F-LE Exponential growth versus polynomial growth

Alignments to Content Standards: F-LE.A.3

Task

The table below shows the values of 2^x and $2x^3 + 1$ for some whole number values of x:

x	2^x	$2x^3+1$
1	2	3
2	4	17
3	8	55
4	16	129
5	32	251

a. The numbers in the third column (values of $2x^3 + 1$) are all larger than the numbers in the second column (values of 2^x). Does this remain true if the table is extended to include whole number values up to ten?

b. Explain how you know that the values of 2^x will eventually exceed those of the polynomial $2x^3 + 1$. What is the smallest whole number value of x for which this happens?

IM Commentary

This problem shows that an exponential function takes larger values than a cubic polynomial function provided the input is sufficiently large.

Solutions

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Solution: Table

(a) The table can be extended for whole number values of x up to $x = 10$ and the values of $2x^3 + 1$ remain larger than those for 2^x:

x	2^x	$2x^3 + 1$
6	64	433
7	128	687
8	256	1025
9	512	1459
10	1024	2001

(b) If the table is continued, for all values of x up to and including 11 the polynomial $2x^3 + 1$ takes a larger value than the exponential 2^x. But

$$2^{12} > 2(12)^3 + 1.$$

x	2^x	$2x^3 + 1$
11	2048	2663
12	4096	3457

We know that the exponential 2^x will eventually exceed in value the polynomial

$2x^3+1$ because its base, 2, is larger than one and an exponential functions grow faster, as the size of x increases, than any particular polynomial function. This is explained in greater detail in the second solution below by examining quotients of 2^x and $2x^3+1$ when evaluated at successive whole numbers.

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Solution: 2. Abstract argument

The argument presented here does not find the smallest whole number (12) where the value of 2^x first exceeds the value of $2x^3+1$ but rather explains why there must be such a whole number. The argument would apply not only to $2x^3+1$ but also to any other polynomial.
Each time the variable x is increased by one unit, the exponential function 2^x doubles:

$$\frac{2^{x+1}}{2^x}=2.$$

For the polynomial function $2x^3+1$, an increase in x by one unit increases the value of the function by a factor of

$$\frac{2(x+1)^3+1}{2x^3+1}=\frac{2x^3+6x^2+6x+7}{2x^3+1}.$$

Unlike the exponential function, these growth factors for the polynomial function depend on the value of x. Notice that as x increases, the expression

$$\frac{2x^3+6x^2+6x+7}{2x^3+1}$$

gets closer and closer to one (because for large positive values of x, the terms $6x^2, 6x, 7,$ and 1 influence the value of the quotient by a small quantity). Thus, as x is continually incremented by one unit, the value of 2^x always doubles while value of $2x^3+1$ only increases by a factor closer and closer to one, thereby allowing the exponential values to eventually surpass the polynomial values.