A-CED Products and Reciprocals

Alignments to Content Standards: A-REI.C A-REI.A.1

Task

The product of two positive numbers is 9. The reciprocal of one of these numbers is 4 times the reciprocal of the other number. What is the sum of the two numbers?

IM Commentary

The purpose of this task is to test student skill at converting verbal statements into two algebraic equations and then solving those equations. The student should assign variables to each number, form the system of the two equations, and solve the system. The final step is to add the values found for the two assigned variables. This problem is different from many that students have seen since the two equations are not linear. One way to solve such a system is by substitution: this is presented in the first solution though students might choose a different way to perform the substitution since either of the given equations can be solved for x in terms of y or for y in terms of x. A second solution is presented where the two equations are manipulated in a different way. The teacher should make sure that students understand the ambiguity present in the task stem: it reads ''The reciprocal of one of these numbers is 4 times the reciprocal of the other number'' and this means that there are two cases to consider. One way of avoiding this ambiguity would be to *choose* x, for example, to denote the large of the two numbers and then we would know that $\frac{1}{x} = \frac{4}{y}$.

This task was adapted from problem #6 on the 2012 American Mathematics Competition (AMC) 10A Test. For the 2012 AMC 10A, which was taken by73703 students, the multiple choice answers for the problem had the following distribution:

Choice	Answer	Percentage of Answers
(A)	$\frac{10}{3}$	4.15
(B)	$\frac{20}{3}$	5.40
(C)	7	2.96
(D)*	$\frac{15}{2}$	49.03
(E)	8	1.97
Omit	--	36.46

Of the 73703 students: 36206, or 49%, were in 10th grade;25498 or 35%, were in 9th grade; and the remainder were below than 9th grade.

Solutions

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Solution: 1 Substitution

Let x denote the first number and y the second number. The product of the two numbers is 9 so this means $xy = 9$. The reciprocal of one of the numbers is four times the other so this means that either $\frac{1}{x} = \frac{4}{y}$ or $\frac{1}{y} = \frac{4}{x}$. We consider first the case where $\frac{1}{x} = \frac{4}{y}$. Multiplying both sides by xy gives $y = 4x$ and we can substitute this into our equation $xy = 9$ to get

$$x(4x) = 9.$$

Solving this gives $x = \pm\sqrt{\frac{9}{4}}$ so $x = \pm\frac{3}{2}$. Since x is positive, we must have $x = \frac{3}{2}$. Substituting this value into the equation $4x = y$ gives $y = 6$. So $x + y = 6 + \frac{3}{2} = \frac{15}{2}$.

We also need to consider the case where $\frac{1}{y} = \frac{4}{x}$. We could repeat the argument of the previous paragraph word for word, with y and x interchanged, or we can observe that

these two sets of equations are identical except that x and y have been interchanged. Interchanging x and y does not impact the second equation $xy = 9$. So the solution to $\frac{1}{y} = \frac{4}{x}$ and $xy = 9$ will be $y = \frac{3}{2}$ and $x = 6$ still having a sum of $\frac{15}{2}$.

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Solution: 2 Manipulating equations

Let x denote the first number and y the second number. The product of the two numbers is 9 so this means $xy = 9$. The reciprocal of one of the numbers is four times the other so this means that either $\frac{1}{x} = \frac{4}{y}$ or $\frac{1}{y} = \frac{4}{x}$. We will assume first that $\frac{1}{x} = \frac{4}{y}$. If we divide both sides of $xy = 9$ by x this gives $y = \frac{9}{x}$. Note that dividing by x is legitimate because x is a non-zero number. We also have $\frac{9}{x} = \frac{36}{y}$. Setting our two values of $\frac{9}{x}$ equal gives

$$y = \frac{36}{y}.$$

We can multiply both sides by y to find $y^2 = 36$ and so $y = 6$ since y is a positive number. Substituting $y = 6$ back into either of our equations gives $x = \frac{3}{2}$ and hence $x + y = 6 + \frac{3}{2} = \frac{15}{2}$.

If it is $\frac{1}{y}$ that is four times $\frac{1}{x}$ the reasoning in the above paragraph can be repeated with x and y interchanged the their sum will still be $\frac{15}{2}$.