1. (12 points) Let $P_2(\mathbb{R})$ be the vector space of all polynomials of degree at most 2 with real coefficients and let

$$<,> : P_2(\mathbb{R}) \times P_2(\mathbb{R}) \to \mathbb{R}, \quad <P(t), Q(t)> := \int_{-1}^{1} (1-t)P(t)Q(t) \, dt$$

(a) Show that $<,>$ is an inner product on $P_2(\mathbb{R})$.

(b) Find a basis for $P_2(\mathbb{R})$ that is orthogonal with respect to $<,>$.

(c) Let $U := \text{span}\{1, 3t - 1\}$. Compute the orthogonal complement $U^\perp$ of $U$ in $P_2(\mathbb{R})$. Justify!

2. (12 points) Let $f(x) = \begin{cases} 0 & \text{for } -\pi < x \leq 0 \\ 1 & \text{for } 0 < x \leq \pi \end{cases}$ extended periodically to a function of period $2\pi$ with domain $\mathbb{R}$.

(a) Compute the Fourier series of $f(x)$. For what values of $x$ does the Fourier series represent $f(x)$? Justify!

(b) Prove the Leibniz formula $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$ by evaluating the Fourier series above at $x = -\frac{\pi}{2}$.

3. (10 points) Let $P \in O(n)$ i.e. let $P$ be an $n \times n$ orthogonal matrix with real coefficients. Prove the following:

(a) $\det P = \pm 1$.

(b) All eigenvalues of $P$ have absolute value 1. (Keep in mind that the eigenvalues may not be real!)

Hint: For an eigenvector $x$ of $P$ consider $||Px||^2$.

4. (16 points) Let $A$ be an $n \times n$ matrix with complex coefficients which is idempotent i.e. $A^2 = A$.

(a) Prove that if $A$ is invertible then $A$ is the identity matrix.

(b) Let $x \in \mathbb{C}^n$ be arbitrary. Show that

(i) the vector $Ax$ is invariant under $A$ i.e. that $A(Ax) = Ax$.

(ii) $x - Ax$ is contained in the nullspace of $A$.

(c) Use the decomposition $x = (x - Ax) + Ax$ to show that $\mathbb{C}^n = E_0(A) \oplus E_1(A)$ i.e. that $\mathbb{C}^n$ is the direct sum of the eigenspaces of $A$ with respect to the eigenvalues 0 and 1. (Note that this implies especially that 0 and 1 are the only possible eigenvalues of $A$.)

(d) Use (c) to prove that $A$ is diagonalizable.

(e) Show that $\text{rank}(A) = \text{trace}(A)$.

—Please turn over!—
5. (18 points) Let \( A = \begin{pmatrix} 0 & -2 & -4 \\ -4 & 2 & 8 \\ 2 & -2 & -6 \end{pmatrix} \).

(a) Compute the eigenvalues and eigenspaces of \( A \).

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( D = P^{-1}AP \).

(c) Compute the minimal polynomial of \( A \).

(d) Use (c) to explicitly compute \( A^n \) for all \( n \in \mathbb{N} \).

(e) Compute the general solution of the differential equation \( \dot{x} = Ax \).

(f) Show that the solutions \( x(t) \) of \( \dot{x} = Ax \) remain bounded as \( t \to \infty \).

(g) Justify in each case whether \( A \) is similar to one of the following matrices:

(i) \( M_1 = \begin{pmatrix} 0 & -2 & -4 \\ -4 & 2 & 8 \\ 2 & -2 & -7 \end{pmatrix} \)  

(ii) \( M_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \)  

(iii) \( M_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

6. (12 points) Consider the linear recurrence relation \( a_{n+2} = -5a_{n+1} + 6a_n \).

(a) Solve the recurrence relation for the initial conditions \( a_0 = 1, a_1 = -6 \).

(b) Show that if \( a_n > 0 \) for all \( n \in \mathbb{N}_0 \) then \( a_n = a_0 \) for all \( n \in \mathbb{N}_0 \) i.e. \( (a_n)_{n\in\mathbb{N}_0} \) is constant.

7. **Bonus Question** (5 points) Prove the Cayley-Hamilton theorem for diagonalizable matrices i.e. prove that if \( A \) is a diagonalizable matrix, then \( \chi_A(A) \) is the zero matrix.  

**Hint:** Prove this first for a diagonal matrix \( D \).