

Abstract

We propose a scheme for implementing a single-ion Stochastic Quantum Processor using a single cold trapped ion's internal state and 3-axis center-of-mass vibrational states as qubits. The processor implements an arbitrary rotation around the z axis of the Bloch sphere of a *data qubit*, given two *program qubits*, that is, the operation realized on the data is determined by using different program qubits and not by varying the gate itself. Unfortunately this cannot be done deterministically, and must be necessarily stochastic. In this proposal the operation is applied successfully with probability $p = 3/4$.

Trapped Ion Quantum Gates

The single ion stochastic quantum processor requires one C-NOT gate and one Toffoli gate. These can be implemented as a particular series of pulses of laser light that subject the ion to dynamics governed by the following four interaction Hamiltonians:

$$\begin{aligned} H_{1C} &= \hbar\Omega_{1C}(a^\dagger\sigma_- + a\sigma_+), & H_{2T} &= i\hbar\Omega_{2T}(a^\dagger b^\dagger\sigma_- - ab\sigma_+), \\ H_{2C} &= -\hbar\Omega_{2C}(a^\dagger b^\dagger\sigma_- + ab\sigma_+), & H_{3T} &= -i\hbar\Omega_{3T}(a^\dagger b^\dagger c^\dagger\sigma_- - abc\sigma_+). \end{aligned} \quad (1)$$

These Hamiltonians may be derived from the general dynamics of a trapped ion interacting with two lasers as follows. We consider an effective three-level ion trapped in a 3D RF Paul trap (Fig. 1). The ion is confined to a three-dimensional harmonic potential, characterized by ν_a , ν_b and ν_c , the trap's frequencies in the x , y and z directions respectively, and is illuminated by two classical external fields of frequencies ω_{12} and ω_{23} respectively (Fig. 2). These are chosen far detuned from the excited state $|2\rangle$, thus generating a stimulated Λ Raman transition between the two states $|1\rangle$ and $|3\rangle$, as depicted in Fig. 3. In this far-detuned case, we can adiabatically eliminate the excited state $|2\rangle$ from the relevant dynamics [1]. After doing so, in the interaction picture we perform the rotating wave approximation to obtain, for the interaction Hamiltonian,

$$H^i = -\hbar\Omega_{13}e^{-\frac{1}{2}(\eta_x^2 + \eta_y^2 + \eta_z^2)} \sum_{m,\mu,n,\nu,l,\lambda=0}^{\infty} \frac{(i\eta_x)^{m+\mu}}{m!\mu!} \frac{(i\eta_y)^{n+\nu}}{n!\nu!} \frac{(i\eta_z)^{l+\lambda}}{l!\lambda!} a^\dagger m a^\mu b^\dagger n b^\nu c^\dagger l c^\lambda e^{i(\nu_a(m-\mu) + \nu_b(n-\nu) + \nu_c(l-\lambda) + \Delta_{13})t} \sigma_{13} + \text{H.c.}, \quad (2)$$

where a, b, c ($a^\dagger, b^\dagger, c^\dagger$) are the harmonic oscillator annihilation (creation) operators in the x, y, z directions, respectively and $\sigma_{13} = |1\rangle\langle 3|$ is the electronic state transition operator between levels $|3\rangle \rightarrow |1\rangle$. Ω_{13} is the new Raman coupling between levels $|1\rangle$ and $|3\rangle$, found after the adiabatic elimination of level $|2\rangle$. The laser detunings are defined as $\Delta_{12} = (\omega_2 - \omega_1) - \omega_{12}$ and $\Delta_{23} = (\omega_2 - \omega_3) - \omega_{23}$. The Lamb-Dicke parameters are given by $\eta_i = \Delta k_i \sqrt{\hbar/2\nu_i m}$, ($i = x, y, z$), with m the mass of the ion and Δk_i the i -component of the difference between the lasers' wave vectors $\mathbf{k}_1 - \mathbf{k}_2$. We have also defined the detuning $\Delta_{13} = \omega_{12} - \omega_{23} - (\tilde{\omega}_3 - \tilde{\omega}_1)$, where $\tilde{\omega}_i$ is the Stark-shifted frequency associated with the electronic state $|i\rangle$.

By choosing suitable frequencies of the lasers we can select stationary terms in Eq. 2, thus engineering the specific Hamiltonians required for our purpose. Indeed, by choosing the laser frequencies as $\omega_{12} = \omega_c - \tilde{\omega}_1 - \nu_a - \nu_b - \nu_c$ and $\omega_{23} = \omega_c - \tilde{\omega}_3$, where ω_c is an arbitrary reference frequency, we select only the resonant terms, that is, those for which the argument of the exponential in Eq. 2 vanishes. In the Lamb-Dicke limit, keeping the lowest-order terms in η_x, η_y and η_z , the Hamiltonian coupling the x, y and z motion of the ion to the electronic state can be written as

$$H_{3T} = i\hbar\Omega_{3T}(a^\dagger b^\dagger c^\dagger \sigma_{13} - abc\sigma_{31}) \quad \text{with} \quad \Omega_{3T} \equiv |\Omega_{13}|e^{-\frac{1}{2}(\eta_x^2 + \eta_y^2 + \eta_z^2)}\eta_x\eta_y\eta_z. \quad (3)$$

H_{1C} , H_{2C} and H_{2T} can be derived in a similar fashion, thus achieving our objective of generating the Hamiltonians in 1.

If we define the operation $U_{iC}(\Omega_{iC}t)$ as shorthand for the application of the time evolution of H_{iC} via suitably-detuned laser pulses (and similarly for the T subscript hamiltonians) during an interval $\Omega_{iC}t$, we can summarize the protocol for the C-NOT gate by the sequential application of $U_{2C}(\pi/2)$, $U_{1C}(\pi/2)$, $U_{2C}(\pi/2)$ to the state vector. This realization uses two vibrational qubits as control and target for the gate. Similarly, the Toffoli gate protocol requires the sequential application of $U_{3T}(\pi/2)$, $U_{2T}(\pi/2)$, $U_{3T}(\pi/2)$, and uses three vibrational qubits, two for the controls and one for the target qubit. We are thus able to perform both C-NOT and Toffoli operations on a single trapped ion, thus paving the way for the Stochastic Programmable Singe-Qubit Processor.

Stochastic Programmable Singe-Qubit Processor

Quantum information may be processed by means of unitary transformations acting on qubits. These transformations are usually implemented with fixed *quantum gate arrays*. Instead of building a different gate array for each required operation, it is possible to build a fixed gate array that takes as inputs not only *data qubits*, but also *program qubits*, defining the operation itself. Unfortunately it is not possible to build a fixed and general-purpose quantum processor which can be programmed to perform an arbitrary quantum computation, since its operation must be necessarily stochastic in nature, as was shown by Nielsen et al.[2]. As an attempt at sidestepping this limitation, Vidal et al. have proposed in Ref. [3] a stochastic programmable gate with a probability of failure $\epsilon = 2^{-N}$, with the number N of *program* qubits that store the unitary transformation. They take as a particular case the one-qubit unitary operation

$$U_\alpha = \exp(i\alpha\sigma_z/2), \quad (4)$$

which corresponds to an α -rotation about the z axis of the Bloch sphere of the data qubit. In this concrete realization of their scheme, using a C-NOT gate and a Toffoli gate, we will limit ourselves to describing the unitary operation U_α using $N = 2$ qubits.

We start by defining the program \mathcal{H}_P and data \mathcal{H}_D Hilbert spaces. Initially, the system is in the state $|d\rangle_D \otimes |\mathcal{P}_U\rangle_P$, where $|d\rangle_D \in \mathcal{H}_D$ and $|\mathcal{P}_U\rangle_P \in \mathcal{H}_P$. The total dynamics of the programmable gate array are described by a fixed unitary operator G , which implements the desired unitary operation U given the program state $|\mathcal{P}_U\rangle_P$, that is,

$$G[|d\rangle_D \otimes |\mathcal{P}_U\rangle_P] = (U|d\rangle_D) \otimes |\mathcal{R}_U\rangle_P, \quad (5)$$

where $|\mathcal{R}_U\rangle_D$ is a residual state shown to be independent of the data state. After application of G , the data state $|d\rangle_D$ has been transformed by the unitary operation U into $U|d\rangle_D$.

To understand the procedure, we will consider the case of a single program register. Let us first define the program $|\alpha\rangle_P$ and data $|d\rangle_D$ states as

$$|\alpha\rangle_P \equiv (e^{i\alpha/2}|0\rangle_P + e^{-i\alpha/2}|1\rangle_P)/\sqrt{2} \quad \text{and} \quad |d\rangle_D \equiv (A|0\rangle_D + B|1\rangle_D)/\sqrt{2}. \quad (6)$$

The operation G that realizes the transformation in Eq. 5 is easily shown to be a C-NOT gate. Indeed, If we represent the C-NOT gate as $G_{\text{C-NOT}}$, with $\sigma_x = (|1\rangle\langle 0| + |0\rangle\langle 1|)_P$, where the data register is the control qubit and the program register is the target qubit, it follows that $\text{C-NOT}[|d\rangle_D|\alpha\rangle_P] = \frac{1}{\sqrt{2}}(U_\alpha|d\rangle_D \otimes |0\rangle_P + U_\alpha^\dagger|d\rangle_D \otimes |1\rangle_P)$. In this case, a measurement on the program register will cause a collapse of the data qubit with outcome $U_\alpha|d\rangle_D$ or $U_\alpha^\dagger|d\rangle_D$, both with probability $p = 1/2$.

To improve upon this scheme, we introduce an additional Toffoli gate, as in Fig. 5. The processor now uses two Hilbert spaces to store the program, \mathcal{H}_{P1} and \mathcal{H}_{P2} . When the output of the C-NOT on the first program register line is $|0\rangle_{P1}$, corresponding to the case where U_α was applied to the data (success), the output on the data line is unchanged. On the other hand, if the output on the first program register line is $|1\rangle_{P1}$, indicating an application of U_α^\dagger (failure), the Toffoli gate effectively acts as a C-NOT gate between the data register line and the second program register line. In this way, in case of failure, we have a chance to correct it. If we select the second program qubit as $|2\alpha\rangle_{P2}$ there is again a one-half chance that, upon performing a measurement on the outcome of both program register lines, $U_{2\alpha}U_\alpha^\dagger = U_\alpha$ will have been applied to $|d\rangle_D$. Thus the total probability of success is $p = 1/2 + 1/4 = 3/4$. The other alternative (the application of $U_\alpha^{\dagger 3}$) occurs with probability $p = 1/4$ and corresponds to the case when one obtains 1 as the final outcome from both program register lines. This is schematically illustrated in Fig 6.

Summary

We show that it is possible to create a Toffoli gate acting on the trapped ion's three center-of-mass vibrational qubits. We make use of this quantum gate in implementing a single-ion stochastic quantum processor that consists of a C-NOT and a Toffoli gate. The C-NOT gate is implemented with two-dimensional vibrational qubits of the ion's center-of-mass motion. Control and coupling to the ion's internal electronic states is achieved via far-detuned lasers exciting a Raman Λ transition. We then implement a second-order scheme for a stochastic one-qubit processor proposed by Vidal et al.[3] The one-qubit processor implements with success probability $p = 3/4$ the unitary operation $U_\alpha = \exp(i\alpha\sigma_z/2)$. corresponding to a rotation for an arbitrary angle $\alpha \in [0, 2\pi)$ around the z axis of the Bloch sphere of the data qubit. The relevant property of this programmable processor is that the unitary operation U desired is specified by input *program states* and not by altering the processor itself. This particular implementation uses a two-qubit program register (states $|\alpha\rangle$ and $|2\alpha\rangle$) and a single-qubit data register (state $|d\rangle$).

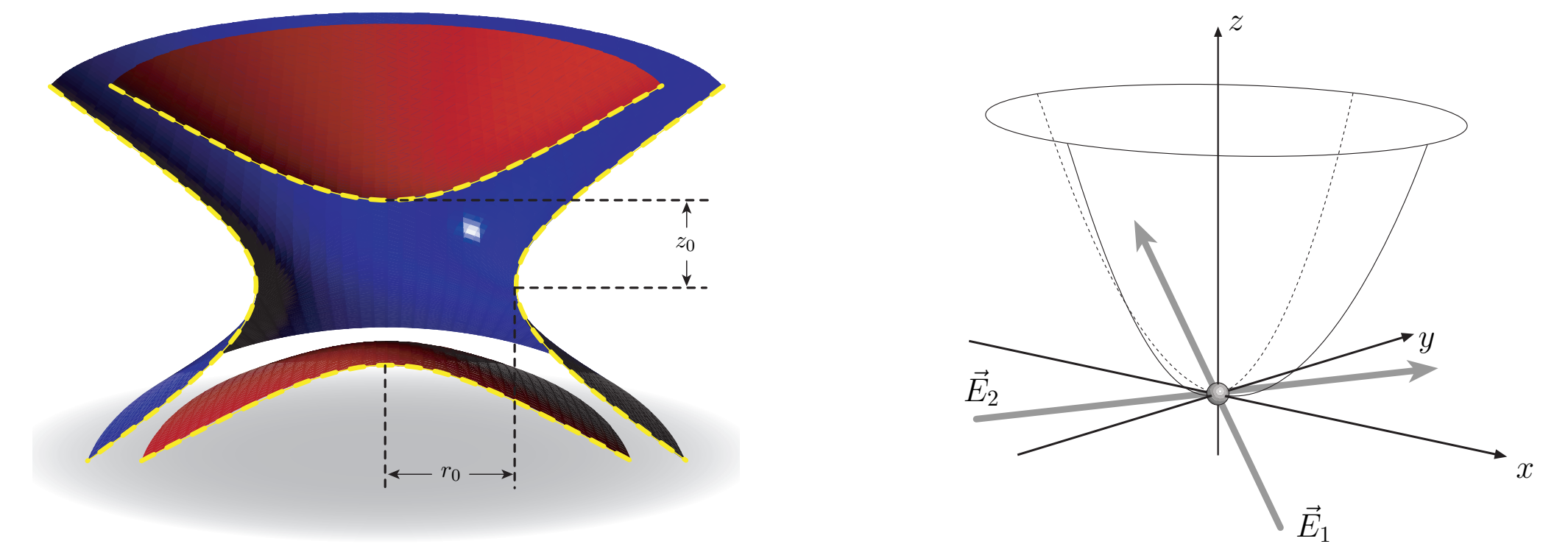


FIGURE 1: (left) 3D RF Paul Trap. An RF field is applied between the hyperboloidal endcaps (red) and ring (blue). A charged particle will experience, to a good approximation, a three-dimensional harmonic potential. FIGURE 2: (right) An ion in a 3D harmonic potential well is illuminated with two lasers.

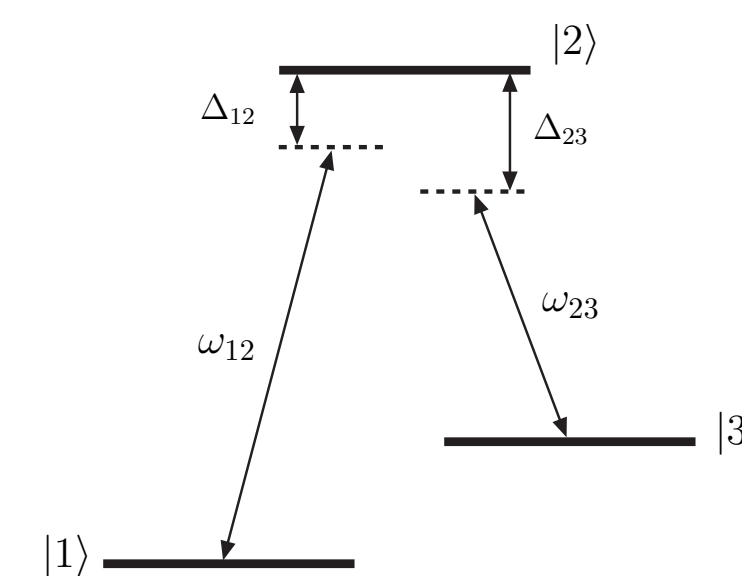


FIGURE 3: Energy level scheme for our system. Classical lasers, of frequency ω_{12} and ω_{23} , detuned from resonance by Δ_{12} and Δ_{23} respectively, induce a Λ Raman transition, coupling its three-dimensional center of mass movement to the ion's internal electronic state.

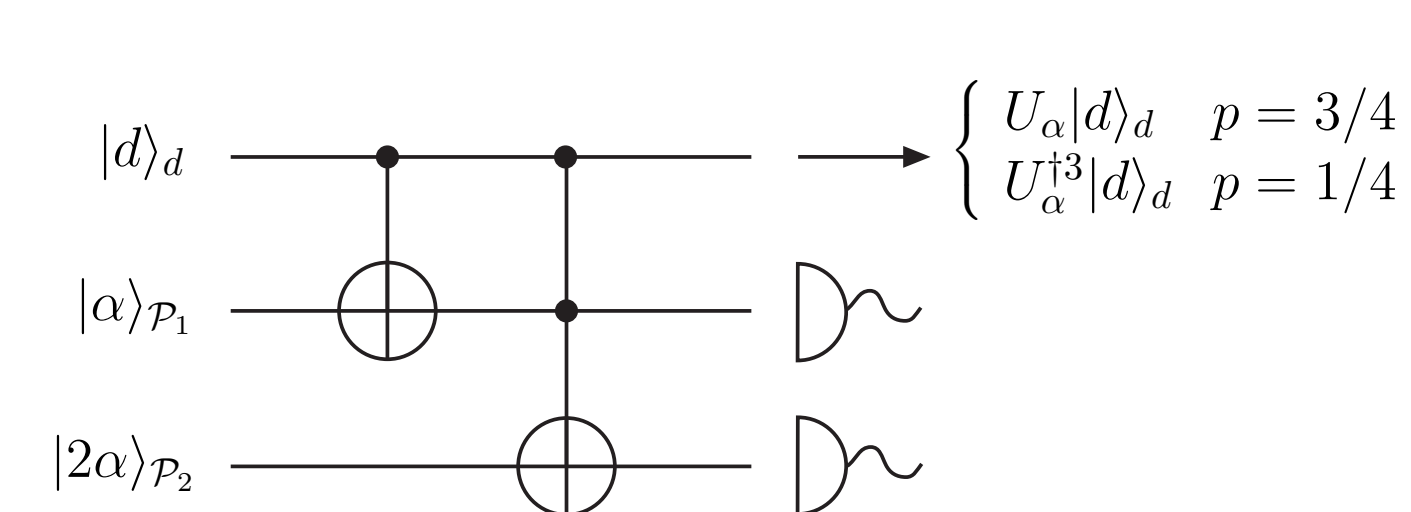
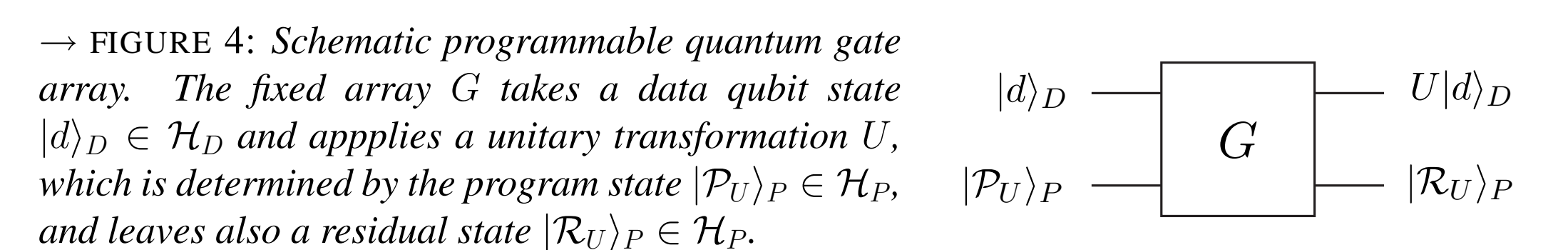


FIGURE 5: One-qubit stochastic quantum processor. It takes, as input, the data state $|d\rangle_D$ and program states $|\alpha\rangle_{P1}$ and $|2\alpha\rangle_{P2}$. An unsuccessful operation will be detected (with probability $p = 1/4$) if

the program registers are measured to be $|1\rangle_{P1}$ y $|1\rangle_{P2}$. In any other case, a successful application of the desired transformation will be detected, and the data state will now be $U_\alpha|d\rangle_D$.

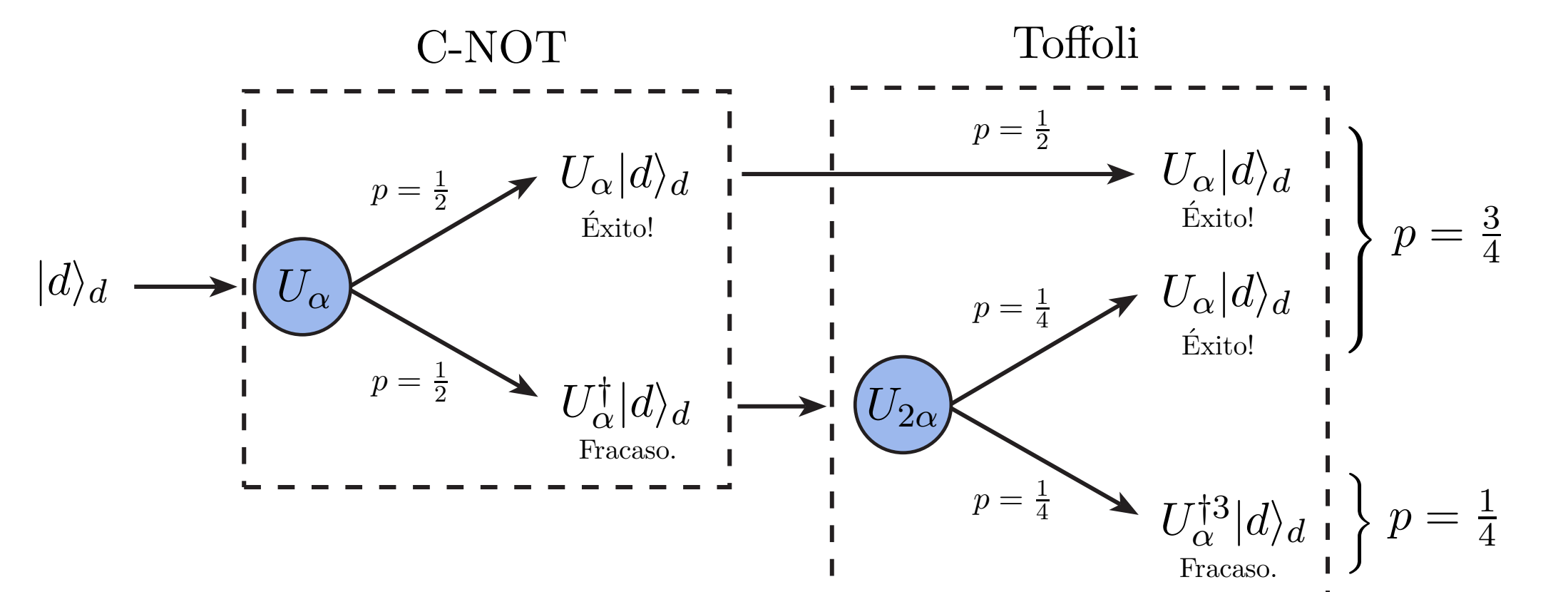


FIGURE 6: Schematic processor operation. (Note: Fracaso = Failure, Éxito=Success).

References

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