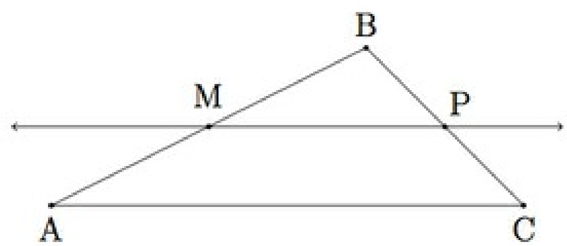
G-CO Midpoints of Triangle Sides

Alignments to Content Standards: G-CO.C.10

Task

Suppose ABC is a triangle. Let M be the midpoint of $\overline{AB}$ and P the midpoint of $\overline{BC}$ as pictured below:



a. Show that $\overleftrightarrow{MP}$ and $\overleftrightarrow{AC}$ are parallel.

b. Show that $|AC| = 2|MP|$.

IM Commentary

The goal of this task is to use similarity transformations to relate two triangles. The triangles in question are obtained by taking midpoints of two sides of a given triangle. In the picture above, $\triangle BAC$ can be seen as a scaled version of $\triangle BMP$, with scale factor 2 and center of dilation B. Equivalently, $\triangle BMP$ is the scaled version of $\triangle BAC$

with scale factor $\frac{1}{2}$ and center of dilation B. The task uses the important fact that a dilation maps a line ℓ not containing the center of dilation to a line parallel to ℓ: this is verified experimentally in G-SRT.1a and is an axiom in the transformational approach to geometry. Also important in this task is G-SRT.1b which allows us to conclude that the dilation with scale factor 2 and center of dilation B doubles the length of $\overline{MP}$.

In many traditional approaches to high school geometry, the result of this task is proven with the SAS similarity theorem: if two triangles share an angle and if the sides making the angle are proportional then the two triangles are similar. This is Book VI, Proposition 6 of Euclid and the proof is very technical, relying on Propositions 1, 2, and 4 of Book VI. The transformational approach to geometry replaces the SAS similarity theorem with properties of dilations: they preserve angle measures, scale all line segment lengths by the same (non-zero) factor, and take lines not passing through the center of dilation to parallel lines.

Edit this solution

Solution

a. Suppose D is the dilation with scale factor 2 and center B. Since M is the midpoint of $\overline{AB}$, we know that $D(M) = A$. Since P is the midpoint of $\overline{BC}$ we know that $D(P) = C$. Since dilations map lines to lines, we know that $D\left(\overleftrightarrow{MP}\right) = \overleftrightarrow{AC}$. Dilations take lines not containing the center of dilation to parallel lines so we can conclude that $\overleftrightarrow{MP}$ is parallel to $\overleftrightarrow{AC}$.

b. Since $D\left(\overline{MP}\right) = \overline{AC}$ and D is a dilation with scale factor 2, this shows that $|AC| = 2|MP|$.