Illustrative Mathematics

N-CN Complex number patterns

Alignments to Content Standards: N-CN.A N-CN.A.1

Task

For this task, the letter i denotes the imaginary unit, that is, $i = \sqrt{-1}$.

a. For each integer k from 0 to 8, write i^k in the form $a + bi$.

b. Describe the pattern you observe, and algebraically prove your observation. In particular, simplify i^{195}.

c. Write each of the following expression in the form $a + bi$:

- $i^2 + i + 1$
- $i^3 + i^2 + i + 1$
- $i^4 + i^3 + i^2 + i + 1$
- $i^5 + i^4 + i^3 + i^2 + i + 1$
- $i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$
- $i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$
- $i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$

d. Describe the pattern you observe, and algebraically prove your observation. In particular, compute

$$i^{195} + i^{194} + \cdots + i^2 + i + 1.$$

IM Commentary

This task serves as a possible first student exploration after an initial introduction to the form and arithmetic of complex multiplication. Students need to understand that every complex number can be expressed in the form $a + bi$, and understand multiplication of complex numbers at least well enough to compute, for example, $i^3 = i^2 \cdot i = -1 \cdot i = -i$ (and understand that this is in the form $a + bi$). The task also (optionally) provides an instance where the formula of a finite geometric series can be used to explain an experimentally observed result.

The task is an excellent model of the Standard for Mathematical Practice MP8 (Look for and express regularity in repeated reasoning), as students are led to make conjectures about patterns based on experimental calculations. Students can then practice their algebraic manipulation in justifying their observation, a potentially very satisfying academic endeavor.

Edit this solution

Solution

Note that writing the numbers ± 1 and $\pm i$ in the form $a + bi$ is deceptively simple as, for example, $i = 0 + 1i$ (that is, we take $a = 0$ and $b = 1$).

a. Below is a table with the requested powers of i. Note that these can be computed by systematically multiplying by increasing powers of i, e.g.,

$$i^3 = i^2 \cdot i = -1 \cdot i = -i,$$

which can be written in the form $-i = 0 + (-1)i$, and then

$$i^4 = i^3 \cdot i = (-i) \cdot i = -i^2 = -(-1) = 1,$$

which similarly can be written as $1 = 1 + 0i$.

i^0	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
1	i	-1	$-i$	1	i	-1	$-i$	1

b. We observe that the pattern of powers of i is cyclical, repeating every 4 exponents. When the exponent is an integer multiple of 4, the result is a 1. Exponents which are one more than a multiple of 4 give a result of i, and so on. To make this precise, we simply observe that any integer can be written as a multiple of 4, plus either 0, 1, 2, or 3. We can justify this pattern as follows: To compute i^n, we write $n = 4k + a$ (where a is 0, 1, 2, or 3), and then observe

$$i^n = i^{4k+a} = (i^4)^k \times i^a = a^k \times i^a = i^a.$$

That is, we can compute i^n by computing i^a, where a is the remainder upon dividing n by 4.

Since 195 is three more than the multiple $192 = 4 \cdot 48$, we have $i^{195} = i^3 = -i$.

c. Here are the algebraic solutions:

- $i^2 + i + 1 = -1 + i + 1 = i$
- $i^3 + i^2 + i + 1 = -i + -1 + i + 1 = 0$
- $i^4 + i^3 + i^2 + i + 1 = 1 + -i + -1 + i + 1 = 1$
- $i^5 + i^4 + i^3 + i^2 + i + 1 = i + 1 + -i + -1 + i + 1 = 1 + i$
- $i^6 + i^5 + i^4 + i^3 + i^2 + i + 1 = -1 + i + 1 + -i + -1 + i + 1 = i$
- $i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1 = -i + -1 + i + 1 + -i + -1 + i + 1 = 0$
- $i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1 = 1 + -i + -1 + i + 1 + -i + -1 + i + 1 = 1$

d. Note that the number of terms is 1 more than the largest exponent in the sum. Using the pattern for i^n observed in the previous parts, we see that the sum of the first 196 powers of i (starting with i^0) is simply

$$\underbrace{\underbrace{1 + -i + -1 + i}_{=0} + \underbrace{1 + -i + -1 + i}_{=0} + \cdots + \underbrace{1 + -i + -1 + i}_{=0}}_{\text{196 total terms, divided into 48 smaller groups that each sum to 0}} = 0.$$

Though not required by the task, this statement generalizes nicely -- whenever the number of terms is a multiple of 4, the corresponding sum will be 0. Whenever the number of terms is one more than a multiple of 4, then there will be one term left over when we are done collecting these groups of 4: The sum

$$\underbrace{1 + -i + -1 + i}_{=0} + \underbrace{1 + -i + -1 + i}_{=0} + \cdots + \underbrace{1 + -i + -1 + i}_{=0} + 1 = 1$$

evaluates to 1 as there will be only a single "1" which does not disappear by virtue of being in a group of 4. Similarly, if the number of terms is two more than a multiple of 4, the result is $1 + i$, and if the number of terms is three more than a multiple of 4, the result is $1 + i + -1 = i$.

Alternatively, we could verify that the formula for a finite geometric series is valid even when the base of the series is i. (This is done identically to the case for real numbers). Applying this formula gives

$$1 + i + i^2 + \cdots + i^n = \frac{1 - i^{n+1}}{1 - i}$$

and then apply our knowledge from above about the powers of i. For example, when n is three more than a multiple of 4, then $n + 1$ is a multiple of 4, so $i^{n+1} = 1$, and we concldue

$$1 + i + i^2 + \cdots + i^n = \frac{1 - i^{n+1}}{1 - i} = \frac{1 - 1}{1 - i} = 0,$$

the same result as above. This applies in particular to the example of $n = 195$ requested in the problem.