A GRAPHICAL LOOK AT THE AIRY INTEGRAL

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[Received 21 January 1980. Revise 15 October 1980]

SUMMARY

As a function of the upper limit s, $\int_0^s e^{if(t)}\,dt$, where f is real-valued, describes a curve in the complex plane. This curve may be studied differential-geometrically in the underlying Euclidean plane. It has arc length s and curvature $f'(s)$ and the transition to its evolute corresponds to integration by parts. This method is used to study the approximation of the Airy integral by a Fresnel integral and yields an estimate for the error.

1. Introduction

THE beauty of the rainbow results partially through a drastic reduction brought about by our eye. The actual physical phenomenon is much more complicated than what we perceive without aids. At each of its points, instead of colour and brightness, the physical rainbow has spectral distributions for the intensities of the two polarization components, the tangential and the radial ones.

A monochromatic rainbow, caused by spherical raindrops of a fixed size and a point sun, still has two polarization components. By circular symmetry, in order to specify a point of the rainbow on the celestial sphere, we need only consider the angle x measuring the arc distance from the centre of the rainbow. In (4) we gave the Airy theory for the intensities of the two polarization components. They were called the *common* rainbow and the *polarized* rainbow, since the radial component is weak compared to the tangential one. Applying a linear approximation to the intensity distribution along the wave front emitted by a raindrop, one finds that the resulting intensities for the two components are certain $\mathbb{R}$-linear combinations of $\mathrm{Ai}(x)^2$ and $\mathrm{Ai}'(x)^2$. Here $\mathrm{Ai}(x)$ is the Airy function, $\mathrm{Ai}'(x)$ its derivative and x is expressed in terms of the radius of the raindrop and the wavelength; moreover, by shifting its value, x is taken to be zero on the Descartes position, i.e., the position of the rainbow according to geometrical optics. The function $\mathrm{Ai}(x)$ is oscillating for $x<0$; this explains the so-called supernumerary bows: the subsequent maxima of intensity beyond the first one.

Bricard (**2**) reports on his observations of rainbows that were produced by a monochromatic searchbeam on fog and were observed through polarizing film. He noted that the maxima of the polarized rainbow alternated with those of the common one. The fact that $\mathrm{Ai}(x)^2$ dominates in the expression for the intensity distribution of the common component and $\mathrm{Ai}'(x)^2$ in that of the polarized component, explains Bricard's observation. The oscillating character of $\mathrm{Ai}(-x)$ and $\mathrm{Ai}'(-x)$ for $x>0$, is indicated most clearly through the well-known asymptotic approximations in terms of cos and sin for $x\to\infty$:

$$\mathrm{Ai}(-x)=\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\cos\left(\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{4}\pi\right)+O(x^{-\frac{7}{4}}),$$
$$\mathrm{Ai}'(-x)=\pi^{-\frac{1}{2}}x^{\frac{1}{4}}\sin\left(\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{4}\pi\right)+O(x^{-\frac{5}{4}}).$$

However, in the case of rainbow observations, x is not large. This raises the question of how good the approximations still are for moderate values of x, say $x\simeq 2$. The question is also of interest in other applications of the Airy function, such as to radio propagation or to quantum mechanics and tunneling (see for instance (**3**), Ch. 15 and (**5**), section 5.13, respectively). We shall give error estimates, here, which allow one to conclude that down to $x=2$ the approximations are good enough for the purposes described above.

Our method is based upon the differential geometry of the plane curves that correspond to integrals of a phase function in the same way that, in a special case, the Cornu spiral corresponds to the Fresnel integral.

We present a few words about our notation, which will require some goodwill. Mostly the letters in the formulae designate *functions*. For instance, if C is a curve embedded in $\mathbb{R}^2$, with arc length variable s measured from a point $A\in C$, we consider C as a 1-dimensional manifold, on which s is a coordinate function. Let α be the direction of the curve, by which we mean the *angle* from the horizontal direction to the direction of the curve. Then α is a function on C and $\alpha(s)=\alpha\circ s^{-1}$ is a function on an $\mathbb{R}$-interval. The curvature is $\kappa=d\alpha/ds$ and we have, for the function α,

$$\alpha=\int_0^s \kappa\, ds \qquad \left(\text{or } \alpha(s)=\int_0^s \kappa(s)\, ds\right).$$

In this seemingly heretical expression the same function s occurs both in the differential form $\kappa\, ds$ and as the upper boundary. A substitution of a particular value s_0 for s takes place only in the upper boundary; thus

$$\alpha(s_0)=\int_0^{s_0} \kappa\, ds.$$

2. Approximation of the Airy function

The integrand of

$$\int_0^\infty e^{if(s)}\,ds,$$

where the phase $f(s)$ is a real-valued function of s, is an infinitesimal complex number, or an infinitesimal vector if the complex plane is viewed as the Euclidean $\mathbb{R}^2$, of length ds and argument $\alpha=f(s)$. Hence, the value of the integral is obtained as the endpoint (for $s_0\to\infty$) of the curve

$$s_0\mapsto\int_0^{s_0} e^{if(s)}\,ds,$$

the shape of which is given by its natural equation $\kappa(s)=f'(s)$, i.e., the curvature as a function of arc length is the derivative of f.

A well-known case is $f(s)=s^2$, where the integral is the Fresnel integral and the curve is the Cornu spiral, which runs from $-(1+i)(\pi/8)^{\frac{1}{2}}$ (for $s=-\infty$) to $(1+i)(\pi/8)^{\frac{1}{2}}$ (for $s=+\infty$). The distance between the two endpoints is $\pi^{\frac{1}{2}}$.

In the case of the integral

$$F(x)=\int_0^\infty e^{i(\frac{1}{3}s^3-xs)}\,ds,$$

one has $f(s)=\frac{1}{3}s^3-xs$, where we shall have to restrict ourselves in this paper to real values of x. Then the real part of the integral is the Airy integral. The corresponding curve

$$C_1: s_0\mapsto\int_0^{s_0} e^{i(\frac{1}{3}s^3-xs)}\,ds,$$

which depends on the real parameter x, has $\kappa(s)=s^2-x$. For a given x, it starts off from the origin in the direction of the positive real axis and, like a roll-tongue, coils up around its ultimate limit point. For the larger negative values of the parameter x, the spiral consists very nearly of circles and the limit may already be estimated by the curvature centre in the starting point $s=0$. If x passes zero and becomes positive, however, the curve also begins to coil around the origin.

Now, by stationary-phase philosophy, one should primarily look at the behaviour of $f'(s)=s^2-x$ in the neighbourhood of its zero $s=b=x^{\frac{1}{2}}$ and a good approximation is obtained by linearizing $f'(s)$ at this point, which gives a Cornu spiral C_2. This approximation becomes better and better as x grows larger and larger, and then our curve C_1 resembles the true Cornu spiral C_2 more and more. The almost-Cornu spiral C_1 starts at the origin for $s=0$, whereas the approximating Cornu spiral C_2 comes, for $s=-\infty$, from a point that again may be approximated fairly well by the curvature centre of C_1

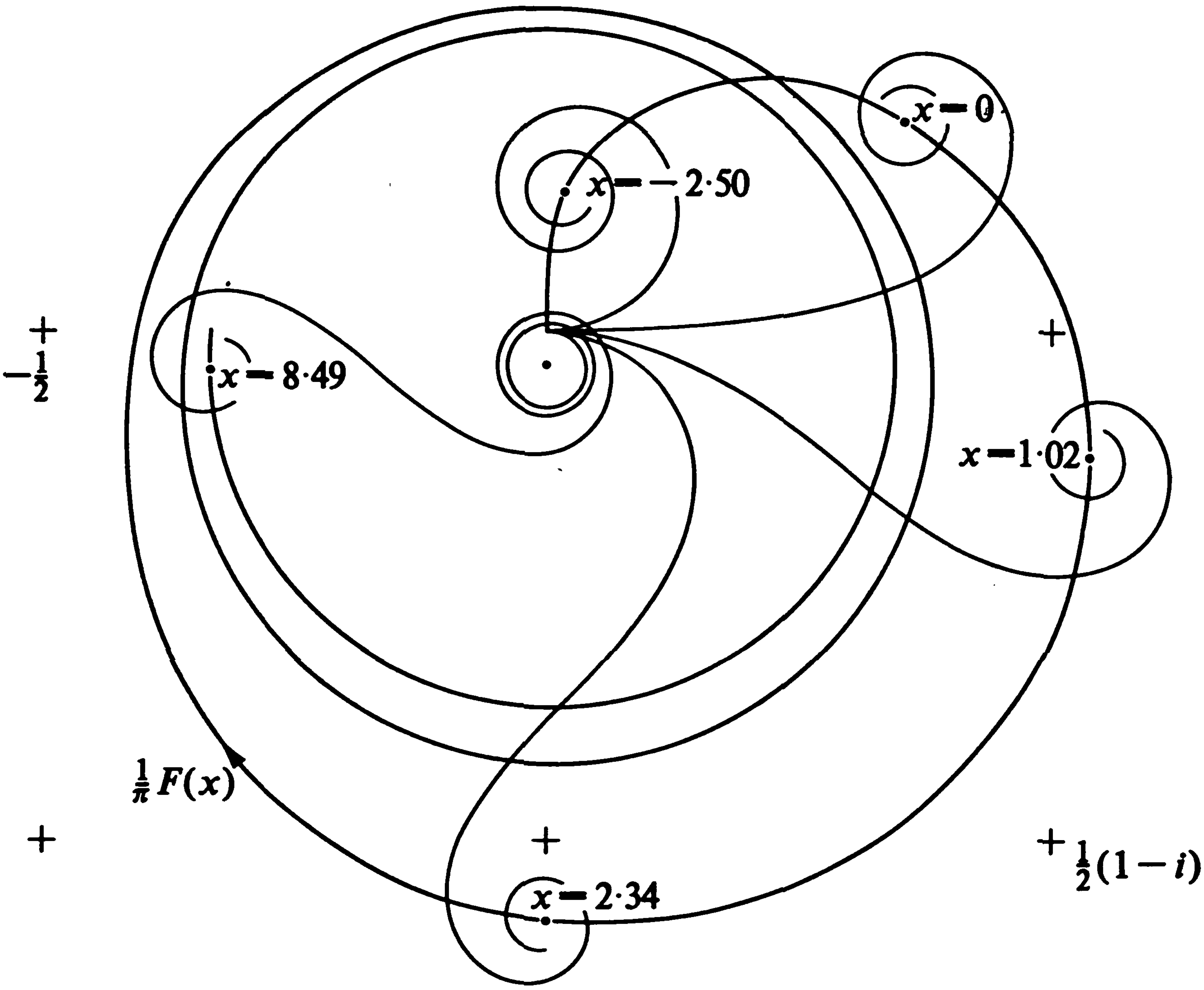

FIG. 1. The spiral with the arrow is the image in the C-plane of the graph of the function

$$x \mapsto \frac{1}{\pi}\int_0^\infty \exp i(\tfrac{1}{3}t^3 - xt)\,dt$$

under the projection $\mathbf{R} \times \mathbf{C} \to \mathbf{C}$. Also drawn are some curves

$$s \mapsto \frac{1}{\pi}\int_0^s \exp i(\tfrac{1}{3}t^3 - xt)\,dt$$

for fixed values of x.

itself at $s = 0$. This curvature centre lies on the negative imaginary axis. The size of the approximating Cornu spiral is known and, if we let the two curves be parallel at their inflection points, its direction as well. The direction of the inflection tangent differs by $\frac{1}{4}\pi$ from the direction at the endpoint of C_2. The curvature radius of C_1 is $1/|s^2 - x|$, i.e. $1/x$ for $s = 0$ and, hence, we let the initial end of C_2 be at the point $-i/x$.

Here is the easy computation for the endpoint of C_2. The linear approximation of $\kappa(s)$ at $s = b$ is $2b(s-b)$. Since we now take the integral over s from $-\infty$ to $+\infty$, a shift of the integration variable has no consequence and only the term $2bs$ is of importance. The straight distance from initial to terminal endpoint of this Cornu spiral is, therefore, $(\pi/b)^{\frac{1}{2}}$. The direction of the tangent at inflection $(s=b)$ is $f(b)=\frac{1}{3}b^3-b^3=-\frac{2}{3}b^3$. Hence, since the Cornu spiral starts at $-i/x$, it ends at the point

$$-i/x+\pi^{\frac{1}{2}}x^{-\frac{1}{4}}e^{i(\frac{1}{4}\pi-\frac{2}{3}x^{3/2})}.$$

Therefore,

$$F(x)/\pi=\mathrm{Ai}\,(-x)+i\,\mathrm{Gi}\,(-x)\sim -i/\pi x+\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}e^{i(\frac{1}{4}\pi-\frac{2}{3}x^{3/2})} \quad \text{as} \quad x\to\infty.$$

For negative x one has, on the other hand,

$$F(x)/\pi=\mathrm{Ai}\,(-x)+i\,\mathrm{Gi}\,(-x)\sim -i/\pi x \quad \text{as} \quad x\to -\infty.$$

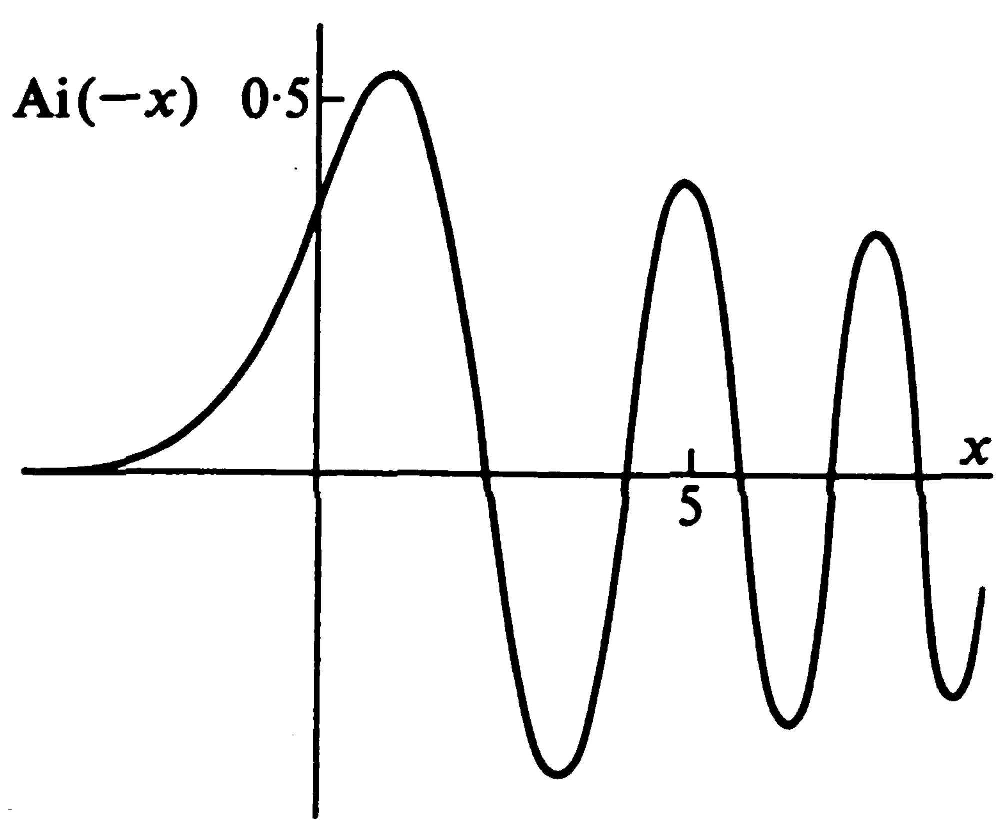

FIG. 2. The Airy function Ai (−x).

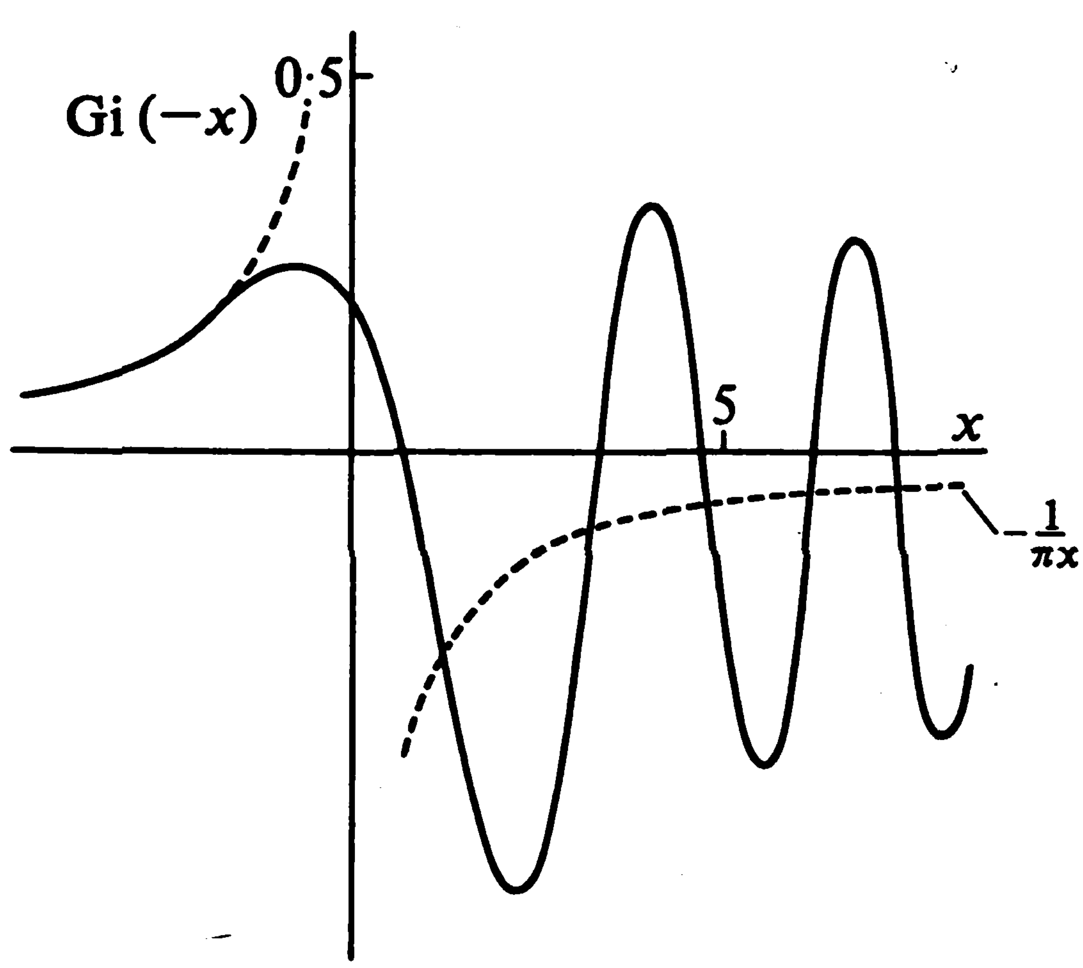

FIG. 3. The function Gi (−x).

One may note that the function $F(x)=\pi\{\mathrm{Ai}(-x)+i\,\mathrm{Gi}(-x)\}$ satisfies

$$Y''+xY=-i$$

and, hence, that the inflection points of the graph of Gi lie on the hyperbola $x\mapsto 1/\pi x$ and those of Ai lie on the axes. As

$$F(0)/\pi=\mathrm{Ai}\,(0)+i\,\mathrm{Gi}\,(0)=(1+i/\sqrt{3})/3^{\frac{2}{3}}\Gamma(\tfrac{2}{3})\simeq 0{\cdot}355(1+i/\sqrt{3}),$$
$$F'(0)/\pi=-\mathrm{Ai}'(0)-i\,\mathrm{Gi}'(0)=(1-i/\sqrt{3})/3^{\frac{1}{3}}\Gamma(\tfrac{1}{3})\simeq 0{\cdot}259(1-i/\sqrt{3})$$

are known values, also in the intermediate region around $x=0$, the two graphs can approximately be drawn, connecting the asymptotic parts.

3. Error estimate

We now want an estimate for the error of the given approximation for $x>0$. We shall denote by s_1, ρ_1 the arc length and curvature radius of C_1 and by s_2, ρ_2 those of C_2, where s_1, s_2 are both measured from the inflection point; thus $s=b+s_1$. If we bring the two curves into correspondence by relating points with the same α, this correspondence is a 1-dimensional variety (in other words, we have 1 degree of freedom) on which s_1, s_2, α, as well as

$$1/\rho_1=\kappa_1=d\alpha/ds_1,$$
$$1/\rho_2=\kappa_2=d\alpha/ds_2,$$

are functions. Furthermore, one has

$$\alpha+\tfrac{2}{3}b^3=bs_1^2+\tfrac{1}{3}s_1^3=bs_2^2,$$
$$1/\rho_1=2bs_1+s_1^2,\qquad s_1\geqslant -b,$$
$$1/\rho_2=2bs_2.$$

We are approximating

$$F(x)=\int_0^\infty e^{i(\frac{1}{3}s^3-xs)}\,ds$$

by

$$-i/x+\int_{-\infty}^{\infty} e^{i(\frac{1}{2}s_2^2)}\,ds_2,$$

i.e., we approximate

$$\int_0^\infty e^{i\alpha}\,ds-[i\rho_1]_{s=0}\ \text{by}\ \int_{-\infty}^{\infty} e^{i\alpha}\,ds_2.$$

After integrating by parts, this means that we approximate

$$\int_{s=0}^{\infty} ie^{i\alpha}\,d\rho_1\ \text{by}\ \int_{s_2=-\infty}^{\infty} ie^{i\alpha}\,d\rho_2.$$

The absolute value of the difference

$$\int_{s=0}^{\infty} ie^{i\alpha}\,d(\rho_1-\rho_2)-\int_{s_2=-\infty}^{-\frac{1}{3}b\sqrt{6}} ie^{i\alpha}\,d\rho_2$$

is the total span $|\int_0^{b^{-2}} e^{i\alpha}\,d\sigma|$ of a curve with arc length variable

$$\sigma=b^{-2}+\rho_1-\rho_2 \quad \text{for} \quad s\geqslant 0$$

and

$$\sigma=-\rho_2 \quad \text{for} \quad s_2\leqslant -\tfrac{1}{3}b\sqrt{6},$$

and phase α, where we have added the term b^{-2} in order to put, somewhat artificially, the two pieces together.

This curve spirals, once again, towards its endpoints. The total length of this correction spiral is b^{-2}; hence

$$|F(x)+i/x-\pi^{\frac{1}{2}}x^{-\frac{1}{4}}e^{i(\frac{1}{4}\pi-\frac{2}{3}x^{3/2})}|<x^{-1}.$$

In order to improve upon this, i.e., to obtain a better estimate for the distance of the endpoints of such a correction spiral, we formulate the following:

LEMMA. *Let $\kappa(s)$, defined on $[0, c]$, be positive and monotonically increasing. Put*

$$\alpha(s)=\int_0^s \kappa(s)\,ds, \qquad g(s)=\int_0^s e^{i\alpha(s)}\,ds$$

and $R(s)=1/\kappa(s)$. Then (i) $|g(c)|\leqslant[s]_{\alpha=2}$; (ii) $|g(c)|\leqslant 2R(0)$.

Proof. (i) Consider the curve drawn at distance $a=[R]_{\alpha=2}$ parallel to the spiral for $0\leqslant\alpha\leqslant 2$; this parallel curve ends with curvature radius zero, cf. Fig. 4. One has $r\leqslant \text{arc } AM+MB$, hence $CD=r+a\leqslant \text{arc } AM+2a$, whereas $[s]_{\alpha=2}=\text{arc } CE=\text{arc } AM+2a$.

(ii) Starting from any point, say $s=0$, let the parameter s increase. Then the infinitesimal change of the curvature circle consists of a rotation and a shrinkage inside itself, since the curvature radius decreases.

Similarly, if $\kappa(s)$ is given on $[c_1, c_2]$ and $0\in[c_1, c_2]$ is an inflection point of $\alpha(s)$ such that $|\kappa(s)|=R(s)^{-1}$ increases to the right and to the left of this point (thus the graph of $\alpha(s)$ becomes steeper and steeper on both sides; we will call it a *shallow inflection point*), then $|g(c_2)-g(c_1)|\leqslant 2R(0)$. Hence, if one puts $R(0)=R$, $|s|_{\alpha=2}=A$, $|s|_{\alpha=-2}=B$, then $|g(c_2)-g(c_1)|\leqslant \min(2R, A+B)$. In case $\alpha=2$ (or $\alpha=-2$) is not reached, we take the total length $|c_1|$ or $|c_2|$ for A or B.

In applying this lemma to an arbitrary curve, where its direction α is given as a function of the arc length σ, we divide the graph of $\alpha(\sigma)$ into segments, which are separated by the extremal points and the inflection points, so that

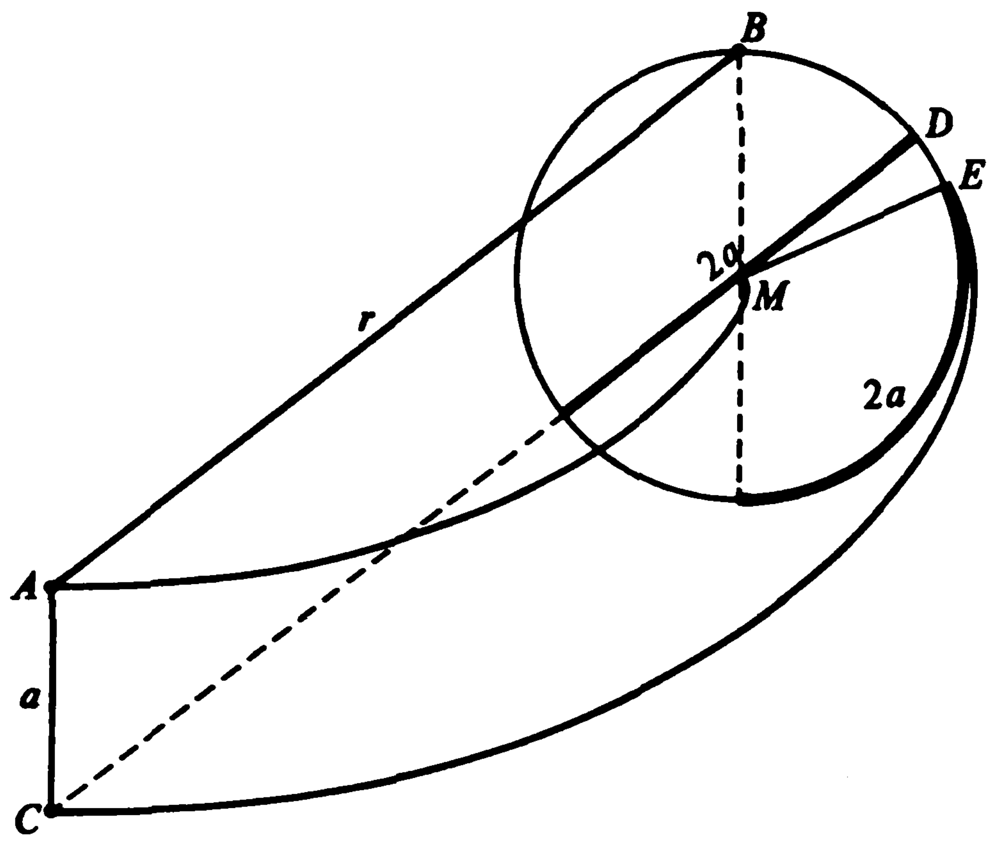

FIG. 4. This supports the argument of the lemma. The arc CE is part of a given curve and the circle is its curvature circle at the point E. If the curvature never decreases along the given curve, its part beyond E will remain within the circle.

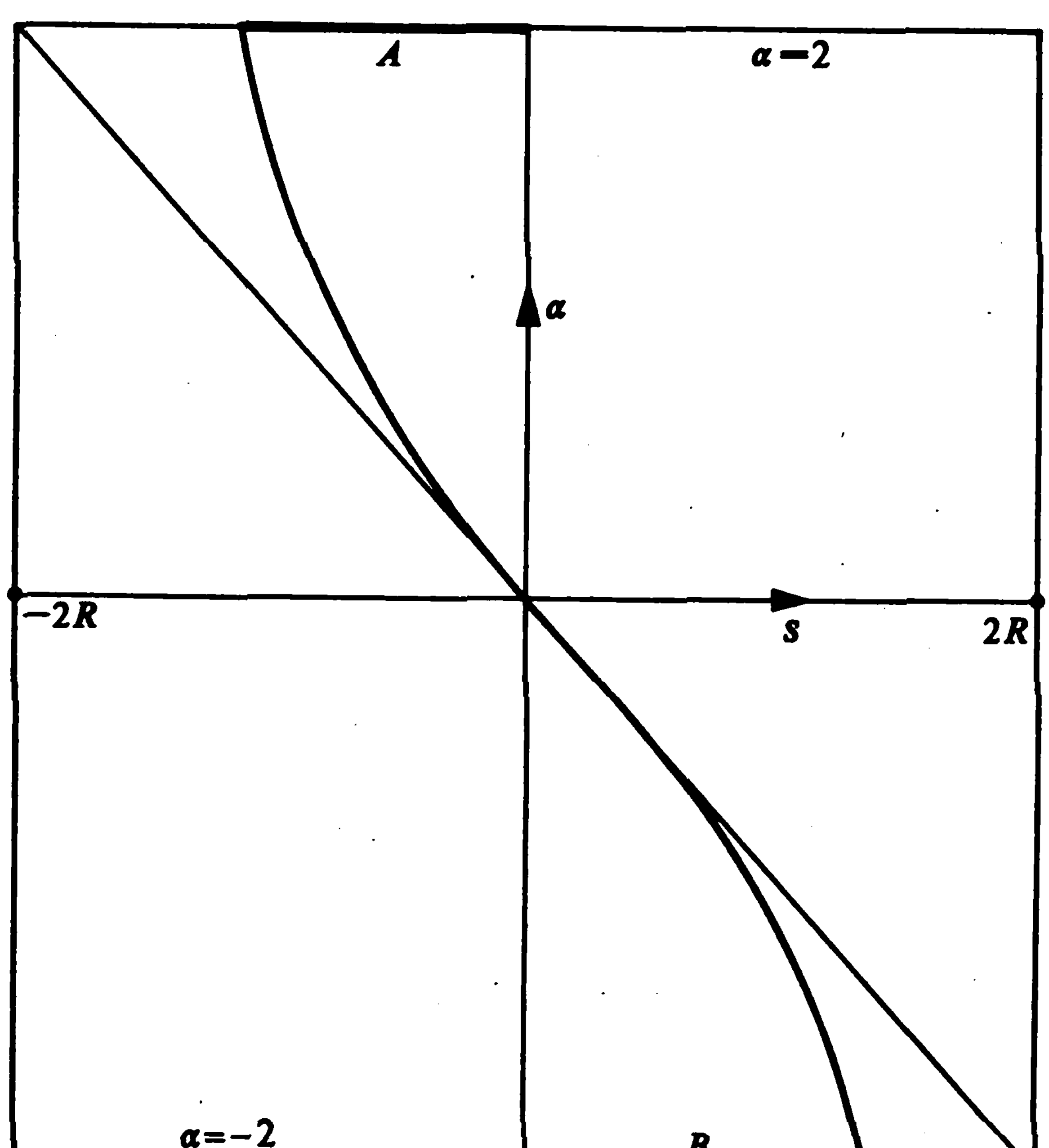

FIG. 5. 'Shallow' inflection point of the graph of a function $\alpha(s)$ that gives the direction of a plane curve as a function of arc length. The figure defines numbers A and B.

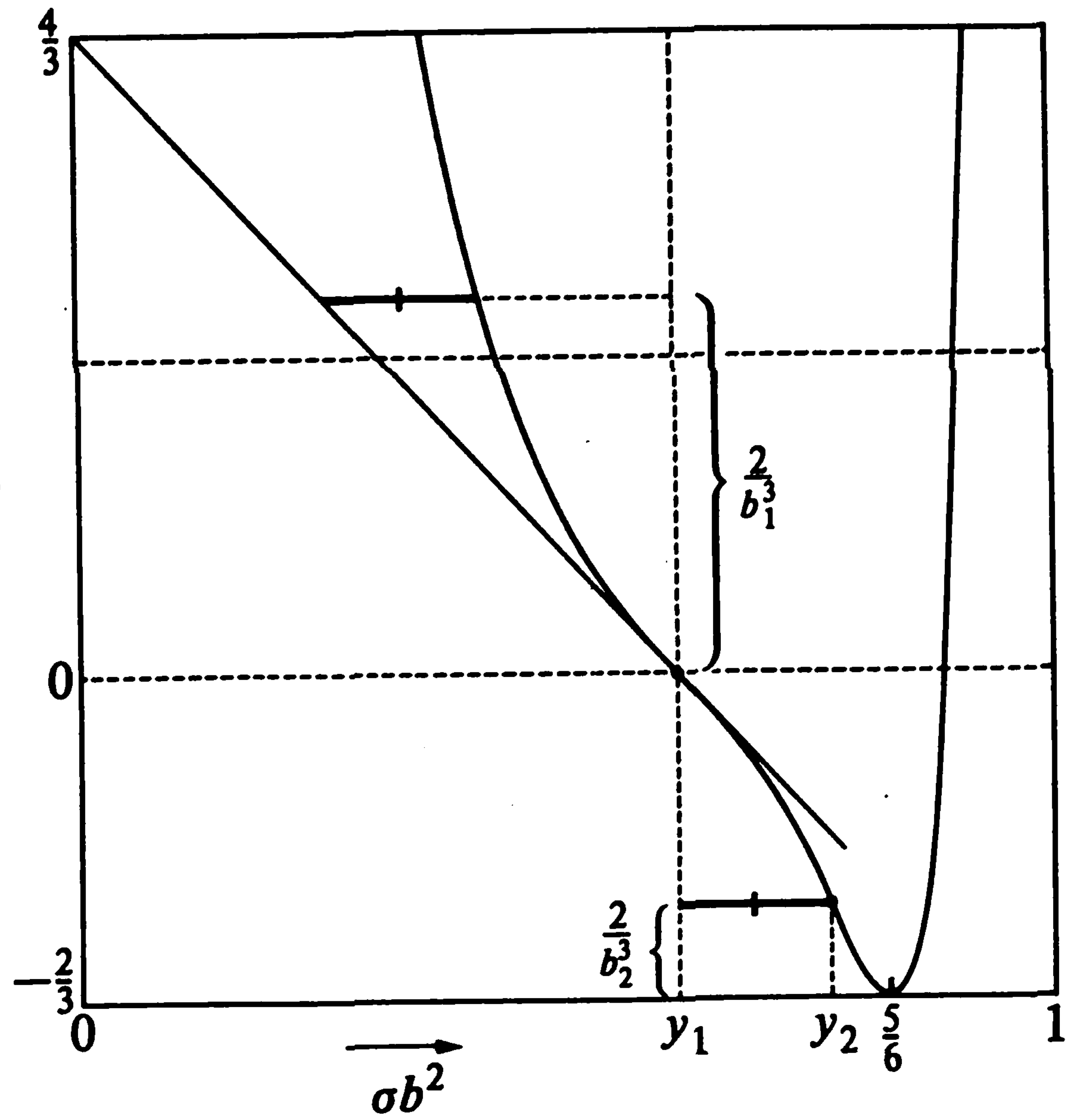

FIG. 6. The function $\alpha(\sigma)$ analogous to Fig. 5, for the case where the plane curve is the correction spiral, the distance of the endpoints of which has to be estimated. The Figure defines numbers b_1, b_2.

on each segment the curvature $d\alpha/d\sigma$ is monotonic. Two adjacent segments separated by a shallow inflection point can be treated simultaneously by the $2R$-method.

To investigate our present $\alpha(\sigma)$, we have plotted α/b^3 against σb^2 in Fig. 6. There is one shallow inflection point, namely the fusion point at $\sigma b^2 = y_1 = \frac{1}{4}\sqrt{6}$ and one steep inflection point at $\sigma b^2 = y_2 \simeq 0{\cdot}766$. Thus, there are four segments, of which segment 1 and segment 2 are separated by the fusion point and may be treated simultaneously by the $2R$-method. One has $R = \frac{3}{16}\sqrt{6}b^{-5}$, the value of $d\sigma/d\alpha$ at the fusion point. Furthermore, the Figure defines a value b_2 (from the ordinate of the inflection point) and a value b_1 (such that the two indicated horizontal distances are equal). One obtains $b_1 \simeq 1{\cdot}38$, $b_2 \simeq 2{\cdot}17$.

For $b < b_1$ we have $A + B < 2R$, so that the estimate $A + B$ is more advantageous. On segment 1, which comes from the incomplete Fresnel integral,

$$\alpha + \tfrac{2}{3}b^3 = (4b\rho_2^2)^{-1} = (4b\sigma^2)^{-1}.$$

To compute the estimate A on segment 1, we must take $\alpha = 2$, whence

$$b\sigma^2 = 3/(8b^3 + 24),$$

and with the corresponding value of σb^2,

$$Ab^2 = \tfrac{1}{4}\sqrt{6} - \sigma b^2 = (3/8)^{\frac{1}{2}}\{1 - b^{\frac{3}{2}}(b^3+3)^{-\frac{1}{2}}\}.$$

For $b \leqslant b_1$ we certainly must take for B the full arc length of segment 2, i.e.,

$$Bb^2 = y_2 - y_1 \simeq 0{\cdot}15.$$

On segment 3,

$$\sigma b^2 = b^2/(s^2 - b^2) + 1/2(\alpha/b^3 + \tfrac{2}{3})^{\frac{1}{2}} + 1,$$
$$\alpha = \tfrac{1}{3}s^3 - b^2 s.$$

To compute the error estimate A, we must take $\alpha + \frac{2}{3}b^3 = 2$, i.e., $b = 2/s_1^2 - \frac{1}{3}s_1$ and with the corresponding value of σb^2,

$$Ab^2 = \tfrac{5}{6} - \sigma b^2 = -b^2/(s_1^2 + 2bs_1) - (b^3/8)^{\frac{1}{2}} - \tfrac{1}{6}.$$

On segment 4,

$$\sigma b^2 = b^2/(s^2 - b^2) - 1/2(\alpha/b^3 + \tfrac{2}{3})^{\frac{1}{2}} + 1,$$
$$\alpha = \tfrac{1}{3}s^3 - b^2 s.$$

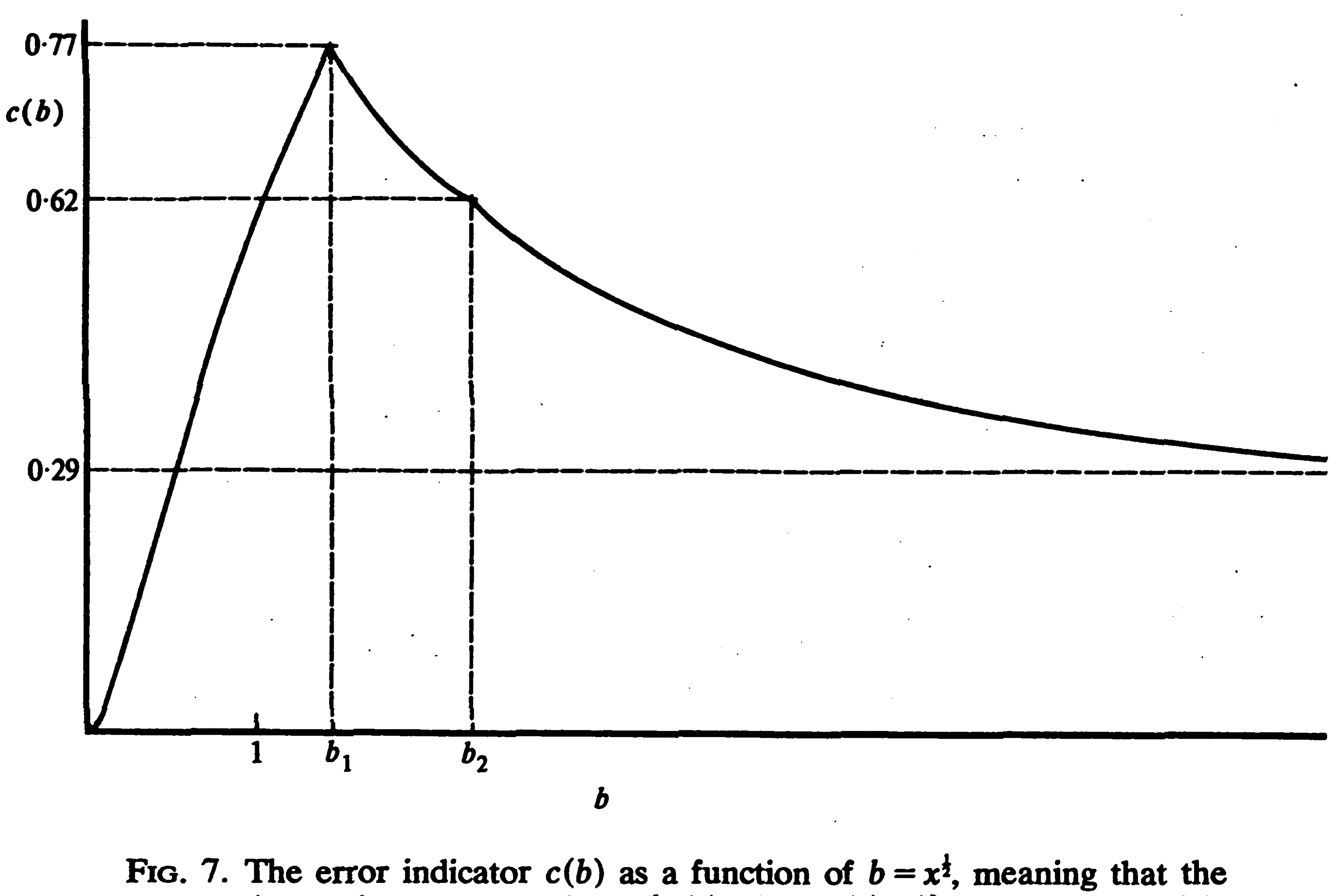

FIG. 7. The error indicator $c(b)$ as a function of $b = x^{\frac{1}{2}}$, meaning that the error estimate, in approximating $\pi\{\mathrm{Ai}(-x) + i\,\mathrm{Gi}(-x)\}$, by our method is $c(x^{\frac{1}{2}})x^{-\frac{7}{4}}$.

To compute the error estimate A, we must again take $\alpha=-\frac{2}{3}b^3+2$, $b=2/s_1^2-\frac{1}{3}s_1$ and with the corresponding value of σb^2,

$$Ab^2=\sigma b^2-\tfrac{5}{6}=b^2/(s_1^2+2bs_1)-(b^3/8)^{\frac{1}{2}}+\tfrac{1}{6}.$$

Writing our total error estimate as $|\text{error}|\leqslant c(b)/b^{\frac{7}{2}}$, we have plotted the 'error indicator' $c(b)$ in Fig. 7. It reaches a maximum at $b=b_1$ with $c(b_1)\simeq 0{\cdot}77$. As a consequence

$$|\text{error}|<0{\cdot}8b^{-\frac{7}{2}}=0{\cdot}8x^{-\frac{7}{4}}.$$

(For $b=1$ we may replace this by $|\text{error}|<0{\cdot}602$, whereas

$$\lim_{b\to\infty} c(b)=5\sqrt{2}/24\simeq 0{\cdot}295.)$$

Summarizing:

$$\left|\int_0^\infty e^{i(\frac{1}{3}s^3-xs)}\,ds+i/x-\pi^{\frac{1}{2}}x^{-\frac{1}{4}}e^{i(\frac{1}{4}\pi-\frac{2}{3}x^{3/2})}\right|<0{\cdot}8x^{-\frac{7}{4}},\qquad x>0.$$

4. Further examples

As exercises one may now apply the same method for $x<0$ and, also, on the derivative $F'(x)$. The results are as follows (decimal expressions are rounded off, but in the final inequalities the margin is ample enough for them to hold precisely):

(i) The error in approximating $F(x)$ for $x<0$ by $-i/x$ is, integrating by parts,

$$F(x)+i/x=\int_{s=0}^{\infty} e^{i\alpha}\,ds+i/x=i\int_{s=0}^{\infty} e^{i\alpha}\,d\rho,$$

where $\alpha=\frac{1}{3}s^3-xs$ and $\rho=ds/d\alpha$. Put $x=-b^2$ and plot α/b^3 against ρb^2. There is one, shallow, inflection point, at $\rho b^2=\frac{5}{6}$, $\alpha/b^3=16\sqrt{5}/75=\mu$, say, and one has two segments, which can be handled simultaneously with the $A+B$, $2R$-method. Writing our error estimate as $|\text{error}|\leqslant c(b)/b^5$, we get for $b\leqslant 1{\cdot}24$, with the $A+B$-method,

$$c(b)=yb^3$$

with

$$b^3=2\{(1-\tfrac{2}{3}y)y^{\frac{1}{2}}(1-y)^{-\frac{3}{2}}-\mu\}^{-1},$$

where $y=1-\rho b^2$ runs from 1 to $0{\cdot}54$ and $c(b)$ increases from 0 to $1{\cdot}04$, and for $b\geqslant 1{\cdot}24$, with the $2R$-method,

$$c(b)=2R=25\sqrt{5}/54\simeq 1{\cdot}04.$$

Thus, uniformly $c(b)<1{\cdot}04$, i.e.,

$$\left|\int_0^\infty e^{i(\frac{1}{3}s^3-xs)}\,ds+i/x\right|<1{\cdot}04\,|x|^{-\frac{5}{2}},\qquad x<0.$$

For $x=1$, the indicator $c(1)$ is $<0{\cdot}7$.

(ii) For the derivative $F'(x)=\int_0^\infty e^{i(\frac{1}{3}s^3-xs)}(-is)\,ds$, we put $s^2=t$, $x=b^2$ and have

$$\alpha=\tfrac{1}{3}t^{\frac{3}{2}}-b^2t^{\frac{1}{2}}=(b^2+t_1)^{\frac{1}{2}}(\tfrac{1}{3}t_1-\tfrac{2}{3}b^2)\quad\text{with}\quad t=b^2+t_1,$$
$$\kappa_1=2\,d\alpha/dt=t^{\frac{1}{2}}-b^2t^{-\frac{1}{2}}.$$

Analogously to the case of $F(x)$ itself, $\kappa_1(t)$ is linearly approximated at $t=b^2$ by $\kappa_2(t_2)=t_2/b$, where we are putting $\alpha=-\frac{2}{3}b^3+t_2^2/4b$. The error in approximating

$$F'(x)=-\tfrac{1}{2}i\int_0^\infty e^{i\alpha}\,dt=-i\int_0^\infty e^{i\alpha}\rho_1\,d\alpha=-\int_{t=0}^\infty d(e^{i\alpha})\rho_1=\int_{t=0}^\infty e^{i\alpha}\,d\rho_1,$$

by

$$-\tfrac{1}{2}i\int_{-\infty}^\infty e^{i\alpha}\,dt_2=-i\int_{t_2=-\infty}^\infty e^{i\alpha}\rho_2\,d\alpha=-\int_{t_2=-\infty}^\infty d(e^{i\alpha})\rho_2=\int_{t_2=-\infty}^\infty e^{i\alpha}\,d\rho_2,$$

consists of two parts:

$$\text{error}=\int_{t=0}^\infty e^{i\alpha}\,d(\rho_1-\rho_2)-\int_{t_2=-\infty}^{-(8/3)^{1/2}b^2} e^{i\alpha}\,d\rho_2.$$

Here

$$\rho_1=dt/2d\alpha=t^{\frac{1}{2}}/(t-b^2),$$
$$\rho_2=dt_2/2d\alpha=b/t_2.$$

Now call $\rho_1-\rho_2=\sigma$ in the first part and $-\rho_2=\sigma$ in the second part and plot α/b^3 against σb. The graphs have the same shape as in the case of F, but the two parts do not fit smoothly. On the contrary, the curve corresponding to the difference integral, with arc length parameter σ and curvature $d\alpha/d\sigma$, has a cusp in the fusion point. Therefore, the two parts are treated separately.

The position of the inflection point of $\alpha(\sigma)$ (which is steep) is at $\sigma b^2\simeq 0{\cdot}43$, $\alpha/b^3\simeq 0{\cdot}22$. So we have four segments in total and on all four we apply the A (or B)-estimate. Writing the error estimate as $c(b)/b^{\frac{5}{2}}$, we get:

On the segment corresponding to part 2,

$$c_1(b)=(\tfrac{3}{8}b^3)^{\frac{1}{2}}\{1-(1+3/b^3)^{-\frac{1}{2}}\}.$$

On segment 2 (from fusion point to inflection point),

$$c_2(b)=(\tfrac{3}{8})^{\frac{1}{2}}b^{\frac{3}{2}}+s^{\frac{1}{2}}b^{\frac{5}{2}}(b^2-s)^{-1}-b^3(\tfrac{8}{3}b^3-8)^{-1}\quad\text{for}\quad b\geqslant 1{\cdot}65,$$

where s runs from $0{\cdot}64$ to 0 and $b^2=\frac{1}{3}s+2s^{-\frac{1}{2}}$ and $c_2(b)=0{\cdot}18b^{\frac{3}{2}}$ for $b\leqslant 1{\cdot}65$.

On segment 3 (from stationary point to inflection point),

$$c_3(b)=b^3/2^{\frac{3}{2}}+(b^{\frac{5}{2}}-b^{\frac{7}{2}}l)/(2b-l)-\tfrac{1}{3}b^{\frac{3}{2}}\quad\text{for}\quad b\geqslant 2{\cdot}09,$$

with $b=2/l^2+l/3$, l running from $0{\cdot}22$ to 0 and $c_3(b)=0{\cdot}10b^{\frac{3}{2}}$ for $b\lesssim 2{\cdot}09$.

On segment 4 (from stationary point to $s/b=\infty$),

$$c_4(b)=(b^{\frac{7}{2}}/k+b^{\frac{5}{2}})/(2b+k)-b^3/2^{\frac{3}{2}}-\tfrac{1}{3}b^{\frac{3}{2}},$$

k running from $\sqrt[3]{6}$ to 0 and $b=2/k^2-\frac{1}{3}k$.

In total, $c(b)=c_1(b)+c_2(b)+c_3(b)+c_4(b)$ reaches a maximum at $b\simeq 1{\cdot}65$ with value 1·03 and comes nearly as high at $b\simeq 2{\cdot}09$ with value 1·00. Hence,

$$|F'(x)+ix^{\frac{1}{4}}\pi^{\frac{1}{2}}e^{i(\frac{1}{4}\pi-\frac{2}{3}x^{3/2})}|<1{\cdot}03x^{-\frac{5}{4}}, \qquad x>0.$$

For $x=0$, the indicator $c(1)$ is less than 0·7.

(iii) A direct confrontation of $F(x)+i/x$ and $ix^{-\frac{1}{2}}F'(x)$ gives the following. With $\alpha=\frac{1}{3}s^3-xs$,

$$F(x)+i/x-ix^{-\frac{1}{2}}F'(x)=\int_0^\infty e^{i\alpha}\,ds+i/x-x^{-\frac{1}{2}}\int_0^\infty e^{i\alpha}s\,ds$$

$$=i\int_{s=0}^\infty e^{i\alpha}\,d\{(1-x^{-\frac{1}{2}}s)\,ds/d\alpha\}.$$

Put $x=b^2$ and $\sigma=(x^{-\frac{1}{2}}s-1)\,ds/d\alpha=(bs+b^2)^{-1}$. Plot α/b^3 against σb^2. There is one inflection point (steep) at $\sigma b^2=\frac{2}{3}$ and there are three segments, for which method A (or B) is to be applied. We write the error estimate as $|\text{error}|\leqslant c(b)/b^{\frac{7}{2}}$ and get:

On segment 1 $(0<y=\sigma b^2\leqslant\frac{1}{2})$,

$$c_1(b)=(\tfrac{1}{2}-y)b^{\frac{3}{2}}, \quad \text{with} \quad b^3=6y^3/(y+1)(2y-1)^2.$$

On segment 2 $(\frac{1}{2}\leqslant y=\sigma b^2\leqslant\frac{2}{3})$,

$$c_2(b)=\begin{cases}\frac{1}{6}b^{\frac{3}{2}} & \text{for} \quad b\leqslant 2{\cdot}13,\\ (\frac{1}{2}-y)b^{\frac{3}{2}}, & \text{with} \quad b^3=6y^3/(y+1)(2y-1)^2, \quad \text{for} \quad b\geqslant 2{\cdot}13.\end{cases}$$

On segment 3 $(\frac{2}{3}\leqslant y=\sigma b^2\leqslant 1)$,

$$c_3(b)=\begin{cases}\frac{1}{3}b^{\frac{3}{2}} & \text{for} \quad b\leqslant 1{\cdot}63\\ (1-y)b^{\frac{3}{2}}, & \text{with} \quad b^3=6y^3/(1-y)(2y^2+2y-1), \quad \text{for} \quad b\geqslant 1{\cdot}63.\end{cases}$$

The total indicator

$$c(b)=c_1(b)+c_2(b)+c_3(b)$$

reaches a maximum at $b\simeq 2{\cdot}13$ with $c(b)\simeq 1{\cdot}33$ (and comes nearly as high at $b\simeq 1{\cdot}63$ with value 1·29); thus

$$|F(x)+i/x-ix^{-\frac{1}{2}}F'(x)|<1{\cdot}4x^{-\frac{7}{4}},\ x>0.$$

For $x=1$, the indicator $c(1)$ is less than 0·7 and $\lim_{b\to\infty}c(b)=\frac{1}{2}\sqrt{2}$.

5. Higher-order approximations

Repeating the process of partial integration and splitting off Fresnel integrals yields the consecutive terms of the asymptotic expansions given in (1) and (6):

$$F(x) \sim \pi^{\frac{1}{2}} x^{-\frac{1}{4}} e^{i(\frac{1}{4}\pi - \frac{2}{3}x^{3/2})}\left(1 + \tfrac{5}{48} i x^{-\frac{3}{2}} - \tfrac{385}{4608} x^{-3} + \ldots\right) +$$

$$+ i\left(-\frac{1}{x} + \frac{2!}{x^4} + \ldots\right) \quad \text{as} \quad x \to \infty,$$

$$F(x) \sim i\left(-\frac{1}{x} + \frac{2!}{x^4} + \ldots\right) \quad \text{as} \quad x \to -\infty.$$

The non-oscillating terms are boundary terms and the oscillating ones come from Fresnel integrals.

As for error estimates, one obtains for instance the following:

(i) For $F(x)$, $x > 0$, the second step consists of replacing σ by $\sigma - \frac{5}{48} s_2$ and differentiating with respect to α. This gives a correction spiral with arc length parameter

$$\tau = -\frac{2s}{(s^2-1)^3} + \frac{1}{4s_2^3} - \frac{5}{96 s_2},$$

where the phase α is considered as a function of τ.

With

$$Fr(x) = \pi^{\frac{1}{2}} x^{-\frac{1}{4}} e^{i(\frac{1}{4}\pi - \frac{2}{3}x^{3/2})},$$

the full length of this correction spiral is $\frac{31\sqrt{6}}{96} b^{-4} < 0{\cdot}8 b^{-4}$, where $x = b^2$, hence,

$$|F(x) - (1 + \tfrac{5}{48} i) Fr(x) + i x^{-1}| < 0{\cdot}8 x^{-2}.$$

(One could make a finer estimate, as before; there are 3 segments. But in the next steps the number of segments grows and so we limit ourselves to taking the full length.)

The third step consists of replacing τ by

$$\omega = \frac{10s^2+2}{(s^2-1)^5} - \frac{3}{8s_2^5} + \frac{5}{192 s_2^3} - \frac{385}{9216 s_2},$$

and the total length of the correcting spiral is $2b^{-7}$, so that

$$|F(x) - i(-x^{-1} + 2x^{-4}) - (1 + \tfrac{5}{48} i x^{-\frac{3}{2}} - \tfrac{385}{4608} x^{-3}) Fr(x)| < 2x^{-\frac{7}{2}}.$$

(ii) For $F(x)$, $x < 0$, the second step gives

$$|F(x) + i x^{-1}| < 1{\cdot}04\, |x|^{-\frac{5}{2}},$$

the total length of the correction spiral being $\frac{25\sqrt{5}}{54} b^{-5} < 1{\cdot}04 b^{-5}$, where

$x = -b^2$. The finer method (3 segments) gives

$$|F(x)+ix^{-1}|<c(b)/b^8 \quad \text{with} \quad c(b)<4{\cdot}8 \quad \text{and} \quad c(1)<1.$$

After the third step one has a correction spiral with total length $\frac{226}{81}b^{-8}<2{\cdot}8b^{-8}$, hence,

$$|F(x)+ix^{-1}-2ix^{-4}|<2{\cdot}8x^{-4}.$$

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