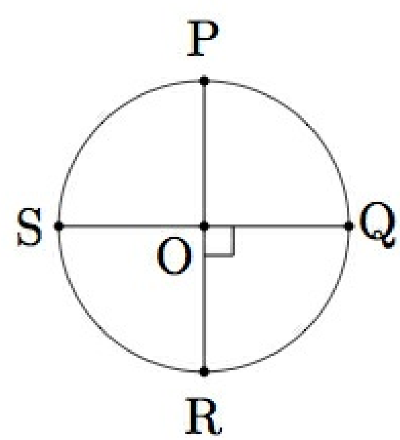
G-CO Inscribing a square in a circle

Alignments to Content Standards: G-CO.D.13

Task

Let C be a circle with center O. Suppose $\overline{PR}$ and $\overline{QS}$ are two diameters of C which are perpendicular to one another at O as pictured below:



a. Explain why triangles POQ, QOR, ROS, and SOP are congruent.

b. Using part (a), deduce that quadrilateral $PQRS$ is a square.

c. What is the area of $PQRS$? Roughly what percent of the area of C is the area of $PQRS$?

IM Commentary

Part (a) of this task could be used for assessment or for instructional purposes. Parts (b) and (c) are mostly for instructional purposes but could be used for assessment. This task provides an opportunity for students to apply triangle congruence theorems in an explicit, interesting context. The problem could be made substantially more demanding by not providing the initial picture and simply prompting students to construct an inscribed square inside a given circle. If used for instructional purposes, the teacher may want to spend some time showing how to construct the two perpendicular diamters of the circle with a straightedge and compass (currently there is no task for this but there will be eventually).

Part 3 of this task is related to a classical computation made by Greek geometers, attempting to estimate the area of a circle by using inscribed polygons with more and more sides. This will be continued in the task ''Inscribing a hexagon in a circle.''

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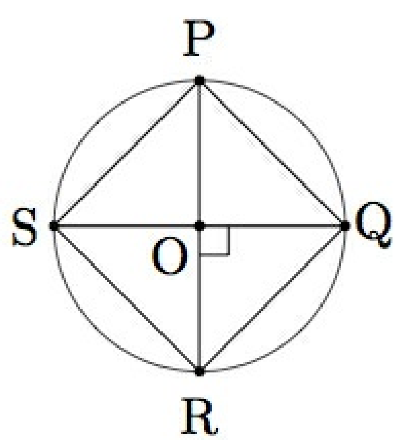
Solution

We draw segments PQ, QR, RS, and S which, collectively, form a square inscribed in the circle as will be shown below:



a. Since lines PR and SQ meet perpendicularly we know that all four angles POQ, QOR,ROS, and SOP are right angles. Furthermore, the four segments OP, OQ, OR,and OS are all congruent because they are radii of the same circle. Thus repeatedly using the SAS criterion for triangle congruence, triangles POQ, QOR, ROS, and SOP are all congruent.

b. The four triangles POQ, QOR,ROS, and SOP are all right isosceles triangles because, as noted above, the two legs of these right triangles are both radii of the same circle. We conclude that all of the base angles in these triangles (OPQ,OQP, OQR, ORQ, ORS, OSR, OSP, OPS) are 45 degree angles. It follows that angles PQR, QRS, RSP, and SPQ are all right angles. Moreover, the triangle congruence shown in part (a) also establishes the congruence of sides PQ, QR, RS, and SP. It follows that quadrilateral $PQRS$ is a square.

c. Let r denote the radius of the circle so its area is πr^2. To find the area of square $PQRS$ we need to calculate the length of one of its sides, say PQ. Using the Pythagorean theorem we find

$$\begin{aligned} |PQ|^2 &= |OP|^2 + |OQ|^2 \\ &= r^2 + r^2 \\ &= 2r^2. \end{aligned}$$

Thus $|PQ| = \sqrt{2}r$. So the area of square $PQRS$ is $2r^2$ versus πr^2, the area of the circle. The square contains slightly more than 63 percent of the area of its circumscribing circle.