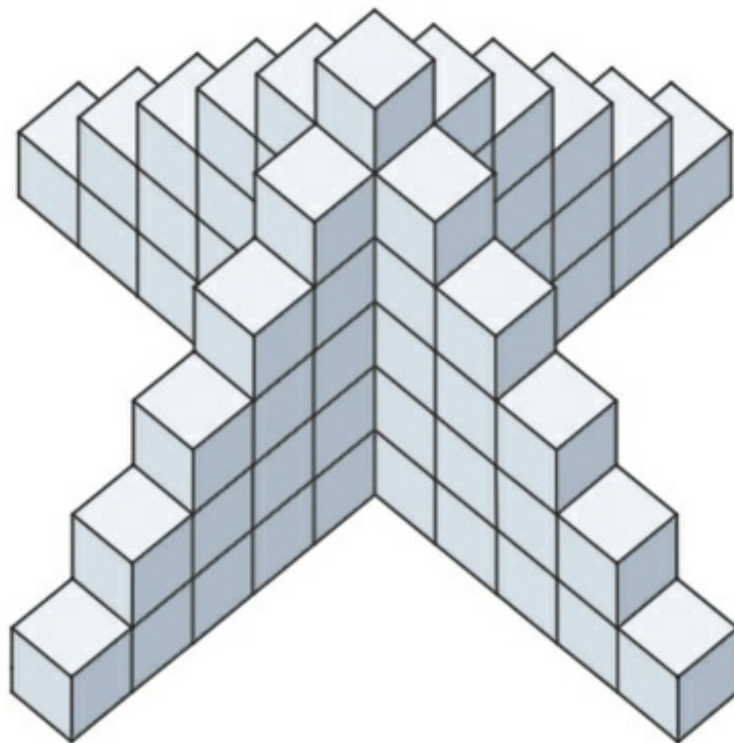


F-BF Skeleton Tower

Alignments to Content Standards: F-BF.A.1

Task



- How many cubes are needed to build this tower?
- How many cubes are needed to build a tower like this, but 12 cubes high? Justify your reasoning.

c. How would you calculate the number of cubes needed for a tower n cubes high?

IM Commentary

This problem is a quadratic function example. The other tasks in this set illustrate F.BF.1a in the context of linear (Kimi and Jordan), exponential (Rumors), and rational (Summer Intern) functions.

This is based on a task developed by the MARS/ Shell Centre team [Mathematics Assessment Resource Service](#) (adapted with permission for redistribution with the Creative Commons License in this form).

Solutions

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Solution: Arithmetic Sequence, Variation 1

In this solution we count the cubes in rows (or layers):

a. The top layer has a single cube. The layer below has one cube beneath the top cube plus 4 new ones making a total of 5. The third layer below has cubes below these 5 plus 4 new ones to make 9. Continuing to add four each time gives a total of

$$1 + 5 + 9 + 13 + 17 + 21 = 66$$

cubes in the skeleton tower with six layers.

b. Building upon the reasoning established in a), the number of cubes in the bottom (12th) layer will be $1 + 4 \times 11$ since it is 11 layers below the top. So for this total, we need to add

$$1 + 5 + 9 + \dots + 45.$$

One way to do this would be to simply add the numbers. A quicker method which will work for any number of layers is the so-called Gauss method (which the great mathematician reportedly used to sum a large list of whole numbers when he was very young). The idea is to rewrite the sum backward as

$$45 + 41 + 37 + \dots + 1.$$

Now if this list is placed below the previous list we can see that each column sums to 46. There are 12 columns so the answer to this problem is half of $12 \times 46 = 552$ or 276.

c. Let $f(n)$ be the number of cubes in the n^{th} layer counting down from the top. Then $f(1) = 1$ is the number of cubes in the top layer, $f(2) = 5$ the number of cubes in the second layer from the top and so on. In this case, the values of the function $f(1), f(2), f(3), \dots$ form an arithmetic sequence, where each term is obtained from the previous one by adding 4. In general, we have $f(n) = 4(n - 1) + 1$. The method used to solve problem b) provides a method both to solve the skeleton tower problem and to add a general arithmetic sequence. For the skeleton tower problem, with n layers in the tower, the sum giving the total number of cubes will be

$$1 + 5 + 9 + \dots + f(n) = 1 + 5 + 9 + \dots + (4(n - 1) + 1).$$

If we use the method of problem b) here, twice this sum will be equal to $n(4(n - 1) + 2)$ and so the general solution for the number of cubes in a skeleton tower with n layers is

$$\frac{n(4(n - 1) + 2)}{2} = n(2n - 1).$$

[Edit this solution](#)

Solution: Arithmetic Sequence, Variation 2

In this solution we count the cubes in columns.

a. The top layer has a single cube and there are five layers of cubes below this one making 6 total cubes in the column with the top one. The next layer introduces 4 new cubes and there are four layers below so this gives a total of 20 cubes in these 4 columns. Similarly, each successive layer introduces 4 new columns, so we calculate the total to be

$$6 + 4 \times 5 + 4 \times 4 + 4 \times 3 + 4 \times 2 + 4 \times 1 = 66.$$

b. Imitating the method used in the first problem, we find that the column under the cube on top has 12 cubes. Then there are four columns which are 11 cubes high, four columns which are 10 cubes high, and so on down to four columns which are one cube high. Adding this all up gives

$$12 + 44 + 40 + \dots + 4.$$

We can simplify this to

$$12 + 4(11 + 10 + \dots + 1).$$

The sum of the first eleven positive integers is found to be 66 (either by hand or by the method indicated in problem c) below). Thus there are $12 + 4 \times 66 = 276$ cubes in this tower.

c. Using the reasoning developed in a) and b), the general skeleton tower problem with n layers has a total of

$$n + 4((n - 1) + (n - 2) + \dots + 1)$$

cubes. In order to evaluate the sum $((n - 1) + (n - 2) + \dots + 1)$ we list the same sum backward, that is $(1 + 2 + \dots + (n - 1))$, beneath it. When we add the columns created by the two sums, each adds to n . Since there are $n - 1$ columns, the total of the two sums is $n(n - 1)$. Thus, the sum $((n - 1) + (n - 2) + \dots + 1)$ will be half this or $\frac{n(n-1)}{2}$. Thus the total number of cubes in the skeleton tower with n layers is

$$n + 4 \times \frac{n(n - 1)}{2} = 2n^2 - n.$$



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