A-REI An Extraneous Solution

Alignments to Content Standards: A-REI.A.2

Task

Megan is working solving the equation

$$\frac{2}{x^2-1} - \frac{1}{x-1} = \frac{1}{x+1}.$$

She says

If I clear the denominators I find that the only solution is $x = 1$ but when I substitute in $x =$ the equation does not make any sense.

a. Is Megan's work correct?

b. Why does Megan's method produce an x value that does not solve the equation?

IM Commentary

The goal of this task is to examine how extraneous solutions can arise when solving ration equations. The task presents an operation, "clearing denominators," which appears to leac to a contradiction. To resolve the contradiction we examine more carefuly what is happening when we clear denominators (MP6). One way to describe the process is that we find a common denominator for both sides and set the numerators equal to each othe This gives solutions to the original equation provided the solutions are in the domain of th rational functions on both sides, that is, provided they are not zeros of one or more of the denominators. In this case, the solution $x = 1$ makes the numerators equal to one another but also makes the denominators of two of the expressions zero, and so $x = 1$ is an extraneous solution.

Another way to think about the process is to connect it to the familiar process students have learned from solving linear equations of multiplying both sides by a non-zero consta to get an equivalent equation. In this case, clearing denominators amounts to mulitplying boths sides by $x^2 - 1$. The problem is that when $x = 1$ or $x = -1$ this is multiplying both sides by zero. So the operation only produces an equivalent equation if you stay away fron those two values, and consider them separately at the end of the process.

Students may also experiment with graphs of the functions on the left hand and right hand side of the equation to see that they are never equal. This confirms Megan's work which shows that if the two expressions are equal to one another then $x = 1$, but this is not possible.

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Solution

a. To check Megan's work, notice that $x^2 - 1 = (x+1)(x-1)$ so $x^2 - 1$ is a common denominator for the three fractions. Using this common denominator we find

$$\frac{2}{x^2 - 1} - \frac{x+1}{x^2 - 1} = \frac{x-1}{x^2 - 1}.$$

Clearing the denominators gives $2 - (x+1) = x - 1$ which means that $x = 1$. So Megan is correct that if we clear denominators and set the two expressions equal to one another we will find $x = 1$. She is also correct that we cannot evaluate the left hand side of the equatio at $x = 1$ since we obtain a zero in the denominators of $\frac{2}{x^2-1}$ and $\frac{1}{x-1}$. Curiously, however, we can evaluate the right hand side of the equation when $x = 1$ and it gives a value of $\frac{1}{2}$.

b. Before she clears denominators, Megan has the equation $\frac{2-(x+1)}{x^2-1} = \frac{x-1}{x^2-1}$ which is the same as

$$\frac{1-x}{x^2 - 1} = \frac{x-1}{x^2 - 1}.$$

In this equation, both numerators and denominators are zero when $x = 1$. So if we remov the denominators then it appears as if $x = 1$ is a solution but it is not because the expressions are not defined when $x = 1$. Clearing denominators is a good technique to fin when two expressions are equal but we have to remember to check that the solutions found are legitimate, that is that the denominator does not give zero when evaluated at these solutions.

Below is an interactive graph of the functions f and g and we can visually see that they do not intersect. The function $g(x) = \frac{1}{x+1}$ is defined and well behaved at and near $x = 1$ but the function $f(x) = \frac{2}{x^2-1} - \frac{1}{x-1}$ is not defined at $x = 1$ and is also not well behaved near the value $x = 1$: