solution to an exponential equation (F-LE.A.4★).
• Practice evaluating log expressions and converting between the exponential form of an equation and the logarithmic form (F-LE.A.4★).

The logarithm can be defined operationally. Just as $\sqrt[3]{2}$ is defined to be the positive real number that when multiplied by itself three times is equal to 2, the solution to $2^x = k$ is defined to be $x = \log_2(k)$. In this section students develop this definition through an exploratory activity. They analyze a number of true statements about logarithms without having been told the meaning of the notation, make conjectures about the pattern they fit using their knowledge of exponents, and express their meaning in terms of an equivalent exponential equation. They come up with a definition of the logarithm by precisely describing what they see and generalizing it. They then practice interpreting and converting logarithmic expressions.

A2.3.3 Use logarithms to solve problems

• Solve exponential equations using logarithms (F-LE.A.4★).
• Understand the natural logarithm as a special case (F-LE.A.4★).

Once students understand what a logarithm is, they need ample opportunity to apply that understanding to solve problems in various contexts. All of the problems in this section should involve exponential functions with base 2, 10, or e and should be solvable by reasoning directly from the definition of the logarithm, using the equivalence between $b^x = y$ and $x = \log_b y$. In particular many of them involve modeling continuous growth using an exponential function with base e, so this is the section where the natural logarithm is introduced. Students do not need to know the property $\log(a^b) = b\log(a)$ or solve equations using this property ("taking logs of both sides").

Note on calculating logarithms: some scientific calculators have buttons for base 10 logarithms and natural logarithms, but not base 2 logarithms. There are many online calculators that can calculate the latter, such as Desmos or the Google calculator, which is activated by typing \log_2(x)$ into the search engine.

Tasks

F-LE Algae Blooms
F-LE Newton's Law of Cooling
F-LE Moore's Law and Computers
F-LE Snail Invasion

A2.3.4 The logarithm function

- **Graph exponential and logarithmic functions, both by hand and using technology (F-IF.7e*, F-BF.B.5(+)).**
- **Verify that $f(x) = 10x$ and $g(x) = log_{10}(x)$ are inverses of one another (F-BF.B.4b(+)).**

The previous sections focus on the definition of the logarithm as a notation for an unknown exponent and on solving equations involving exponentials and logarithms. In this section students start to view the logarithm as a function. This is analogous to the progression from learning about the square root symbol to studying the square root function. Students create graphs of logarithm functions by viewing them as the inverse of exponential functions, with inputs and outputs reversed. They apply their previous understanding of the nature of exponential functions to draw conclusions about the behavior of the graphs of logarithmic functions. For example, students who understand debt as an exponential function of time view the same situation as time being a logarithmic function of debt.

Tasks

F-BF Exponentials and Logarithms II
F-LE Exponential Kiss

A2.3.5 Going deeper: properties of logarithms (optional)

- **Understand and explain the properties of logarithms.**
- **Use properties of logarithms to solve problems.**

The sections previous to this one are sufficient to meet the standards about logarithms in CCSSM including the (+) standards. Students pursuing STEM careers will need to go further. This section and the next one show the bridging material needed for STEM readiness.

The properties of logarithms arise naturally from the properties of exponents and the fact that logarithmic functions are inverse to exponential functions. In this section students are given opportunities to notice patterns when combining logarithmic expressions with operations (for example, $\log(A) + \log(B)$ is equal to $\log(AB)$), conjecture that these patterns could be useful shortcuts, and then justify why the patterns always work.

A2.3.6 Going deeper: using the properties of logarithms (optional)

Solve problems using the properties of logarithms.

Students solve problems involving exponential functions with bases other than 2, e, or 10, or involving more than one exponential function with different bases, so that it is natural to use the property $\log(a^b) = b\log(a)$. The problems here are more complex than the ones in Section 3 and are suitable for students who are preparing for STEM majors in college.

Tasks
F-LE Rumors
F-LE Comparing Exponentials

A2.3.7 Summative assessment

Assess students' ability to
• solve equations with unknown exponents using logarithms (F-BF.B.5(+), F-LE.A.4*);
• graph a simple logarithmic function by hand showing intercepts and end behavior (F-IF.C.7e*);
• demonstrate understanding of the inverse nature of exponential and logarithmic functions (F-BF.B.4(+));
• solve a real-world problemw ith an unknown exponent using logarithms (F-LE.A.4*).

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