Illustrative Mathematics

N-RN Evaluating Exponential Expressions

Alignments to Content Standards: N-RN.A.1

Task

Henry explains why $4^{3/2} = 8$:

I know that 4^3 is 64 and the square root of 64 is 8.

Here is Henrietta's explanation for why $4^{3/2} = 8$:

I know that $\sqrt{4} = 2$ and the cube of 2 is 8.

a. Are Henry and Henrietta correct? Explain.

b. Calculate $4^{5/2}$ and $27^{2/3}$ using Henry's or Henrietta's strategy.

c. Use both Henry and Henrietta's reasoning to express $x^{m/n}$ using radicals (here m and n are positive integers and we assume $x > 0$).

IM Commentary

The goal of this task is to use properties of exponents for whole numbers (in particular the fact that $(a^m)^n = a^{mn}$) in order to explain how expressions with fractional exponents are defined. There are some situations where fractional exponents result in whole number answers: for example, $4^{1/2} = \sqrt{4} = 2$. Parts (a) and (b) begin here as this should be a familiar situation for students, and part (c) then asks them to develop the more general situation where the exponential expression is not equal to a whole number.

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Solution

a. Henry's reasoning is correct. We can show this with equations:

$$\begin{aligned} 4^{3/2} &= (4^3)^{1/2} \\ &= \sqrt{4^3} \\ &= \sqrt{64} \\ &= 8. \end{aligned}$$

The first equation is true because $\left(4^{3/2}\right)^2 = 4^{2\times 3/2} = 4^3$ and $\sqrt{4^3}$ is the only positive number whose square is 4^3. We can also reverse the series of equalities, starting with the right hand side and manipulating to obtain the left hand side:

$$\begin{aligned} 8 &= \sqrt{64} \\ &= \sqrt{4^3} \\ &= 4^{3/2} \end{aligned}$$

with the last equality coming from the fact that $4^{3/2}$ and $\sqrt{4^3}$ are both the only positive number whose square is 64. In all of the series of equalities that follow in the solution, students can similarly reverse the order.

Henrietta's reasoning is also correct. We can show this with equations:

$$\begin{aligned} 4^{3/2} &= (4^{1/2})^3 \\ &= (\sqrt{4})^3 \\ &= 2^3 \\ &= 8. \end{aligned}$$

The first equation is again true because because both sides are positive with the same square.

b. Henry's reasoning can be adapted to find $4^{5/2}$ as follows:

$$\begin{aligned} 4^{5/2} &= \sqrt{4^5} \\ &= \sqrt{1024} \\ &= 32. \end{aligned}$$

Alternatively, knowing that $4^{3/2} = 8$, we could reason that

$$\begin{aligned} 4^{5/2} &= 4^{3/2} \times 4 \\ &= 8 \times 4 \\ &= 32. \end{aligned}$$

Using Henrietta's reasoning to find $4^{5/2}$, we have

$$\begin{aligned} 4^{5/2} &= (\sqrt{4})^5 \\ &= 2^5 \\ &= 32. \end{aligned}$$

c. Applied to $x^{m/n}$, Henry's idea is to first find x^m and then extract the n^{th} root:

$$x^{m/n} = \sqrt[n]{x^m}.$$

The equation is true because the n^{th} power of both sides is equal to x^m: since both sides are positive when n is even, this means that both sides of the equation are equal.

Henrietta's idea is to first take the n^{th} root of x and then raise this to the m^{th} power:

$$x^{m/n} = \left(\sqrt[n]{x}\right)^m.$$

The equation is also true because the n^{th} power of both sides is equal to x^m.