Non-diagonalizable maps/matrices

Recall

* Spectral theory: $A = UDU^*$ (D Diag. with $\mathrm{Eig}(A)$, U unitary ($U^* = U^{-1}$) with normalized eigenvectors of A as columns) $\Leftrightarrow$ A normal ($AA^* = A^*A$)

D real $\Leftrightarrow$ A hermitian ($A = A^*$)

* Maps with distinct eigenvalues can be diagonalized ($A = UDU^{-1}$, U not unitary in general).
* Maps with multiple eigenvalues (alg. mult. > 1) cannot be diagonalized if alg. mult. $>$ geo. mult. (dim. of solutionspace to $A\bar{x} = \lambda\bar{x}$) for some eigenvalue.
* Schur: Any A can be written $A = UTU^*$ with U unitary and T upper triangular (with eigenvalues of A on diagonal).

If A $n \times n$ matrix (or map $V \to V$), $A^2, A^3, \ldots$ are also $n \times n$ matrices. If $q(z) = a_0 + a_1 z + \ldots + a_m z^m$ is a polynomial then $q(A) = a_0 I + a_1 A + \ldots + a_m A^m$ is a $n \times n$-matrix.

If $A = VBV^{-1}$ then $A^2 = VB\underbrace{V^{-1}V}_{I}BV^{-1} = VB^2V^{-1}$; $A^3 = VB^3V^{-1}, \ldots$ so $q(A) = V(a_0 I + a_1 B + \ldots + a_m B^m)V^{-1} = Vq(B)V^{-1}$

Also implies $\det q(A) = \det q(B)$

Cayley-Hamilton Theorem

If $p(z) = \det(A - zI)$ then $p(A) = 0$. zero-matrix

Proof

Use schur: $A = UTU^* \Rightarrow p(A) = U\, p(T)\, U^*$

$$T = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad p(z) = \pm (z - \lambda_1)(z - \lambda_2) \cdot \ldots \cdot (z - \lambda_n)$$

2. $p(z) = k(z)m(z)$ so $m(z) = 0 \Rightarrow p(z) = 0$.

If $p(\lambda) = 0$ then $A\bar{v} = \lambda\bar{v}$ has solution $\bar{v} \neq 0$.

$\Rightarrow A^2\bar{v} = \lambda^2\bar{v}$, $A^3\bar{v} = \lambda^3\bar{v}$, — so

$$\bar{0} = \underset{\text{zeromatrix}}{0}\,\bar{v} = m(A)\bar{v} = \underbrace{m(\lambda)}_{\text{number}}\underbrace{\bar{v}}_{\neq\bar{0}} \Rightarrow m(\lambda) = 0.$$

QED

Ex (from lecture 3)

$$A_1 = \begin{bmatrix} -1 & 4 & 4 \\ -8 & 11 & 8 \\ 8 & -8 & 5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 & 2 \\ -8 & 11 & 8 \\ 10 & -11 & -7 \end{bmatrix}$$

$p_1(z) = p_2(z) = -(z+1)(z-3)^2$, $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 3$

for both A_1 and A_2

A_1 diagonalizable: $A_1 = TDT^{-1}$, $D = \begin{bmatrix} -1 & & \\ & 3 & \\ & & 3 \end{bmatrix} \Rightarrow$

$$\Rightarrow (A_1 - \lambda_1 I)(A_1 - \lambda_2 I) = (A_1 + I)(A_1 - 3I) =$$

$$= T(D+I)(D-3I)T^{-1} = T\underbrace{\begin{pmatrix} 0 & & \\ & 4 & \\ & & 4 \end{pmatrix}\begin{bmatrix} -4 & & \\ & 0 & \\ & & 0 \end{bmatrix}}_{0}T^{-1} = \underset{\text{zero-matrix}}{0}$$

$$\Rightarrow m_1(A_1) = A_1^2 - 2A_1 - 3I = 0$$

so $m_1(z) = \underset{(z+1)(z+3)}{z^2 - 2z - 3}$ is the minimal pol. of A_1

A_2 not diagonalizable, Calculation shows that

$(A_2 + I)(A_2 - 3I) \neq 0$ so $m_2(z) = (z+1)(z-3)^2 = -p(z)$ is the min. pol. of A_2.

Generalized eigenvectors

The geom. mult. of eigenvalue λ of A is the dim of the solution space to $A\bar{v} = \lambda\bar{v}$ (eigenspace), i.e. the dim. of the nullspace $N(A - \lambda I)$. If geom. mult. < alg. mult., then A is not diagonalizable.

We now introduce gen. eigenvectors and prove that their dimension is = alg. mult.

The dimension of the null spaces cannot begin to increase if it has stopped to increase at some step. Since the dim $\le n$ there must be smallest number $d(\lambda)$ s.t. $E_\lambda = N((A-\lambda I)^{d(\lambda)})$.

Since null spaces are vector spaces, E_λ is a vector space

Note $(A-\lambda I)^{d(\lambda)}(A\bar{v}) \underset{\uparrow \text{commute}}{=} A\underbrace{(A-\lambda I)^{d(\lambda)}\bar{v}}_{=\bar{0}} = \bar{0}$

so $\bar{v} \in E_\lambda \Rightarrow A\bar{v} \in E_\lambda$. We write $AE_\lambda \subset E_\lambda$ and say E_λ is A-invariant. The restriction of A to E_λ is a map $A|_{E_\lambda} : E_\lambda \to E_\lambda$. Let $N_\lambda = A|_{E_\lambda} - \lambda I|_{E_\lambda} : E_\lambda \to E_\lambda$

$N_\lambda^{d(\lambda)}\bar{v} = \bar{0} \quad \forall \bar{v} \in E_\lambda \Rightarrow N_\lambda^{d(\lambda)} = 0$ so N_λ nilpotent and can only have eigenvalue 0.

Next time: $\begin{cases} \dim(E_{\lambda_i}) = \alpha_i = \text{alg. mult.} \\ E_1 \oplus \dots \oplus E_r = V \end{cases}$