McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 317

NUMERICAL ANALYSIS

Examiner: Professor A. Humphries
Associate Examiner: Professor N. Nigam

Date: Thursday December 9, 2004
Time: 2:00 P.M – 5:00 P.M

INSTRUCTIONS

1. All questions carry equal weight
2. Credit will be given for **6** best answers
3. **Non-Programmable** calculators are permitted
4. This is a closed book exam
5. Notes are not permitted
6. Please answer all questions in the examination booklet
7. This exam consists of the cover page and 4 pages of 8 questions

1. (a) Use the Taylor expansion

$$f(x^*) = f(x) + (x^* - x)f'(x) + \frac{(x^* - x)^2}{2} f''(\xi)$$

for $f \in C^2[a, b]$ to derive Newton's method for approximating a root x^* of the equation $f(x) = 0$.

(b) Show that Newton's method can be written as a fixed point iteration

$$x_{n+1} = g(x_n),$$

for a suitable choice of $g(x)$.

(c) Furthermore, show that $g'(x^*) = 0$ provided that $f'(x^*) \neq 0$.

(d) Find $\lim_{x \to x^*} g'(x)$ for Newton's method when $f'(x^*) = 0$ but $f''(x^*) \neq 0$.

(e) The root $x^* = 5$ of

$$f(x) = x^3 - 9x^2 + 15x + 25$$

is approximated using Newton's method with $x_0 = 3$.

i. Compute x_5.

ii. What is the apparent rate of convergence?

2. Let $g(x)$ be a continuous function on the real line that maps the interval $[a, b]$ into itself, such that

$$|g(x) - g(y)| \leqslant \gamma |x - y| \qquad \forall x, y \in (-\infty, \infty)$$

where $\gamma < 1$, and suppose the sequence $\{x_n\}_{n=0}^{\infty}$ is generated iteratively via $x_{n+1} = g(x_n)$.

(a) Show that there exists $x^* \in [a, b]$ such that $x^* = g(x^*)$.

(b) Show that x^* is the unique real fixed point of g.

(c) Show that $|x_{j+1} - x^*| \leqslant \gamma |x_j - x^*|$, and deduce that $x^* = \lim_{n \to \infty} x_n$.

(d) The usual "Fixed Point Theorem" makes one additional assumption on g that we did not make here; what is it? Hence give an example of a function for which the usual fixed point theorem does not apply, but for which the assumptions of this question imply existence of and convergence to a fixed point.

3. Assume that $x_0 < x_1 < \ldots < x_n$. Then divided differences can be defined recursively using the formula

$$f[x_i, x_{i+1}, \ldots, x_{i+j}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+j}] - f[x_i, x_{i+1}, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}.$$

   (a) Define the zeroth divided differences $f[x_j]$ for $j = 0, 1, \ldots, n$.

   (b) Let

$$p_n(x) = \sum_{j=0}^{n} c_j w_j(x)$$

   be the Newton form of the interpolating polynomial based on $x_0, x_1, \ldots, x_n$. Define the polynomials $w_j(x)$ for $j = 0, 1, \ldots, n$. State c_j in terms of the appropriate divided difference(s) for $j = 0, 1, \ldots, n$.

   (c) Assuming that f is n-times differentiable, show that there exists $\xi \in [x_0, x_n]$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

   (d) Given

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3,$$
$$f(x_0) = 3, \quad f(x_1) = 5, \quad f(x_2) = 9, \quad f(x_3) = 17,$$

   construct the appropriate table of divided differences and hence state

      i. the polynomial of degree 3 which interpolates at x_0, x_1, x_2, x_3.

      ii. the polynomial of degree 2 which interpolates at x_1, x_2, x_3.

4. Consider the Forward Difference Approximation

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h},$$

   and the data

x	0	0.1	0.2
$f(x)$	0	0.090483	0.163746

   (a) Use the forward difference formula with $h = 0.1$ and $h = 0.2$ to obtain two approximations to $f'(0)$.

   (b) Using Taylor Series or otherwise find the error term for the forward difference approximation.

   (c) Use Richardson extrapolation to obtain a better approximation to $f'(0)$.

   (d) State a finite formula for approximating $f''(x_0)$, and use it to obtain an approximation to $f''(0.1)$.

5. (a) Define the degree of accuracy (also known as the degree of precision) of a quadrature formula $I_h(f)$ for approximating the integral

$$I(f) = \int_a^b f(x)dx.$$

(b) Find the degree of accuracy of the quadrature formula

$$I_h(f) = \frac{3h}{4}[f(a) + 3f(a+2h)]$$

where $h = (b-a)/3$.

(c) Given that $I(f) = I_h(f) + kh^{p+2}f^{(p+1)}(\xi)$, where p is the degree of accuracy of the method find k.

(d) Use this method and the fact that

$$\ln(2) = \int_1^2 \frac{1}{x}dx$$

to obtain an approximation to $\ln(2)$. Use the error formula from above to derive an upper bound for the error in this approximation.

6. Simpson's Rule $J_h(f) = (h/3)[f_0 + 4f_1 + f_2]$, where $f_i = f(x_i)$ for approximating $I(f) = \int_{x_0}^{x_2} f(x)dx$ has the error formula

$$I(f) - J_h(f) = -\frac{h^5}{90}f^{(4)}(\eta)$$

where $\eta \in [x_0, x_2]$.

(a) Let n be even, $x_0 = a$, $x_n = b$, $h = (b-a)/n$ and $x_j = a + jh$. Derive the Composite Simpson's Rule for approximating $I(f) = \int_a^b f(x)dx$ by applying Simpson's Rule to appropriate subintervals.

(b) Assuming that $f \in C^4[a,b]$ prove that the Composite Simpson's Rule satisfies

$$I(f) - I_h(f) = -\frac{(b-a)}{180}h^4 f^{(4)}(\xi)$$

for some $\xi \in [a,b]$.

(c) Let $I_h(f)$ be the Composite Simpson's Rule approximation to

$$I(f) = \int_0^1 x^3 + \sin(\pi x)dx.$$

Show that $I(f) \leqslant I_h(f)$ and give an upper bound on $|I(f) - I_h(f)|$ when $h = 1/10$. What value of h is required to ensure that $|I(f) - I_h(f)| \leqslant 10^{-4}$?

7. Consider the initial value problem

$$y' = f(t, y) = \lambda y, \qquad 0 \leqslant t \leqslant T, \quad y(0) = \alpha > 0, \quad \lambda < 0.$$

Suppose you approximate the solution $y(t)$ using the Runge-Kutta method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \tfrac{1}{4}hf(t_i, w_i) + \tfrac{3}{4}hf\left(t_i + \tfrac{2}{3}h, w_i + \tfrac{2}{3}hf(t_i, w_i)\right), \quad i = 0, \ldots N$$

with time-step h.

   (a) Show that $y(t_{i+1}) = e^{h\lambda}y(t_i)$,

   (b) and that $w_{i+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_i$.

   (c) Under what conditions on h does $\lim_{i\to\infty} w_i = 0$?

   (d) Define the local truncation error $\tau_{i+1}(h)$ and show that for this problem

$$\tau_{i+1}(h) = \frac{1}{6}(\xi\lambda)^3 w_i,$$

   where $\xi \in (0, h)$ and hence that $0 < y(t_i) < w_i$ for all $i > 0$.

8. Use the linear shooting method to approximate the solution $y'(1)$ of the boundary value problem

$$y'' = -3y' + 2y + 2t + 3, \qquad 0 \leqslant t \leqslant 1, \qquad y(0) = 2, \quad y(1) = 1,$$

using $h = \frac{1}{2}$, and the (Forward) Euler method.