McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 317

NUMERICAL ANALYSIS

Examiner: Professor A. Humphries
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Date: Wednesday December 14, 2005
Time: 9:00AM - 12:00PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.
2. All questions carry equal weight.
3. Credit will be given for **6** best answers
4. This is a closed book exam.
5. Notes are not permitted.
6. Non-Programmable calculators are permitted.
7. This exam comprises the cover page, and 3 pages of 8 questions.
8. EXAM IS PRINTED DOUBLE-SIDED

1. (a) State the "Fixed Point Theorem," which gives sufficient conditions for an iteration $x_{n+1} = g(x_n)$ to converge to a fixed point.

   (b) Find an interval and a starting point x_0 on which the iterative scheme to find $\sqrt{2}$;
$$x_{n+1} = x_n - \frac{1}{3}(x_n^2 - 2),$$
   satisfies the conditions of the theorem. What is the rate of convergence of the iterative scheme?

   (c) Aitken's Δ^2 method to speed up convergence of a sequence $\{x_n\}$ can be written as
$$\hat{x}_n = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}.$$
   Find the iterates of *Steffensen's method* for the problem in (b), up to and including the second application of Aitken's Δ^2 formula, using a suitable starting point.

2. (a) Let $f(x)$ be $n+1$ times continuously differentiable on $[a, b]$ and $x_0, x_1, \dots, x_n$ be distinct interpolation points in $[a, b]$. Define the fundamental Lagrange polynomials $l_0(x), l_1(x), \dots, l_n(x)$ for the interpolation points and show that
$$p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x)$$
   interpolates f at $x_0, x_1, \dots, x_n$.

   (b) Show that $p_n(x)$ is the unique interpolating polynomial of degree n.

   (c) Suppose that $n = 3$
$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3,$$
$$f(x_0) = 0, \quad f(x_1) = 0, \quad f(x_2) = 4, \quad f(x_3) = 6.$$
   Find $p_3(x)$ and evaluate $p_3(2.5)$. Find a bound for the error in this approximation of $f(2.5)$, when $\max_{x\in[0,3]} |f^{(4)}(x)| \leqslant 10$, using the error formula
$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

3. (a) What is the key difference between Lagrange and Hermite interpolants? What is the difference between a clamped and a natural cubic spline?

   (b) A natural cubic spline S on $[0, 2]$ has the formula
$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \leqslant x < 1 \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \leqslant x \leqslant 2. \end{cases}$$
   Find a, b, c, d.

   (c) A cubic Bezier curve $\mathbf{B}(t)$ has end points $\mathbf{b}_0 = (0, 0)$ and $\mathbf{b}_3 = (1, 0)$ and guide points $\mathbf{b}_1 = (0, 1/2)$ and $\mathbf{b}_2 = (1, 1/2)$. What is the role of the guide points and what properties does the curve have with respect to the four given vectors? State the formula of the curve $\mathbf{B}(t)$.

4. (a) Using the formula for roots of a quadratic equation and 3-digit decimal chopping compute approximations to the roots of $x^2 - 1000x - 1 = 0$. Organise your calculations so as to minimise the effect of the errors.

   (b) Consider the centered-difference expression for approximating $f''(x_0)$:

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - \frac{h^2}{12}f^{(4)}(\xi), \qquad x_0 - h < \xi < x_0 + h.$$

   Suppose $|f^{(4)}(x)| \leqslant M$ for all $x \in [x_0 - h, x_0 + h]$, and that $h > 0$. If we encounter roundoff errors δ_1, δ_2 in computing $f(x_0+h), f(x_0-h)$ respectively, and $|\delta_1|, |\delta_2| < \delta$, find an upper bound on the total error in the approximation. Determine the value of h which minimises this bound when $M = 150$ and $\delta = 10^{-10}$, and state the bound.

5. (a) Define the degree of accuracy (also known as the degree of precision) of a quadrature formula $I_h(f)$ for approximating the integral

$$I(f) = \int_a^b f(x)dx.$$

   (b) Find constants α, β and γ such that the degree of accuracy of the quadrature formula

$$I_h(f) = h[\alpha f(a) + \beta f(a + \gamma h)]$$

   is as large as possible, where $h = (b - a)$.

   (c) What is the degree of accuracy p of the method in (b)? Given that $I(f) = I_h(f) + kh^{p+2}f^{(p+1)}(\xi)$, find k.

6. (a) Let $I_h(f)$ be the Composite Trapezoidal Rule approximation to

$$I(f) = \int_0^1 e^{x^2} dx.$$

   Evaluate $I_h(f)$ when $h = 0.5$ and when $h = 0.25$.

   (b) Derive the error bound

$$I(f) - I_h(f) = -\frac{(b-a)}{12}h^2 f''(\xi)$$

   for some $\xi \in [a, b]$, for the Composite Trapezoidal rule, from the error bound for the Trapezoidal rule.

   (c) Use the error bound in (b) to obtain upper bounds on the errors for the approximations in (a).

   (d) Apply one-step of Richardson extrapolation to the approximations in (a), to find a better approximation to $I(f)$.

7. Consider the initial value problem

$$y' = f(y), \qquad 0 \leqslant t \leqslant T, \quad y(0) = \alpha.$$

Suppose you approximate the solution $y(t)$ using the Runge-Kutta method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + hf(w_i + \tfrac{h}{2} f(w_i)), \quad i = 0, \ldots N \end{aligned}$$

with time-step $h > 0$.

(a) Define the local truncation error $\tau_{i+1}(h)$ and use it to determine the order of this method.

(b) Consider the case where

$$f(y) = \lambda y, \quad \lambda < 0,$$

and

i. show that $w_{i+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_i$.

ii. Under what conditions on h does $\lim_{i\to\infty} w_i = 0$?

8. (a) State sufficient conditions on $p(t)$, $q(t)$, $r(t)$, to ensure that the boundary value problem

$$y'' = p(t)y' + q(t)y + r(t), \qquad a \leqslant t \leqslant b, \qquad y(a) = \alpha, \quad y(b) = \beta,$$

has a unique solution.

(b) Use the linear shooting method to approximate the solution $y(0.5)$ of the boundary value problem

$$y'' = -2y' + ty + 3, \qquad 1 \leqslant t \leqslant 2, \qquad y(1) = 1, \quad y(2) = 2,$$

using $h = \frac{1}{2}$, and the (Forward) Euler method.