

F-LE Do two points always determine a linear function?

Alignments to Content Standards: F-LE.A.2 F-IF.A.1

Task

- a. Suppose $P_1 = (0, 5)$ and $P_2 = (3, -3)$. Sketch P_1 and P_2 . For which real numbers m and b does the graph of a linear function described by the equation $f(x) = mx + b$ contain P_1 ? Explain. Do any of these graphs also contain P_2 ? Explain.
- b. Suppose $P_1 = (0, 5)$ and $P_2 = (0, 7)$. Sketch P_1 and P_2 . Are there real numbers m and b for which the graph of a linear function described by the equation $f(x) = mx + b$ contains P_1 and P_2 ? Explain.
- c. Now suppose $P_1 = (c, d)$ and $P_2 = (g, h)$ and c is not equal to g . Show that there is only one real number m and only one real number b for which the graph of $f(x) = mx + b$ contains the points P_1 and P_2 .

IM Commentary

In a geometric context, two distinct points P_1 and P_2 always determine a unique line in the Cartesian plane (this is one of the basic postulates of Euclidean geometry). Only the non-vertical lines, however, can be described by the graph of a function $f(x) = mx + b$. This task focuses on producing an explicit function $f(x)$ in the case where the line is not vertical.

This problem allows the student to think geometrically about lines and then relate this geometry to linear functions. Or the student can work algebraically with equations in

order to find the explicit equation of the line through two points (when that line is not vertical).

Solutions

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Solution: 1

a. The graph of the linear function described by $f(x) = mx + b$ contains $(0, 5)$ when the y -intercept of the graph, given by b , is equal to 5. So the linear functions whose graph contains $(0, 5)$ are those of the form

$$f(x) = mx + 5$$

where m is a real number.

In order for the graph of f to contain $(3, -3)$ we need

$$-3 = f(3) = 3m + 5.$$

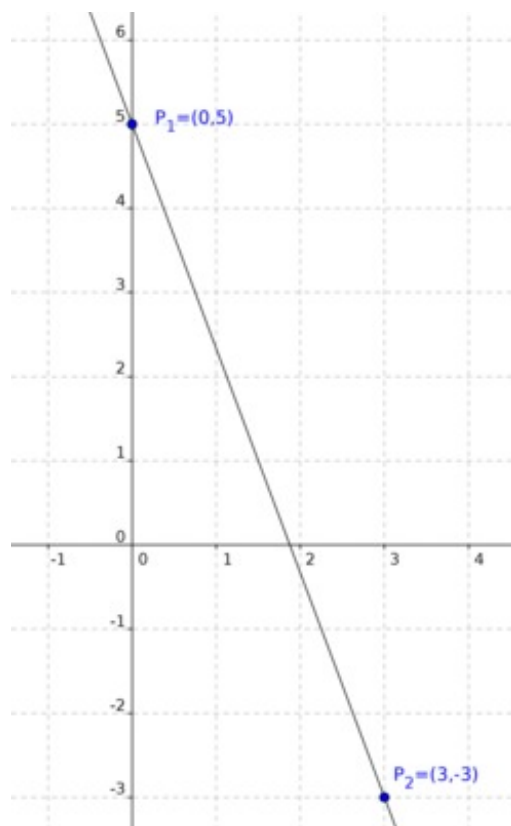
Solving for m gives

$$m = -\frac{8}{3}.$$

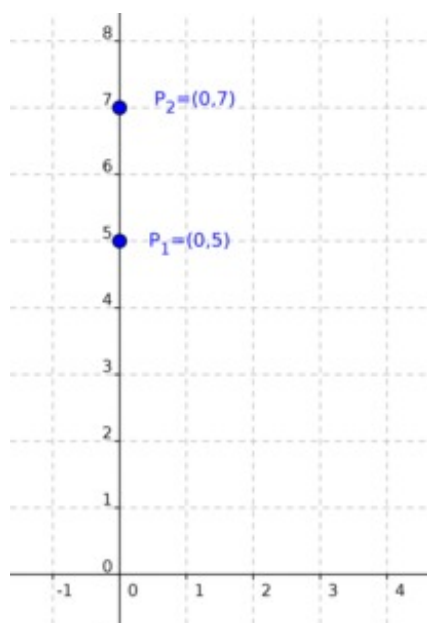
So there is only one linear function whose graph passes through $(0, 5)$ which also passes through $(3, -3)$, namely the one described by

$$f(x) = -\frac{8}{3}x + 5,$$

depicted below.



b. If f is any function, then its graph consists of the pairs $(x, f(x))$. So for each x -value there can only be one associated $f(x)$ -value which is on the graph of f . In this case $(0, 5)$ and $(0, 7)$ both have x -value zero and so these two points cannot be on the graph of any function, whether or not the function is linear.



c. Asking for the point $P_1 = (c, d)$ to be on the graph of $f(x) = mx + b$ means that

$$d = mc + b.$$

Similarly asking for the point $P_2 = (g, h)$ to be on the graph of $f(x) = mx + b$ means that

$$h = mg + b.$$

Subtracting the second equation from the first gives

$$d - h = m(c - g)$$

so

$$m = \left(\frac{d - h}{c - g} \right).$$

Note that the expression for the slope m only makes sense if $c \neq g$, which has been assumed in the problem. Once we have found m we can plug this into one of the equations (either for d or for h) and then solve for b :

$$d = \left(\frac{d - h}{c - g} \right) c + b.$$

Solving for b gives

$$b = d - \left(\frac{d - h}{c - g} \right) c.$$

So with the hypothesis that $c \neq g$ there is a linear function whose graph contains P_1 and P_2 namely

$$f(x) = \left(\frac{d - h}{c - g} \right) x + d - \left(\frac{d - h}{c - g} \right) c.$$

Finally, we note that we have also that there is a *unique* such line, as any line containing P_1 and P_2 must have the slope and y-intercept as calculated above. Explicitly: If there were a different linear function $g(x) = m_1x + b_1$ whose graph contained P_1 and P_2 , then by virtue of it containing P_1 we must have

$$d = m_1c + b_1.$$

Similarly, since the graph of g contains P_2 , we must have

$$h = m_1 g + b_1.$$

These two allow us to solve for m_1 , as above, and we find the same solution, namely

$$m_1 = \left(\frac{d - h}{c - g} \right).$$

We then find, also as above, that $b_1 = b$ and so $g(x) = f(x)$: there is only one linear equation of the form $f(x) = mx + b$ whose graph contains P_1 and P_2 .

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Solution: 2 Thinking Geometrically

From a geometric point of view, we can see the answer to part (c) as follows. Given the two points P_1 and P_2 we can draw the unique line containing these two points. If $f_1(x) = m_1x + b_1$ and $f_2(x) = m_2x + b_2$ were two equations whose graphs describe this line, this means that

$$m_1x + b_1 = m_2x + b_2$$

for all values of x . This is only possible if $m_1 = m_2$ and $b_1 = b_2$ as we can see, for example, by rewriting this equation as

$$(m_1 - m_2)x = b_2 - b_1.$$

The right hand side of this equation does not depend on x so this forces $m_1 - m_2 = 0$ and once $m_1 = m_2$ then the left hand side is zero, forcing $b_1 = b_2$.



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