G-CO Defining Reflections

Alignments to Content Standards: G-CO.A.4

Task

Carlos finds the following definition of a reflection in a math book:

The reflection r_ℓ about a line ℓ takes each point P on ℓ to itself and takes each point Q not on ℓ to the point $r_\ell(Q)$ such that ℓ is the perpendicular bisector of $\overline{Qr_\ell(Q)}$.

Carlos does not find this definition very helpful. He says "the reflection about a line ℓ sends each point to its mirror image on the other side of ℓ."

a. In what ways is Carlos' definition of reflection more helpful than one from the math book?

b. In what ways is the math book definition of reflection more helpful than Carlos' definition?

IM Commentary

The goal of this task is to compare and contrast the visual intuition we have of reflections with their technical mathematical definition. As is often the case, giving a precise mathematical definition of a concept is an extremely difficult process which takes hundreds (or sometimes thousands) of years. The transformations of the plane grow from practical experience: for reflections, we often model them by flipping a piece of paper (which is modeling the plane). In addition, there is another very important physical representation of reflections, namely a mirror. Mirrors also show us in a very concrete way one of the vital properties of reflections: when we look in a mirror and move our right arm, the mirrored image moves its left arm. So although the mirror image ''looks identical'' it is different in an important way as it has a different orientation (a technical mathematical term for saying that the notions of left and right are reversed).

This task requires time and patience and is ideally suited for in class group work. If there are mirrors present in the classroom the teacher may wish to have students experiment so that they can see first-hand how the mirror image is similar and how it differs from the original. They should also test their intuition for the mirror image of figures by folding (relatively transparent) paper.

Having this concrete visual and spatial model for reflections is vital as it helps guide and build our intuition. On the other hand, we can ask many mathematical questions about reflections such as:

- Does a reflection change the distance between two points or does this distance stay the same before and after the reflection?
- Is the image of a triangle ABC under reflection congruent to ABC?
- Given a pair of points P and Q, how many lines ℓ can I find so that reflection about ℓ sends P to Q?

Carlos' physical model with the mirror may give some intuition for the answers to these questions but to address them properly we need a clear mathematical definition of reflections.

Like its partner task www.illustrativemathematics.org/tasks/1509 work on this task exemplifies MP6, ''Attend to Precision.'' Indeed the subtitle of this mathematical practice is ''Communicate Precisely'' which requires using language carefully and, in particular, defining concepts crisply and precisely. The main mathematical flaw in Carlos' intuitive notion of reflection is its lack of precision.

Edit this solution

Solution

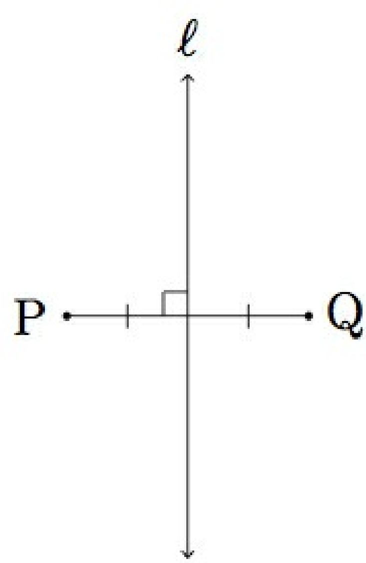
a. If we model the plane with a piece of paper (imagining that it continues on indefinitely in all directions) modeling translations and rotations is straightforward: for a translation we move the paper (or plane) a fixed distance in some direction and for a rotation we fix a point (the center of the rotation) and then rotate the paper about that point through a fixed angle. Reflections can also be modeled nicely with the piece of paper as we can choose the line of reflection and then flip the piece of paper over while leaving this line fixed. The problem with this model is that if we have points labelled and shapes drawn in the plane it is difficult to ''see'' what happens to them because they are now invisible, facing the table.

Carlos' intuition for a reflection helps circumvent this problem and more effectively visualize the impact of a reflection. Moreover, the trick of flipping over the piece of paper only works for the plane: if we are asked to think about reflections of 3 dimensional Euclidean space then this is no longer possible while the mirror is actually a 3 dimensional model. So the first advantage to Carlos' notion of a reflection is that it links the geometric notion of reflection to common every day experience.

Next, by looking at what happens in a mirror, we can visualize one of the most important properties of reflections, namely that they switch notions of left and right, clockwise and counterclockwise. If we look in a mirror and raise our right arm, our ''reflection'' is raising its left arm. Similarly, if we observe the reflection of a clock in the mirror, two things are apparent: first, the numbers on the clock are difficult to read because they are reflected but secondly and more importantly, the reflected clock is moving in the opposite direction!

The mathematical definition does not provide this intuition or link with experience. It is abstract and is only arrived at after doing concrete work with reflections.

b. Below is a picture of how reflection about ℓ works according to the mathematical definition, in the case where P does not lie on ℓ:



In this case, the reflection of P about ℓ is the point Q because ℓ is the perpendicular bisector of $\overline{PQ}$.

One advantage to the mathematical definition of reflection about ℓ is that it brings out right away the fact that reflections separate the plane into two pieces: the points on ℓ which remain in place after the reflection and the two halves of the plane which are swapped by the reflection. It is for this reason that the definition offers two separate scenarios which are analyzed separately. The same issue arises in the mirror image model but is not addressed since the model only gives intuition for what happens to the part of the plane in front of the mirror. A similar structure holds also in three dimensional space but here it requires a plane to separate the space into two ''half spaces'' and accordingly reflections in three space are about planes.

Next, the mathematical definition makes it clear that, for P not on ℓ, if $r_\ell(P) = Q$ then $r_\ell(Q) = P$: in other words the reflection interchanges pairs of points. We can see this because if ℓ is the perpendicular bisector of $\overline{PQ}$ then it is also the perpendicular bisector of $\overline{QP}$. This is not possible to see in the physical model because we would have to place a second mirror in front of the first mirror and, although an interesting experiment, this is awkward to do and also very disorienting if studied closely (because it leads to an infinite regression of reflected images).

Finally and perhaps most importantly, the mathematical definition allows us to study the properties of reflections in more depth. Do reflections preserve distance? What happens if I compose two different reflections? How does moving the line ℓ impact the result of the reflection? These questions can be and should be studied with physical models such as Carlos' but in order to make clear arguments and establish results, a precise mathematical definition is required.