#1 $\lim\limits_{x\to-\infty} \dfrac{\ln(6e^{12x}+6x^2)}{\ln(2e^{4x}+3x^2)}$ $\quad\dfrac{\infty}{\infty}$ $\quad$ (Note: $e^{-\infty}=0$)

$\overset{HR}{=} \lim\limits_{x\to-\infty} \dfrac{72e^{12x}+12x}{6e^{12x}+6x^2} \div \dfrac{8e^{4x}+6x}{2e^{4x}+3x^2}$ $\quad$ (since $e^{-\infty}=0$)

$= \lim\limits_{x\to-\infty} \dfrac{12x}{6x^2} \div \dfrac{6x}{3x^2} = \lim\limits_{x\to-\infty} \dfrac{2}{x} \div \dfrac{2}{x} = 1 \Rightarrow$ (a)

#2 $\lim\limits_{x\to0} \dfrac{1-\cos x}{x} \overset{HR}{=} \lim\limits_{x\to0} \dfrac{\sin x}{1} = 0 \Rightarrow$ (b)

#3 $\lim\limits_{x\to\infty} \dfrac{\sqrt{25+2x}-5}{x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x\left(\frac{25}{x}+2\right)}-5}{x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{2x}-5}{x}$

$\overset{HR}{=} \lim\limits_{x\to\infty} \dfrac{\frac{2}{2\sqrt{2x}}}{1} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{2x}} = 0 \Rightarrow$ (a) $\quad$ (Note: you can also multiply by the conjugate)

#4 at $x=-2$

$\lim\limits_{x\to-2^-} f(x) = -2+3 = 1$

$\lim\limits_{x\to-2^+} f(x) = -3-(-2) = -1$

$f(-2) = -3-(-2) = -1$

$\Rightarrow f$ NOT CONT at $x=-2$

at $x=5$

$\lim\limits_{x\to5^-} f(x) = -3-5 = -8$

$\lim\limits_{x\to5^+} f(x) = 2(5)-18 = -8$

$f(5) = -3-5 = -8$

f CONT at $x=5$

$\Rightarrow$ (a)

#5 "MVT": f cont, diff on $(a,b) \Rightarrow \exists\, c$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$f(x) = 4x^2+2x+3$ $\quad f'(x) = 8x+2$ $\quad$ MVT on $[3,X]$ gives:
(polyn $\Rightarrow$ DIFF on $\mathbb{R}$)

$$f'(x) = 38 = \frac{f(X)-f(3)}{X-3} \Rightarrow 4X^2+2X+3-45 = 38(X-3)$$

$$\Rightarrow 4X^2+2X-42 = 38X-114 \Rightarrow 4X^2-36X+72 = 0$$

$\Rightarrow 4(x^2-9x+18)=0$

$\Rightarrow 4(x-3)(x-6)=0 \Rightarrow \underset{\text{Rejected}}{x=3}$ or $\boxed{x=6}$

$\Rightarrow$ (e)

#6 "IVT": f cont $[a,b] \Rightarrow$ for every N between $f(a)$ & $f(b)$ there is at least one x_0 in (a,b) s.t. $f(x_0)=N$.

$f(x)$ polyn $\Rightarrow$ cont $\mathbb{R}$.

$f(-2) = 5(-2)^3+(-2)^2+(-2)+3 = 5(-8)+4-2+3 = -40+5 = -35$

$f(2) = 5(2)^3+2^2+2+3 = 40+9 = 49$

$\Rightarrow f(x_0) = -9$ (since -9 is the only nbr in $(-35,49)$) $\Rightarrow$ (e)

#7 <u>at $x=0$</u>

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{(3)\sin 6x}{(3)2x} = \lim_{x\to0^-} 3\,\frac{\sin 6x}{6x} = 3$$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} a\cos(4x) = a$$

$$f(0) = a\cos(0) = a$$

$\Rightarrow f$ cont at $x=0$ and thus everywhere else if $a=3 \Rightarrow$ (a)

#8 $f'(x) = 4x^3\ln(x^2+9) + x^4\cdot\dfrac{2x}{x^2+9}$

$$f'(2) = 4(8)\ln(13) + 16\cdot\frac{4}{13} = 32\ln(13)+\frac{64}{13} \Rightarrow \text{(e)}$$

#9 $$\lim_{x\to-\infty}\frac{12x^3+3x}{\sqrt{9x^6+2}} = \lim_{x\to-\infty}\frac{x^3\left(12+\frac{3}{x^2}\right)}{\sqrt{x^6\left(9+\frac{2}{x^6}\right)}} = \lim_{x\to-\infty}\frac{x^3\left(12+\frac{3}{x^2}\right)}{|x^3|\sqrt{9+\frac{2}{x^6}}}$$

$$= \lim_{x\to-\infty}\frac{x^3\left(12+\frac{3}{x^2}\right)}{-x^3\sqrt{9+\frac{2}{x^6}}} = \frac{12}{-\sqrt{9}} = -4$$

$$\lim_{x\to\infty} \frac{x^3\left(12+\frac{3}{x^2}\right)}{|x^3|\sqrt{9+\frac{2}{x^6}}} = \lim_{x\to\infty} \frac{x^3\left(12+\frac{3}{x^2}\right)}{x^3\sqrt{9+\frac{2}{x^6}}} = \frac{12}{\sqrt{9}} = 4$$

$\Rightarrow y=-4$ HA at $-\infty$ AND $y=4$ HA at ∞

$\Rightarrow$ 2 HA $\Rightarrow$ (d)

#10 $f(x) = \ln\left|\frac{x+1}{(x-3)(x-4)}\right|$

$\lim_{x\to3} f(x) = \ln(\infty) = \infty$ $\qquad$ $\lim_{x\to4} f(x) = \ln(\infty) = \infty$

$\lim_{x\to-1} f(x) = \ln(0^+) = -\infty$ $\Rightarrow$ $x = 3, 4, -1$ V.A

$\Rightarrow$ (d)

#11 Range of $\tanh(x)$ is $(-1, 1)$ $\Rightarrow$ (c)

#12 $y' = 6x^2+4$ at $x=-2$ $\qquad$ $y' = 6(4)+4 = 28$

$y = mx+b \Rightarrow -24 = 28(-2) + b \Rightarrow -24+56 = b \Rightarrow b = 32$

$y = 28x + 32$

$(2, 88)$ belongs to $y = 28x+32$ $\Rightarrow$ (b)

#13 $5\cos x - 2y'\sin y = 0 \Rightarrow y' = \frac{5\cos x}{2\sin y}$ at $x=0$ $\quad y' = \frac{5\cos(0)}{2\sin\left(\frac{\pi}{6}\right)} = 5$

$\Rightarrow$ (b)

#14 $f'(x) = 5\cos x + 2\sin x$

$f'(x) = 0 \Rightarrow 2\sin x = -5\cos x \Rightarrow \tan x = -\frac{5}{2} < 0$

$\Rightarrow$ angle NOT in $\left[0, \frac{\pi}{2}\right]$

$f(0) = 5\sin(0) - 2\cos(0) = -2$; $\quad f\left(\frac{\pi}{2}\right) = 5\sin\frac{\pi}{2} - 2\cos\frac{\pi}{2} = 5$

$\Rightarrow$ Abs max value is 5 $\Rightarrow$ (a)

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#15 $f'(x) = 4x^3 e^{-3x} + (1+x^4)(-3)e^{-3x}$
$= e^{-3x}(-3x^4 + 4x^3 - 3)$
$= 0$

either $e^{-3x} = 0$ (NO SOL.) OR $-3x^4 + 4x^3 - 3 = 0$ (NO SOL.)

$\Rightarrow$ NO MIN $\Rightarrow$ (e)

#16 $f'(x) = 3x^2 \ln x + x^3 \frac{1}{x} = 3x^2 \ln x + x^2$
$f''(x) = 6x \ln x + 3x^2 \frac{1}{x} + 2x = 6x \ln x + 5x$
$f''(1) = 6(1)(0) + 5(1) = 5 \Rightarrow$ (e)

#17 $f'(x) = 3x^2 - 6x - 189$
$f''(x) = 6x - 6$
$f''(x) = 0 \Rightarrow x = 1 \Rightarrow$ (b)

#18 $f'(x) = \frac{1}{1+x^{12}} \cdot 6x^5$ $\quad f'(1) = \frac{6}{1+1} = 3 \Rightarrow$ (a)

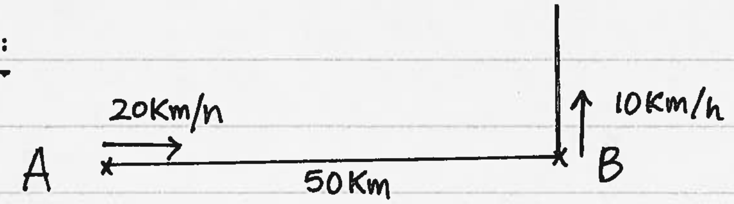
#19 $f(x) = \ln(x^6) - \ln(x^8+1) = 6\ln x - \ln(x^8+1)$
$f'(x) = \frac{6}{x} - \frac{8x^7}{x^8+1}$ $\quad f'(1) = \frac{6}{1} + \frac{8}{2} = 10 \Rightarrow$ (d)

#20 $f'(x) = 3e^{-x} + (3x-9)(-e^{-x}) = e^{-x}(3 - 3x + 9) = e^{-x}(-3x+12)$
$f''(x) = -e^{-x}(-3x+12) + e^{-x}(-3)$
$= e^{-x}(3x - 12 - 3) = e^{-x}(3x - 15)$

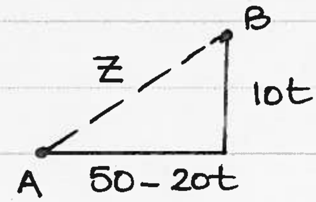
for $x < 5$ $f''(x)$ −ve ; $x > 5$ $f''(x)$ +ve $\Rightarrow$ (a)

PART B :

#21

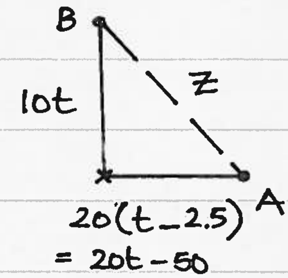


at noon.



If $0 \leq t \leq 2.5$ hours then at time t

$$\Rightarrow z = (50-20t)^2 + (10t)^2$$
$$= 2500 - 2000t + 500t^2$$



If $t > 2.5$ hours then at time t

$$\Rightarrow z = (10t)^2 + (20t-50)^2 = 500t^2 - 2000t + 2500$$

$\therefore \; z = 500t^2 - 2000t + 2500$

At 3 pm $\Rightarrow$ $t = 3$

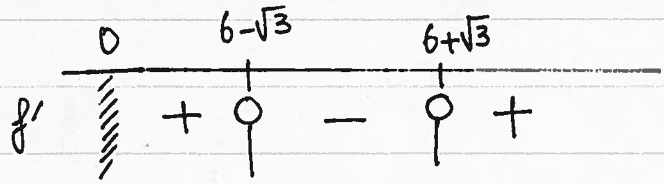
$\dfrac{dz}{dt} = 1000t - 2000$ at $t=3$ $\dfrac{dz}{dt} = 1000$ Km/hr.

#22 $f'(x) = 2x - 24 + \dfrac{18}{x}$

$f''(x) = 2 - \dfrac{18}{x^2}$

$f'(x) = 0$ $\dfrac{2x^2 - 24x + 18}{x} = 0 \Rightarrow x^2 - 12x + 9 = 0$

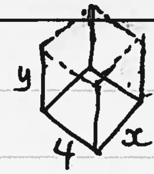
$\Rightarrow x = \dfrac{12 \pm \sqrt{144-36}}{2} = 6 \pm \sqrt{3}$



f decreasing on $(6-\sqrt{3}, 6+\sqrt{3})$

$f''(x) = 0 \Rightarrow \dfrac{2x^2 - 18}{x^2} = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow \underset{\text{rejected}}{x = -3}, \; \underset{\text{I.P.}}{\underline{\underline{x = 3}}}$

#23



base material $6\,\$/cm^2$

sides material $3\,\$/cm^2$

$V = 84\ cm^3$

$$V = 4xy = 84 \Rightarrow y = \frac{84}{4x} = \frac{21}{x}$$

Area of base $= 4x$ Cost of base $= 4x(6) = 24x\ \$$

Area of sides $= 4y + xy + 4y + xy$

$$= 8y + 2xy = y(8+2x) = \frac{21}{x}(8+2x) = \frac{168}{x} + 42$$

Cost of sides $= \left(\frac{168}{x} + 42\right)(3) = \frac{504}{x} + 126$

COST of the box $= 24x + \frac{504}{x} + 126 = C(x)$

$C'(x) = 24 - \frac{504}{x^2}$ $C'(x) = 0 \Rightarrow 24 = \frac{504}{x^2} \Rightarrow x^2 = \frac{504}{24} = 21$

$\Rightarrow x = \sqrt{21}$ cm.

$C''(x) = \frac{1008}{x^3}$ $C''(\sqrt{21}) = \frac{1008}{(\sqrt{21})^3} > 0 \Rightarrow x = \sqrt{21}$ cm is the most economical Length for the base.

#24 $x^2 - 5x - 6 = (x-6)(x+1)$

for $(-\infty,-1) \cup (6,\infty)$ $x^2 - 5x - 6 > 0$

for $(-1,6)$ $x^2 - 5x - 6 < 0$

-6 -1 3 6

$+\ 0\ -\ \ +$

$$f(x) = \begin{cases} x^2 - 5x - 6 + 2x^2 + 17x = 3x^2 + 12x - 6 & -6 \le x \le -1 \\ -x^2 + 5x + 6 + 2x^2 + 17x = x^2 + 22x + 6 & -1 \le x \le 3 \end{cases}$$

NOTE: at -1 we use both $=$ because $f(-1)$ is the same.

on $[-6,-1]$ $f'(x) = 6x + 12$ $f'(x) = 0 \Rightarrow x = -2$

-6	-1	-2	x
30	-15	-18	$f(x)$

on $[-1,3]$ $f'(x) = 2x + 22$ $f'(x) = 0 \Rightarrow x = -11$ rejected.

-1	3	x
-18	81	$f(x)$

Abs max = 81 at $x=3$

Abs min = -18 at $x=-1$

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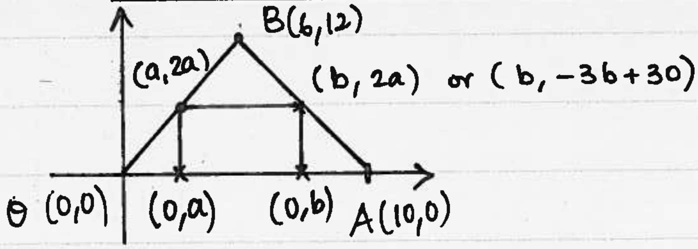
#25 $\lim_{x\to 1} \frac{(x-1)-x^6\ln x}{\ln x\,(x-1)} \overset{HR}{=} \lim_{x\to 1} \frac{1-6x^5\ln x - x^6\cdot\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}$

$= \lim_{x\to 1} \frac{1-6x^5\ln x - x^5}{x-1+x\ln x}\cdot\frac{x}{1} = \lim_{x\to 1}\frac{x-6x^6\ln x - x^6}{x-1+x\ln x}$

$\overset{HR}{=} \lim_{x\to 1} \frac{1-36x^5\ln x - 6x^6\cdot\frac{1}{x} - 6x^5}{1+\ln x + x\frac{1}{x}}$

$= \lim_{x\to 1}\frac{1-36x^5\ln x - 12x^5}{2+\ln x} = \frac{1-12}{2} = \frac{-11}{2}.$

#1



Equation of line (OB) : $m = \frac{12-0}{6-0} = 2 \Rightarrow y = 2x$

Equation of line (AB) : $m = \frac{12-0}{6-10} = -3$ $\qquad 0 = -3(10) + b \Rightarrow b = 30$

$y = -3x + 30$

We can now write that $2a = -3b + 30 \Rightarrow b = \frac{30-2a}{3} = 10 - \frac{2}{3}a$

The width of the rectangle is $b - a = 10 - \frac{2}{3}a - a = 10 - \frac{5}{3}a$

The height of the rectangle is $2a$

$\Rightarrow$ Area $= \left(10 - \frac{5}{3}a\right)(2a) = -\frac{10}{3}a^2 + 20a$ for $a \geq 0$

$\Rightarrow$ The question is now :

Find the Abs. max of $f(x) = 20x - \frac{10}{3}x^2$ for $x \geq 0$

$f'(x) = 20 - \frac{20}{3}x$ $\qquad f'(x) = 0 \Rightarrow x = 3$

$f''(x) = -\frac{20}{3} \Rightarrow f''(3) = -\frac{20}{3} < 0$

$\therefore$ $x = 3$ gives Abs max $f(x)$.

SOL: $a = 3$; $b = 10 - \frac{2}{3}(3) = 8$ $\Rightarrow$ width $= 8 - 3 = 5$ height $= 6$.

Area $= 30$.

#2 (a) $\lim_{x \to 1^+} \arctan\left(\frac{1}{0^-}\right) =$ "$\arctan(-\infty)$" $= -\frac{\pi}{2}$

(b) $\lim_{x \to 1} \frac{\ln x}{x-1} \overset{HR}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1.$

(c) $\lim\limits_{x\to 0} \dfrac{1-\cos x^2}{x^2 \sin x^2} \overset{HR}{=} \lim\limits_{x\to 0} \dfrac{2x \cdot \sin(x^2)}{2x \sin(x^2) + 2x^3 \cos(x^2)}$

$= \lim\limits_{x\to 0} \dfrac{\sin(x^2)}{\sin(x^2) + x^2\cos(x^2)} \overset{HR}{=} \lim\limits_{x\to 0} \dfrac{2x\cos(x^2)}{2x\cos(x^2) + 2x\cos(x^2) - 2x^3\sin(x^2)}$

$= \lim\limits_{x\to 0} \dfrac{2x\cos(x^2)}{4x\cos(x^2) - 2x^3\sin(x^2)} = \lim\limits_{x\to 0} \dfrac{\cos(x^2)}{2\cos(x^2) - x^2\sin(x^2)} = \dfrac{1}{2-0} = \dfrac{1}{2}$

(d) $\lim\limits_{x\to\infty} \dfrac{\sinh^2 x}{\ln(\cosh 2x)} \overset{HR}{=} \lim\limits_{x\to\infty} \dfrac{2\sinh(x)\cosh(x)}{\frac{1}{\cosh(2x)} \cdot \sinh(2x) \cdot 2}$

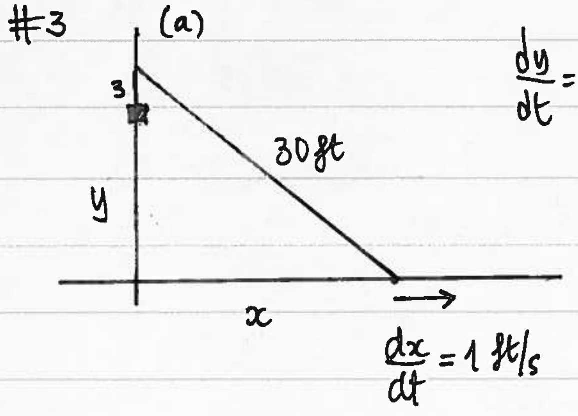
$= \lim\limits_{x\to\infty} \dfrac{\sinh(2x) \cdot \cosh(2x)}{2\sinh(2x)} = \lim\limits_{x\to\infty} \dfrac{\cosh(2x)}{2} = \infty.$

Note: $\sinh(2x) = 2\sinh x \cosh x$.

(e) 0^0 so let $y = x^{\frac{2}{1+\ln x}}$ $\quad \ln y = \dfrac{2}{1+\ln x} \cdot \ln x = \dfrac{2\ln x}{1+\ln x}$.

$\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{2\ln x}{1+\ln x} \overset{HR}{=} \lim\limits_{x\to 0^+} \dfrac{2/x}{1/x} = 2 \quad \therefore \lim\limits_{x\to 0^+} y = e^2.$

#3 (a)



$\dfrac{dy}{dt} = ?$ when $x = 10$

$(y+3)^2 + x^2 = 30^2$

(Derive) $\quad 2(y+3)\dfrac{dy}{dt} + 2x\dfrac{dx}{dt} = 0$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{-x\, dx/dt}{y+3}$

When $x = 10$

$(y+3)^2 + 10^2 = 30^2$

$(y+3)^2 = 800$

$y+3 = 20\sqrt{2} \Rightarrow y = 20\sqrt{2} - 3$

$= \dfrac{-10(1)}{20\sqrt{2} - 3 + 3} = \dfrac{-1}{2\sqrt{2}}$ ft/s.

(b) Note: a must be +ve.

(If a were −ve we would have $\infty + \infty = \infty \neq 0$)

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$$\lim_{x\to\infty} \sqrt{x^2-x+1} - (ax+b) \cdot \frac{\sqrt{x^2-x+1} + (ax+b)}{\sqrt{x^2-x+1} + (ax+b)} = 0$$

$$\lim_{x\to\infty} \frac{x^2-x+1-(ax+b)^2}{\sqrt{x^2(1-\frac{1}{x}+\frac{1}{x^2})}+ax+b} = 0$$

$$\lim_{x\to\infty} \frac{x^2-a^2x^2-x-2abx+1-b^2}{|x|\sqrt{1-\frac{1}{x}+\frac{1}{x^2}} + x\left(a+\frac{b}{x}\right)} = 0$$

Note : $x \to \infty$
$|x| = x$

$$\lim_{x\to\infty} \frac{(1-a^2)x^2-(1+2ab)x+(1-b^2)}{x\left(\sqrt{1-\frac{1}{x}+\frac{1}{x^2}} + a + \frac{b}{x}\right)} = 0$$

Since the limit $=0$
Degree of the top cannot be bigger than deg bottom
$\Rightarrow 1-a^2=0$
$\Rightarrow a^2=1 \Rightarrow \boxed{a=1}$
(a must be +ve)

$$\lim_{x\to\infty} \frac{-(1+2b)x+1-b^2}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+1+\frac{b}{x}} = 0$$

(Note $\frac{\#}{\infty} \to 0$)

$$\lim_{x\to\infty} \frac{-(1+2b)x+1-b^2}{\sqrt{1}+1} = 0 \Rightarrow \frac{-(1+2b)}{2} = 0 \Rightarrow b = -\frac{1}{2}.$$

(c) for $x<2$ $\quad f'(x) = c \Rightarrow f'(1) = c = 3 \quad \therefore \quad c=3$

$$f(x) = \begin{cases} 3x+K & x \leq 2 \\ x^2+bx & x>2 \end{cases}$$

for $x<2$ & $x>2$ f is a polynomial $\Rightarrow$ DIFF

at $x=2$ (Continuity)

$$\lim_{x\to 2^-} f(x) = 3(2)+K = 6+K$$

$$\lim_{x\to 2^+} f(x) = 2^2 + 2b = 4+2b$$

$$f(2) = 3(2) + K = 6+K$$

$\Rightarrow$ f cont at $x=2$ for $4+2b = 6+K \Rightarrow 2b - K = 2$

(or $K = 2b-2$)

Differentiability

$$* \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{3x+K-(6+K)}{x-2} = \lim_{x\to 2^-} \frac{3x-6}{x-2} = \lim_{x\to 2^-} \frac{3(x-2)}{(x-2)} = 3$$

$$* \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{x^2+bx-(6+K)}{x-2} = \lim_{x\to 2^+} \frac{x^2+bx-6-(2b-2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{x^2+bx-2b-4}{x-2} = \lim_{x\to 2^+} \frac{(x^2-4)+b(x-2)}{(x-2)} = \lim_{x\to 2^+} \frac{(x-2)(x+2)+b(x-2)}{(x-2)}$$

$$= \lim_{x\to 2^+} \frac{x+2+b}{1} = 4+b$$

$\therefore$ $f'(2)$ exists (f diff at 2) for $4+b=3 \Rightarrow b=-1$

so $K = 2b-2 = -4$.

#4(a) $f'(x) = \dfrac{2x}{1+x^2}$ $\quad f''(x) = \dfrac{2(1+x^2)-2x(2x)}{(1+x^2)^2} = \dfrac{2-2x^2}{(1+x^2)^2}$

$$= \frac{2(1-x)(1+x)}{(1+x^2)^2}$$

		-1		1	
f''	$-$	0	$+$	0	$-$

f concave down on $(-\infty,-1) \cup (1,\infty)$

(b) $\frac{d}{dx} f^{-1}(0) = \frac{1}{f'(f^{-1}(0))}$

$f'(x) = \frac{2\cosh(2x)(x+1) - \sinh(2x)}{(x+1)^2}$ AND $f^{-1}(0) = 0$ since $f(0) = 0$.

so $f'(0) = \frac{2(1)(2) - 0}{1^2} = 4$

$\Rightarrow \frac{d}{dx} f^{-1}(0) = \frac{1}{4}$.

(c) $F(t) = 2\arctan(t) + C$

$F(0) = -5 \Rightarrow 2\arctan 0 + C = -5 \Rightarrow C = -5$

$\Rightarrow F(t) = 2\arctan t - 5$.

(d) Let $y = \left(\frac{\sin x}{e}\right)^{\sin x}$ then $\ln y = \sin x \ln\left(\frac{\sin x}{e}\right) = \sin x\,[\ln(\sin x) - 1]$

so $\frac{1}{y} y' = \cos x\,[\ln(\sin x) - 1] + \sin x\left[\frac{\cos x}{\sin x}\right]$

$y' = y\left[\cos x(\ln(\sin x) - 1) + \cos x\right] = \frac{(\sin x)^{\sin x}}{e^{\sin x}}\left[\cos x(\ln(\sin x) - 1) + \cos x\right]$

at $x = \pi/2$ $\quad y' = \frac{1^1}{e^1}\left[0(\ln 1 - 1) + 0\right] = \frac{1}{e}(0) = 0$.

#5 (a) Let $f(x) = x^5 + 7x^3 + 13x - 18$ (polyn $\Rightarrow$ cont, diff on $\mathbb{R}$)

$f(0) = -18 < 0$

$f(1) = 1 + 7 + 13 - 18 = 3 > 0$

Since 0 is between -18 and 3 and f is cont., by IVT, there is a least one value x_0 in $(0,1)$ such that $f(x_0) = 0$.

Suppose another value $c \neq x_0$ exists such that $f(c)=0$.

We have : f cont, diff AND $f(c) = f(x_0)$

$\Rightarrow$ by Rolle's Theorem there must be a value a between c and x_0 such that $f'(a) = 0$.

BUT: $f'(x) = 5x^4 + 21x^2 + 13 > 0$ for all x

$\therefore$ No other value exists s.t. $f(x) = 0$

$\Rightarrow$ $f(x)$ has a unique root

$\Rightarrow$ the eq. has a unique solution.

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(b) $f'(x) = 1 - \frac{1}{x^2}$ $\quad f'(x) = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0$

$\Rightarrow x = -1$ or $x = 1$
(rejected) ✓

x	0.01	100	1
$f(x)$	100.01	100.01	2

Abs max = 100.01 at $x = 0.01$ and $x = 100$

Abs min = 2 at $x = 1$.

#6 (a) $\frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{y'x - y}{x^2} = \frac{1}{2}\frac{1}{x^2+y^2}(2x + 2yy')$

$\Rightarrow \frac{y'x - y}{x^2+y^2} = \frac{x + yy'}{x^2+y^2}$

$\Rightarrow y'(x-y) = x + y \Rightarrow y' = \frac{x+y}{x-y}$ at $(1,0)$ $y' = \frac{1+0}{1-0} = 1$

(b) $y'' = \frac{(1+y')(x-y) - (x+y)(1-y')}{(x-y)^2} \underset{\text{at }(1,0)}{=} \frac{(1+1)(1-0) - (1+0)(1-1)}{(1-0)^2} = 2$

Estimate the slope (y') when $x = 0.95$ (near $x=1$)

$L(x) = y''(x-1) + y'_{\text{at }(1,0)} = 2(x-1) + 1 = 2x - 1$

y' at $x = 0.95 \approx 2(0.95) - 1 = 1.9 - 1 = 0.9$

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#7 $f(x) = \dfrac{x^2}{(x-2)^2}$

(a) DOM : $x-2 \neq 0 \Rightarrow x \neq 2$ $\quad (-\infty,2)\cup(2,\infty)$

(b) candidate for V.A. is $x=2$

$$\lim_{x\to 2^-} \frac{x^2}{(x-2)^2} = \frac{4}{0^+} = +\infty \qquad \lim_{x\to 2^+} \frac{x^2}{(x-2)^2} = \frac{4}{0^+} = +\infty$$

$\Rightarrow x=2$ (V.A.)

$$\lim_{x\to -\infty} \frac{x^2}{(x-2)^2} \overset{HR}{=} \lim_{x\to -\infty} \frac{2x}{2(x-2)} \overset{HR}{=} \lim_{x\to -\infty} \frac{2}{2} = 1$$

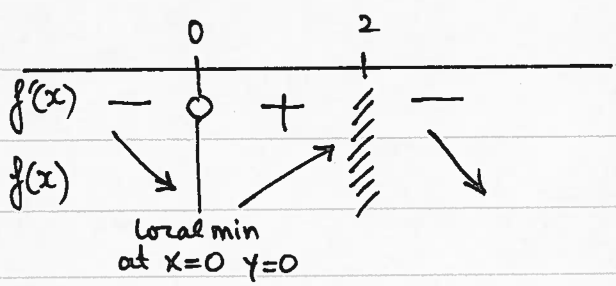
$$\lim_{x\to \infty} \frac{x^2}{(x-2)^2} \overset{HR}{=} \lim_{x\to \infty} \frac{2x}{2(x-2)} \overset{HR}{=} \lim_{x\to \infty} \frac{2}{2} = 1$$

$\Rightarrow y=1$ H.A.

(c) $$f'(x) = \frac{2x(x-2)^2 - x^2 \cdot 2(x-2)}{(x-2)^4} = \frac{2x(x-2)(x-2-x)}{(x-2)^4} = \frac{-4x}{(x-2)^3}$$

$f'(x)=0 \Rightarrow x=0$ (critical value)

$f'(x)$ DNE $\Rightarrow x=2$ (V.A.)

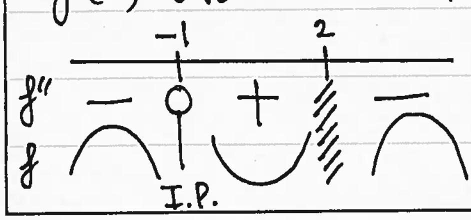


f decreasing on $(-\infty,0)\cup(2,\infty)$; f increasing on $(0,2)$;
Local min $=0$ at $x=0$.

(d) $$f''(x) = \frac{-4(x-2)^3 + 4x\cdot 3(x-2)^2}{(x-2)^6} = \frac{-4(x-2)^2(x-2-3x)}{(x-2)^6} = \frac{8(x+1)}{(x-2)^4}$$

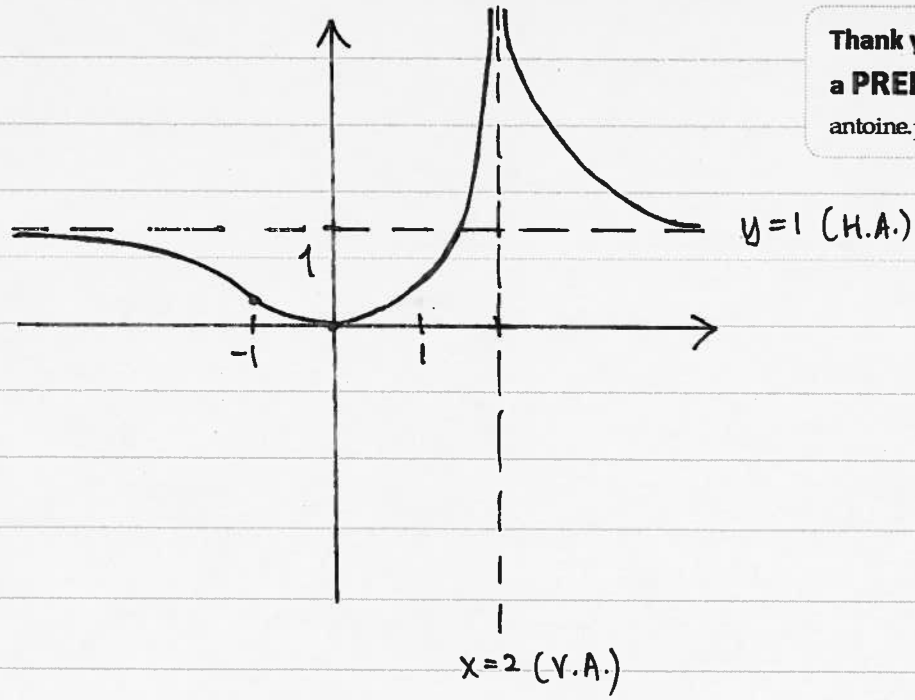
$f''(x)=0 \Rightarrow x=-1$

$f''(x)$ DNE $x=2$ V.A.



f concave down on $(-\infty,-1)\cup(2,\infty)$
f concave up on $(1,2)$
f has an inflection pt at $x=-1$ $y=f(-1)=\frac{1}{9}$

(e)



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McGill MATH 140 – Final Exam December 2008

#1 (a) $\lim_{x\to\frac{\pi}{8}} \dfrac{f(x)-f(\pi/8)}{x-\pi/8} = f'(\pi/8)$ (DEF of the derivative)

$$f'(x) = 2\sec^2(2x) \Rightarrow f'(\pi/8) = 2\sec^2(\pi/4) = 2(\sqrt{2})^2 = 4$$

(b) $\lim_{x\to 0} \dfrac{x-\sinh x}{x-\operatorname{sech} x} = \dfrac{0-0}{0-1} = 0$

(c) $\lim_{x\to\infty} \arcsin(-\cos x)$ DNE $\quad$ ($\cos\infty$ DNE)

(d) Let $z = (1-15y)^{\frac{1}{y}} \Rightarrow \ln z = \frac{1}{y}\ln(1-15y) = \dfrac{\ln(1-15y)}{y}$

$$\lim_{y\to 0^-} \ln z = \lim_{y\to 0^-} \frac{\ln(1-15y)}{y}$$

$$\overset{HR}{=} \lim_{y\to 0^-} \frac{\frac{-15}{1-15y}}{1} = \lim_{y\to 0^-} \frac{-15}{1-15y} = -15$$

$$\Rightarrow \lim_{y\to 0^-} (1-15y)^{\frac{1}{y}} = e^{-15}$$

(e) $\lim_{x\to-\infty} 5+\sqrt{x^2+4x+5}+x = 5+\lim_{x\to-\infty} \sqrt{x^2+4x+5}+x \cdot \dfrac{\sqrt{x^2+4x+5}-x}{\sqrt{x^2+4x+5}-x}$

$$= 5+\lim_{x\to-\infty} \frac{x^2+4x+5-x^2}{\sqrt{x^2+4x+5}-x} = 5+\lim_{x\to-\infty} \frac{4x+5}{\sqrt{x^2(1+\frac{4}{x}+\frac{5}{x^2})}-x}$$

$$= 5+\lim_{x\to-\infty} \frac{x(4+5/x)}{|x|\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}-x} = 5+\lim_{x\to-\infty} \frac{4x}{-x-x} = 5+(-2) = 3.$$

#2 $f(2) = 3 \Rightarrow \boxed{2c+d=3}$

<u>f cont at $x=1$</u>

$\lim_{x\to 1^-} f(x) = 1+b \qquad \lim_{x\to 1^+} f(x) = c+d \qquad f(1) = 1+b$

$$\therefore\ c+d = 1+b \Rightarrow \boxed{b-c-d=-1}$$

f differentiable at $x=1$

(since f continuous we use the formulas for the derivative)

$x<1 \quad f'(x) = 2x+b \quad$ at $x=1 \quad 2+b$

$x>1 \quad f'(x) = c \quad$ at $x=1 \quad c$

$$\Rightarrow \quad 2+b=c \quad \Rightarrow \quad \boxed{b-c=-2}$$

$$\therefore \begin{cases} 2c+d=3 & (1) \\ b-c-d=-1 & (2) \\ b-c=-2 & (3) \end{cases}$$

(2) AND (3) $\Rightarrow d-1=-2 \Rightarrow d=-1$

Replace in (1) $\Rightarrow 2c-1=3 \Rightarrow c=2$

Replace in (3) $\Rightarrow b-2=-2 \Rightarrow b=0$

NOTE: Because in this exam they ask for ANSWER ONLY you can skip some details while solving it to gain some time.

(b) $$\lim_{x\to\infty} \frac{x^2+1-(Lx+M)(x+1)}{x+1} = \lim_{x\to\infty} \frac{x^2+1-Lx^2-Lx-Mx-M}{x+1}$$

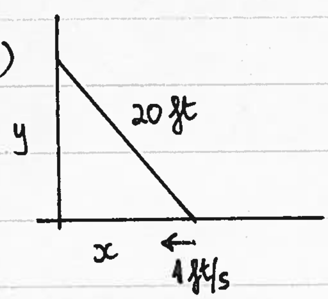
$$= \lim_{x\to\infty} \frac{(1-L)x^2-(L+M)x+1-M}{x+1}$$

since the limit is $=3$ and NOT ∞,
$1-L=0 \Rightarrow L=1$

$$\therefore \lim_{x\to\infty} \frac{-(1+M)x+1-M}{x+1} = 3$$

$$\Rightarrow \frac{-(1+M)}{1} = 3 \Rightarrow M=-4.$$

(c)



$\frac{dx}{dt} = -1$

$\frac{dy}{dt} = ?$ when $x=12$

$$x^2+y^2=400$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x\, dx/dt}{y} = \frac{-12(-1)}{16} = \frac{3}{4} \text{ ft/s}$$

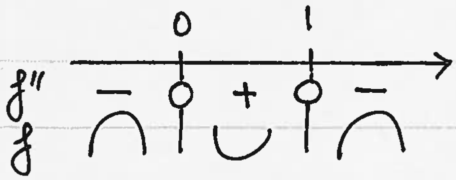
$12^2+y^2=20^2$

$y^2 = 400-144 = 256 \Rightarrow y=16$

#3(a) $f(x) = 2x^3 - x^4$

$f'(x) = 6x^2 - 4x^3$ $\qquad$ $f''(x) = 12x - 12x^2 = 12x(1-x)$

$f''(x) = 0 \implies x = 0\,,\ x = 1$



f concave down on $(-\infty, 0) \cup (1, \infty)$

(b) $1 = 2\dfrac{dx}{dy}\cosh(3x) + 2x\sinh(3x)\cdot 3\dfrac{dx}{dy}$

When $x = 0$:

$$1 = 2\frac{dx}{dy}(1) + 0 \implies \frac{dx}{dy} = \frac{1}{2}.$$

(c) $G(t) = -6\arcsin t + C$

$G(0) = -6\arcsin(0) + C = 12 \implies C = 12$

$\therefore\quad G(t) = -6\arcsin t + 12.$

(d) Let $y = f(x) = \dfrac{(\ln x)^x}{x^{\ln x}}$ $\qquad$ $\ln y = \ln(\ln x)^x - \ln(x)^{\ln x}$

$$= x\ln(\ln x) - \ln x\cdot\ln x$$

$$= x\ln(\ln x) - (\ln x)^2$$

$$\frac{1}{y}y' = \ln(\ln x) + x\cdot\frac{1}{\ln x}\cdot\frac{1}{x} - 2(\ln x)\cdot\frac{1}{x}$$

$$\implies y' = y\left[\ln(\ln x) + \frac{1}{\ln x} - \frac{2\ln x}{x}\right] = \frac{(\ln x)^x}{x^{\ln x}}\left[\ln(\ln x) + \frac{1}{\ln x} - \frac{2\ln x}{x}\right].$$

#4 Let $f(x) = 2x - 3\tan x + 1$ $\quad (-\pi/2 < x < \pi/2)$

f cont on $(-\frac{\pi}{2}, \frac{\pi}{2})$

$f(0) = +1 > 0$ $\qquad$ $f(\pi/4) = +\frac{\pi}{2} - 3 + 1 = \frac{\pi}{2} - 2 < 0$

Since f cont on $(-\pi/2, \pi/2)$ and 0 is between 1 and $\frac{\pi}{2} - 2$ then there is at least one value x_0 in $(0, \pi/4)$ s.t. $f(x_0) = 0$.

Now suppose another value $c \in (-\pi/2, \pi/2)$ exists s.t. $f(c) = 0$.
Since f cont, diff on $(-\pi/2, \pi/2)$ AND $f(c) = f(x_0) = 0$
by Rolle's thm there is a value between c & x_0 s.t. $f'(x) = 0$.
BUT $f'(x) = 2 - 3\sec^2 x = 0$

$\Rightarrow \quad \sec^2 x = \frac{2}{3} \quad \Rightarrow \quad \cos^2 x = \frac{3}{2}$ NOT POSSIBLE

$\Rightarrow$ No value exists where $f'(x) = 0$ in $(-\pi/2, \pi/2)$

$\Rightarrow$ No other value c exists s.t. $f(c) = 0$ in $(-\pi/2, \pi/2)$

$\Rightarrow$ $f(x) = 0$ has exactly one sol in $(-\pi/2, \pi/2)$

$\Rightarrow$ $y = 2x - 3\tan x + 1$ crosses the x-axis exactly once.

(b)

		-5		-1		4	
f'	+	0	−	0	+	0	+
f	↗		↘		↗		↗

Critical pts $x = -5$, $x = -1$, $x = 4$
at $x = -5$ f has a local max.
at $x = -1$ f has a local min.

#5 (a) we write y' for $\frac{dy}{dx}$.

$$1 + y' = 3(x-y)^2(1-y') \quad \Rightarrow \quad 1 + y' = 3(x-y)^2 - 3y'(x-y)^2$$

$$\therefore \quad y'\left(1 + 3(x-y)^2\right) = 3(x-y)^2 - 1$$

$$\Rightarrow \quad y' = \frac{3(x-y)^2 - 1}{1 + 3(x-y)^2} \quad \text{at } (1,0) \quad y' = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$$

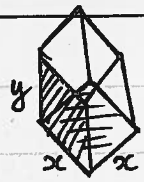
(b) $$y'' = \frac{6(x-y)(1-y')\left[1 + 3(x-y)^2\right] - \left[3(x-y)^2 - 1\right] \cdot 6(x-y)(1-y')}{\left[1 + 3(x-y)^2\right]^2}$$

at $(1,0)$ $$y'' = \frac{6(1)(\frac{1}{2})[1+3] - [3-1] \cdot 6(1)(\frac{1}{2})}{[1+3]^2} = \frac{3}{8}.$$

(c) $$L(x) = y''(x-1) + y' = \frac{3}{8}(x-1) + \frac{1}{2}$$

y' at $x = 0.95$ $\approx L(0.95) = \frac{3}{8}(-0.05) + \frac{1}{2} = -\frac{0.15}{8} + \frac{1}{2} = \frac{3.85}{8}.$

#6



bottom + back $= x^2 + xy$ (PINE)

Top + 3 sides $= x^2 + 3xy$ (OAK)

Volume $= x^2 y = 6 \Rightarrow y = \frac{6}{x^2}$ $(x>0)$

bottom + back : $x^2 + x\frac{6}{x^2} = x^2 + \frac{6}{x}$

Top + 3 sides : $x^2 + 3x \cdot \frac{6}{x^2} = x^2 + \frac{18}{x}$

COST: $C(x) = \frac{1}{2}\left(x^2 + \frac{6}{x}\right) + x^2 + \frac{18}{x} = \frac{x^2}{2} + \frac{3}{x} + x^2 + \frac{18}{x} = \frac{3x^2}{2} + \frac{21}{x}$

pine costs half as much as OAK.

$C'(x) = 3x - \frac{21}{x^2} = 0 \Rightarrow 3x = \frac{21}{x^2} \Rightarrow x^3 = 7 \Rightarrow x = \sqrt[3]{7}$

$C''(x) = 3 + \frac{42}{x^3}$ $\quad C''(\sqrt[3]{7}) > 0 \Rightarrow x = \sqrt[3]{7}$ gives Abs. Min Cost

Dimensions: $\sqrt[3]{7}, \sqrt[3]{7}, \frac{6}{7^{2/3}}$.

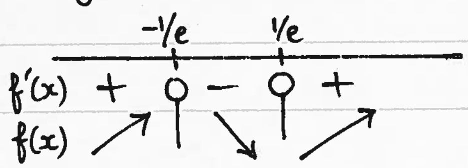
#7 $(x \neq 0)$ $\quad f(x) = x \ln|x|$

(a) $\lim\limits_{x\to 0} x\ln|x| = \lim\limits_{x\to 0} \frac{\ln|x|}{1/x} \overset{HR}{=} \lim\limits_{x\to 0} \frac{1/x}{-1/x^2} = \lim\limits_{x\to 0} -x = 0$

$\Rightarrow$ The discontinuity can be removed by setting $f(0) = 0$.

(b) $f'(x) = \ln|x| + x \cdot \frac{1}{x} = \ln|x| + 1$

$f'(x) = 0 \Rightarrow \ln|x| = -1 \Rightarrow |x| = \frac{1}{e} \Rightarrow x = \pm\frac{1}{e}$



f has a local max at $x = -\frac{1}{e}$ $\quad y = \frac{1}{e}$

f has a local min at $x = \frac{1}{e}$ $\quad y = -\frac{1}{e}$

$f''(x) = \frac{1}{x}$ $\quad f''(x) = 0$ (NO SOL.) $\quad f''(x)$ undef for $x = 0$.

		0	
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	|	$\smile$
		I.P.	

f concave down on $(-\infty, 0)$

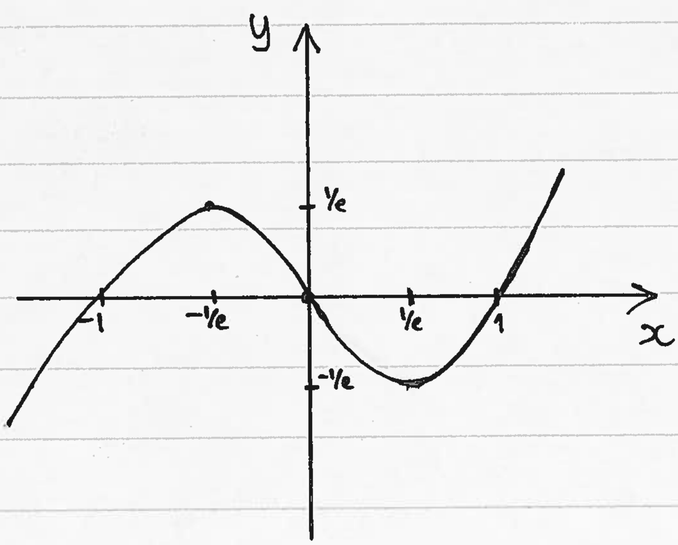
f concave up on $(0, \infty)$

f has an inflection pt at $x=0$ $y=0$.

(c) NO V.A. (no value of x where $\lim_{x \to a} f(x) = +\infty$ or $-\infty$)

Horizontal Asymptote:

$\lim_{x \to -\infty} x \ln|x| = -\infty$ $\qquad$ $\lim_{x \to \infty} x \ln|x| = \infty$ $\qquad$ NO H.A.



McGill MATH 140 – Supplemental/Deferred Final Exam May 2008

#1 (a) $\displaystyle \lim_{x\to-\infty} \frac{x^3\left(\frac{1}{x^3}+1\right)}{x^3\left(\frac{1}{x^3}-5\right)} = -\frac{1}{5}$

(b) $\displaystyle \lim_{x\to 0} \frac{\sin(\sec x)}{\sec(\sin x)} = \frac{\sin(1)}{\sec(0)} = \sin(1).$

(c) $\displaystyle \lim_{x\to 0^+} \frac{\tanh 2x}{\arctan(2x)} \overset{HR}{=} \lim_{x\to 0^+} \frac{2(1-\tanh^2(2x))}{\frac{2}{1+4x^2}} = \frac{2}{\frac{2}{1+0}} = 1.$

(d) $\displaystyle \lim_{u\to 3} \frac{\ln(e^u)}{e^{\ln u}} = \lim_{u\to 3} \frac{u}{u} = 1.$

(e) $\displaystyle \lim_{u\to-\infty} \sqrt{u^2+4u+1}+u+3 \cdot \frac{\sqrt{u^2+4u+1}-(u+3)}{\sqrt{u^2+4u+1}-(u+3)} = \lim_{x\to-\infty} \frac{u^2+4u+1-(u+3)^2}{\sqrt{u^2+4u+1}-u-3}$

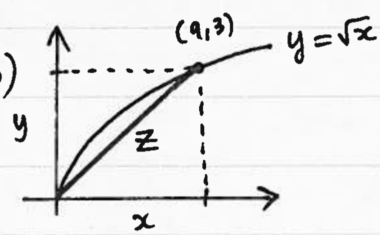
$$= \lim_{u\to-\infty} \frac{u^2+4u+1-u^2-6u-9}{|u|\sqrt{1+\frac{4}{u}+\frac{1}{u^2}}-u\left(1+\frac{3}{u}\right)} = \lim_{u\to-\infty} \frac{-2u-8}{-u\sqrt{1+\frac{4}{u}+\frac{1}{u^2}}-u\left(1+\frac{3}{u}\right)}$$

$$= \lim_{u\to-\infty} \frac{-u\left(2+\frac{8}{u}\right)}{-u\left[\sqrt{1+\frac{4}{u}+\frac{1}{u^2}}+\left(1+\frac{3}{u}\right)\right]} = \frac{2}{\sqrt{1}+1} = 1.$$

#2 (a) $\displaystyle \lim_{x\to\infty} (e^x-3)^2(e^{-x^2}+4) = \infty\cdot 4 = \infty$

$\displaystyle \lim_{x\to-\infty} (e^x-3)^2(e^{-x^2}+4) = 9\cdot(4) = 36 \quad \Rightarrow \quad y = 36$ is H.A. at $-\infty$

(b)



$x=9 \quad y=3$

$\frac{dy}{dt} = 1$ cm/sec.

$z^2 = x^2 + y^2$

$z^2 = (y^2)^2 + y^2 = y^4 + y^2$

$z = \sqrt{y^4+y^2}$

Since $y=\sqrt{x}$

$\Rightarrow x = y^2$

$$\frac{dz}{dt} = \frac{1}{2}(y^4+y^2)^{-1/2}\cdot\left(4y^3\frac{dy}{dt}+2y\frac{dy}{dt}\right)$$
$$= \frac{1}{2}\left(81+9\right)^{-1/2}\cdot\left(4(27)+2(3)\right)$$
$$= \frac{57}{\sqrt{90}} \text{ cm/sec}$$

(c) $f(x) = 3x^5 - 3 - 10x^4 + 10x^3$

$f'(x) = 15x^4 - 40x^3 + 30x^2$

$f''(x) = 60x^3 - 120x^2 + 60x = 60x\,(x^2 - 2x + 1) = 60x\,(x-1)^2$

$f''(x) = 0 \implies x = 0, \quad x = 1.$

	0		1	
$f''(x)$ $-$		$+$		$+$
	I.P.			

$f(x)$ has an I.P. at $x=0$ $\quad y = -3 \qquad (0,-3)$

#3 (a) f cont. at $x=4$

$\lim_{x\to 4^+} f(x) = 16 - 8 + 2 = 10 \qquad \lim_{x\to 4^-} f(x) = -16 + 4A + B$

$f(4) = -16 + 4A + B.$

f cont. $\quad -16 + 4A + B = 10 \implies 4A + B = 26.$

f diff at $x=4$

Since f cont: $\quad x<4 \quad f'(x) = -2x + A \quad$ at $x=4 \quad -8+A$

$x>4 \quad f'(x) = 2x - 2 \quad$ at $x=4 \quad 8-2 = 6$

$-8 + A = 6 \implies A = 14$

$\therefore \quad 4(14) + B = 26 \implies B = 26 - 56 = -30.$

(b) $\frac{d}{du} \ln|\csc u + \cot u| = \frac{1}{\csc u + \cot u} \cdot -\csc u \cot u - \csc^2 u$

$= \frac{-\csc u\,(\cot u + \csc u)}{\csc u + \cot u} = -\csc u.$

(c) $f'(x) = -\frac{1}{x} + C \qquad f'(2) = -\frac{1}{2} + C = 0 \implies C = \frac{1}{2}.$

$\implies f'(x) = -\frac{1}{x} + \frac{1}{2}$

$f(x) = -\ln|x| + \frac{1}{2}x + D \qquad f(1) = -\ln 1 + \frac{1}{2} + D = 0 \implies D = -\frac{1}{2}$

$f(x) = -\ln|x| + \frac{1}{2}x - \frac{1}{2}.$

(d) $y = \ln(-x e^{x^2})$ $\quad y' = \dfrac{1}{-x e^{x^2}} \cdot \left(-e^{x^2} - x\, e^{x^2} \cdot 2x\right)$

$$= \frac{-e^{x^2}(1+2x^2)}{-x\, e^{x^2}} = \frac{1+2x^2}{x}$$

at $(-1,1)$ $\quad y' = \dfrac{1+2}{-1} = -3.$ (SLOPE of tan. line)

$y = mx + b$ $\qquad 1 = -3(-1) + b \Rightarrow 1 = 3 + b \Rightarrow b = -2.$

$y = -3x - 2.$

#4 (a) $f(x) = 2\arctan(\sqrt{x}) - \arcsin\left(\dfrac{x-1}{x+1}\right)$

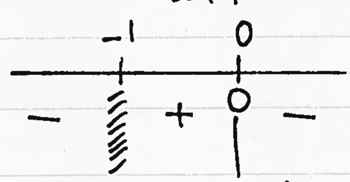
$*\ x \geq 0$ (because of $\sqrt{x}$)

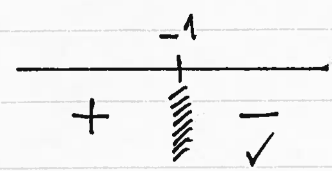
$*\ -1 \leq \dfrac{x-1}{x+1} \leq 1$

$$-1 - \frac{x-1}{x+1} \leq 0 \qquad\qquad \frac{x-1}{x+1} - 1 \leq 0.$$

$$\frac{-x-1-x+1}{x+1} \leq 0 \qquad\qquad \frac{x-1-x-1}{x+1} \leq 0$$

$$\frac{-2x}{x+1} \leq 0 \qquad\qquad \frac{-2}{x+1} \leq 0$$





$(-\infty, -1) \cup [0, \infty)$ $\qquad\qquad (-1, \infty)$

$[0, \infty)$

$\Rightarrow$ overall domain is $[0, \infty)$

(b) $f'(x) = \dfrac{2}{1+x} \cdot \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{1 - \dfrac{(x-1)^2}{(x+1)^2}}} \cdot \dfrac{x+1-(x-1)}{(x+1)^2}$

$$f'(x) = \frac{1}{\sqrt{x}(1+x)} - \frac{x+1}{\sqrt{(x+1)^2-(x-1)^2}} \cdot \frac{2}{(x+1)^2}$$

$$= \frac{1}{\sqrt{x}(1+x)} - \frac{2}{\sqrt{4x}(x+1)} = \frac{1}{\sqrt{x}(1+x)} - \frac{2}{2\sqrt{x}(x+1)} = 0.$$

$$f'(x) = 0 \implies f(x) = C \quad \text{(constant)}.$$

(c) $f(9) = f(1) = 2\arctan(1) - \arcsin(0) = 2\frac{\pi}{4} - 0 = \frac{\pi}{2}.$

#5(a) $x^3 + y^3 + 3(x^2+y^2) + xy = 5$

$$3x^2 + 3y^2y' + 6x + 6yy' + y + xy' = 0$$

$$y'(3y^2+6y+x) = -3x^2 - 6x - y \implies y' = \frac{-3x^2-6x-y}{(3y^2+6y+x)}$$

at $(-1,-1)$ $\quad y' = \frac{-3+6+1}{3-6-1} = \frac{4}{-4} = -1.$

(b) $$y'' = \frac{(-6x-6-y')(3y^2+6y+x) + (3x^2+6x+y)(6yy'+6y'+1)}{(3y^2+6y+x)^2}$$

at $(-1,-1)$ $\quad y'' = \frac{(6-6+1)(3-6-1) + (3-6-1)(6-6+1)}{(3-6-1)^2} = \frac{-4+(-4)}{16} = -\frac{1}{2}.$

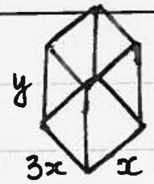
(c) $L(x) = y'(-1)(x+1) + (-1)$

$= -1(x+1) - 1$

$= -x - 2$

y at $x = -0.99 \approx L(-0.99) = +0.99 - 2 = -1.01$

#6



$V = 1000 \text{ cm}^3$

$\Rightarrow \quad 3x^2 y = 1000 \quad \Rightarrow \quad y = \dfrac{1000}{3x^2}.$

$(x > 0)$

Base : $3x^2(4) = 12x^2$ \$

Sides : $(3xy + xy + 3xy + xy)(9) = 72xy = 72x\dfrac{1000}{3x^2} = \dfrac{24000}{x}$ \$

Cost $= C(x) = 12x^2 + \dfrac{24000}{x} \quad x > 0$

$C'(x) = 24x - \dfrac{24000}{x^2} = 0 \quad \Rightarrow \quad 24x = \dfrac{24000}{x^2} \quad \Rightarrow \quad x^3 = 1000$

$\Rightarrow \quad x = 10$ cm.

$C''(x) = 24 + \dfrac{48000}{x^3} \qquad C''(10) = 24 + \dfrac{48000}{1000} > 0$

$\Rightarrow x = 10$ cm gives Abs. min cost.

$C(10) = 12(10)^2 + \dfrac{24000}{10} = 1200 + 2400 = 3600$ \$.

#7 if $x < 0 \quad f(x) = x - 4$ is increasing.

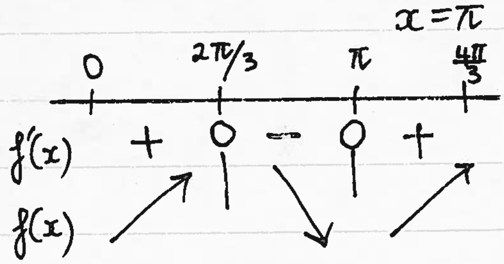
if $x > \frac{4\pi}{3} \quad f(x) = \left(\frac{15}{4\pi}\right)x$ is increasing

for $0 < x < \frac{4\pi}{3}$:

$f'(x) = 4(2\sin x \cos x + \sin x) = 4\sin x(2\cos x + 1)$

$f'(x) = 0 \quad \Rightarrow \quad \sin x = 0 \quad$ or $\quad \cos x = -\frac{1}{2}$

$x = \pi \qquad\qquad x = \frac{2\pi}{3}$



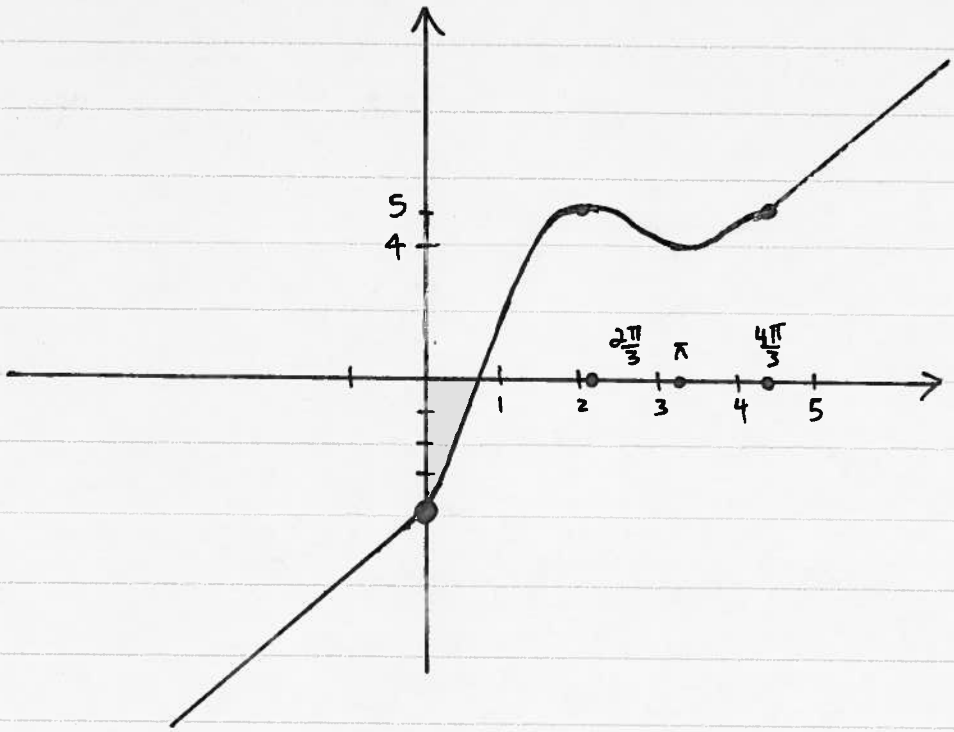
f increasing on $\left(0, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{4\pi}{3}\right)$

$\Rightarrow$ overall f increasing on $\left(-\infty, \frac{2\pi}{3}\right) \cup (\pi, \infty)$.

(b) f has a local max at $x = \frac{2\pi}{3} \quad y = 4\left(\sin^2\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}\right)\right) = 5.$

f has a local min at $x = \pi \quad y = 4$

(c)



McGill MATH 140 – Final Exam December 2007

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#1(a) $\lim_{x\to\infty} \frac{x+x^2}{1-2x^2} = \lim_{x\to\infty} \frac{x^2(\frac{1}{x}+1)}{x^2(\frac{1}{x^2}-2)} = \frac{-1}{2}$

(b) $\lim_{x\to 0} \frac{\sin(3x^2)}{(\sin 3x)^2} = \lim_{x\to 0} \underbrace{\frac{\sin(3x^2)}{3x^2}}_{1} \cdot 3x^2 \cdot \underbrace{\frac{(3x)^2}{(\sin 3x)^2}}_{1} \cdot \frac{1}{(3x)^2} = \lim_{x\to 0} \frac{3x^2}{9x^2} = \frac{1}{3}.$

(c) $\lim_{x\to 0^+} \arctan(-\frac{1}{x}) = \lim_{x\to 0^+} \arctan(-\infty) = -\frac{\pi}{2}.$

(d) $\lim_{u\to 3} \frac{\ln(u/3)}{u-3} \overset{HR}{=} \lim_{u\to 3} \frac{\frac{1}{u/3}\cdot\frac{1}{3}}{1} = \lim_{u\to 3} \frac{1}{u} = \frac{1}{3}.$

(e) $\lim_{u\to -\infty} \sqrt{u^2+2u+4} - \sqrt{u^2-3u+1} \cdot \frac{\sqrt{u^2+2u+4}+\sqrt{u^2-3u+1}}{\sqrt{u^2+2u+4}+\sqrt{u^2-3u+1}}$

$= \lim_{x\to -\infty} \frac{u^2+2u+4-(u^2-3u+1)}{\sqrt{u^2+2u+4}+\sqrt{u^2-3u+1}} = \lim_{x\to-\infty} \frac{5u+3}{|u|\sqrt{1+\frac{2}{u}+\frac{4}{u^2}}+|u|\sqrt{1-\frac{3}{u}+\frac{1}{u^2}}}$

$= \lim_{x\to -\infty} \frac{u(5+3/u)}{-u\left(\sqrt{1+\frac{2}{u}+\frac{4}{u^2}}+\sqrt{1-\frac{3}{u}+\frac{1}{u^2}}\right)} = \frac{5}{-(\sqrt{1}+\sqrt{1})} = -\frac{5}{2}.$

#2 (a) $\lim_{x\to -\infty} 2\arctan x - 1 = 2\arctan(-\infty) - 1 = 2(-\frac{\pi}{2}) - 1 = -\pi - 1$

$\lim_{x\to\infty} 2\arctan x - 1 = 2\arctan(\infty) - 1 = 2(\frac{\pi}{2}) - 1 = \pi - 1$

$y = -\pi - 1$ H.A at $-\infty$; $y = \pi - 1$ H.A. at ∞.

(b) $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x^2-4} = \frac{1}{0^+} = +\infty$

$x = -2$ V.A.

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{1}{x^2-4} = \frac{1}{0^-} = -\infty$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x^2-4} = \frac{1}{0^-} = -\infty$

$$\lim_{x\to 2^+} \frac{1}{x^2-4} = \frac{1}{0^+} = +\infty$$

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(c) $V = \frac{4}{3}\pi R^3$ $\qquad \frac{dV}{dt} = 10 \text{ cm}^3/s$ $\qquad \frac{dr}{dt} = ?$ when $r = 12$ cm.

$$\frac{dV}{dt} = \frac{4}{3}\pi . 3R^2 . \frac{dR}{dt} = 4\pi R^2 \frac{dR}{dt} \implies \frac{dR}{dt} = \frac{dV/dt}{4\pi R^2} = \frac{10}{4\pi (12)^2}$$

$$\frac{dR}{dt} = \frac{5}{288\pi} \text{ cm/s.}$$

#3. (a) Let $y = x^{\frac{x}{\ln x}}$ $\qquad \ln y = \frac{x}{\ln x} . \ln x = x$

$$\frac{1}{y} y' = 1 \implies y' = y = x^{\frac{x}{\ln x}}.$$

<u>NOTE:</u> $x^{\frac{x}{\ln x}} = e^{\ln x^{x/\ln x}} = e^{\frac{x}{\ln x} . \ln x} = e^x \quad \therefore \quad y' = e^x.$

(b) $$\frac{d}{du} \cos(\arcsin u) = -\sin(\arcsin u) . \frac{1}{\sqrt{1-u^2}}$$

<u>NOTE:</u> for $u \in [-1,1]$ $\quad \sin(\arcsin u) = u \implies \frac{d}{du}\cos(\arcsin u) = \frac{-u}{\sqrt{1-u^2}}.$

(c) $F(x) = \cosh x + C$ $\qquad F(0) = \cosh 0 + C = -1$

$\therefore \quad C = -1-1 = -2.$

$\therefore \quad F(x) = \cosh x - 2.$

(d) $$f(t) = \frac{t^2}{1-t}\sqrt{\frac{3-t}{(3+t)^2}}$$

$$f'(t) = \frac{2t(1-t) - t^2(-1)}{(1-t)^2} . \sqrt{\frac{3-t}{(3+t)^2}} + \frac{t^2}{1-t} . \frac{1}{2}\left[\frac{3-t}{(3+t)^2}\right]^{-\frac{1}{2}} . \frac{-1(3+t)^2 - (3-t).2(3+t)}{(3+t)^4}$$

$$f'(2) = \frac{4(-1)+4}{(1-2)^2} . \sqrt{\frac{3-2}{(3+2)^2}} + \frac{4}{-1} . \frac{1}{2}\left[\frac{1}{25}\right]^{-1/2} . \frac{-25-10}{5^4}$$

$$= 0 + (-2)(5) . \frac{-35}{25(25)} = \frac{14}{25}.$$

Note: could also use Logarithmic differentiation.

#4(a) $f(x) = 4x^5 + x^3 + 2x + 1$

f is a polynomial $\Rightarrow$ cont. & diff. on $\mathbb{R}$

$f(-1) = -4 - 1 - 2 + 1 = -6 < 0$

$f(1) = 4 + 1 + 2 + 1 = 8 > 0$

Since 0 is between -6 and 8 AND $f(x)$ cont. $\Rightarrow$ by I.V.T. there is an x_0 in $(-1, 1)$ s.t. $f(x_0) = 0$.

Suppose another value α exists s.t. $f(\alpha) = 0$

$\Rightarrow$ Since f diff & $f(\alpha) = f(x_0)$ by Rolle's thm there must be a value c between α and x_0 s.t. $f'(c) = 0$.

BUT $f'(x) = 20x^4 + 3x^2 + 2 > 0$ for all x.

$\therefore$ NO such value α exists $\Rightarrow$ x_0 is unique.

$\Rightarrow$ $f(x) = 0$ has exactly one solution.

(b) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{Kx^2}{1 - \cos x} \overset{HR}{=} \lim\limits_{x \to 0^+} \dfrac{2Kx}{\sin x} \overset{HR}{=} \lim\limits_{x \to 0^+} \dfrac{2K}{\cos x} = 2K$

$\lim\limits_{x \to 0^-} f(x) = 8$ $\qquad f(0) = 8$.

$f(x)$ cont at $x = 0$ (and thus everywhere) if $2K = 8 \Rightarrow K = 4$.

#5. $x^5 + x^2 y + y^3 = 4y + 3$

(a) $5x^4 + 2xy + x^2 y' + 3y^2 y' = 4y'$

$\Rightarrow y'(x^2 + 3y^2 - 4) = -5x^4 - 2xy$

$\Rightarrow y' = \dfrac{-5x^4 - 2xy}{x^2 + 3y^2 - 4}$ at $(1,2)$ $y' = \dfrac{-5 - 4}{1 + 12 - 4} = \dfrac{-9}{9} = -1$.

(b)
$$y'' = \frac{(-20x^3 - 2y - 2xy')(x^2+3y^2-4) - (-5x^4 - 2xy)(2x+6yy')}{(x^2+3y^2-4)^2}$$

$x=1 \quad y=2$
$y'=-1$

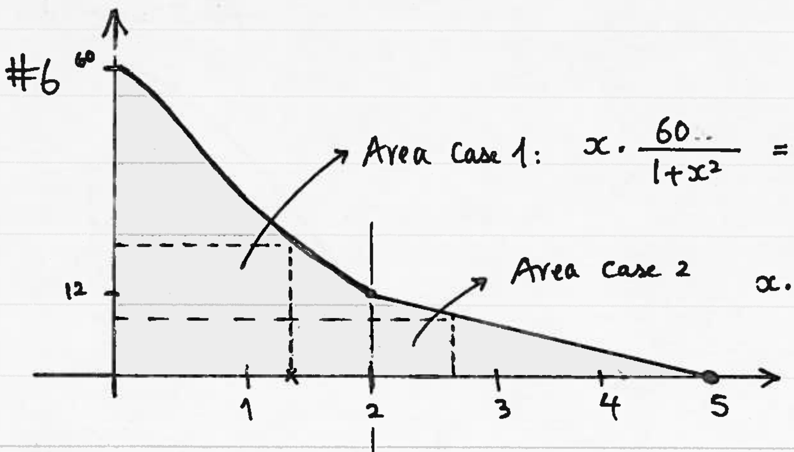
at $(1,2)$
$$y'' = \frac{(-20-4+2)(1+12-4)+(5+4)(2-12)}{(1+12-4)^2}$$

$$= \frac{-22(9)-90}{9^2} = -\frac{32}{9}.$$

(c) $L(x) = y'|_{(1,2)}(x-1) + 2 = -(x-1)+2 = -x+3$

at $x = 0.97$ $\quad y' \approx L(0.97) = -0.97+3 = 2.03.$

#6



Area case 1: $x \cdot \frac{60}{1+x^2} = \frac{60x}{1+x^2}$ $\quad [0,2]$

Area case 2 $\quad x.(20-4x) = 20x-4x^2$ $\quad [2,5]$

it's ok to consider it closed because of cont.

<u>CASE 1:</u> $\quad A(x) = \frac{60x}{1+x^2}$ $\quad [0,2]$

$$A'(x) = \frac{60(1+x^2) - 60x \cdot 2x}{(1+x^2)^2} = \frac{60+60x^2-120x^2}{(1+x^2)^2} = \frac{60(1-x^2)}{(1+x^2)^2}$$

$A'(x) = 0$

$\Rightarrow x=-1$ (rejected) $\quad x = 1$

x	0	1	2
$A(x)$	0	30	24

<u>CASE 2:</u> $\quad A(x) = 20x-4x^2$ $\quad [2,5]$

$A'(x) = 20-8x \quad A'(x)=0 \Rightarrow 20-8x=0 \Rightarrow x = 5/2.$

x	2	2.5	5
$A(x)$	24	25	0

$\Rightarrow$ MAX AREA is 30 when $x=1$.

#7. $x \geq 0$ $\quad f(x) = x e^{-2x^2}$

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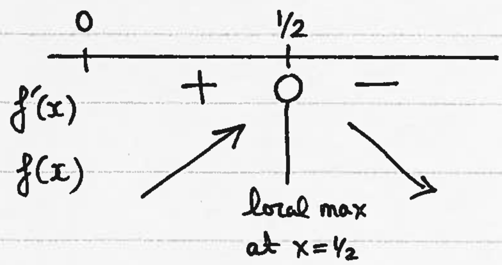
(a) $f'(x) = e^{-2x^2} + x(-4x)e^{-2x^2} = e^{-2x^2}(1-4x^2)$

$f'(x) = 0 \Rightarrow 1 - 4x^2 = 0 \Rightarrow x = -\frac{1}{2}$ (rejected) or $x = \frac{1}{2}$.



f increasing on $(0, \frac{1}{2})$

f decreasing on $(\frac{1}{2}, \infty)$

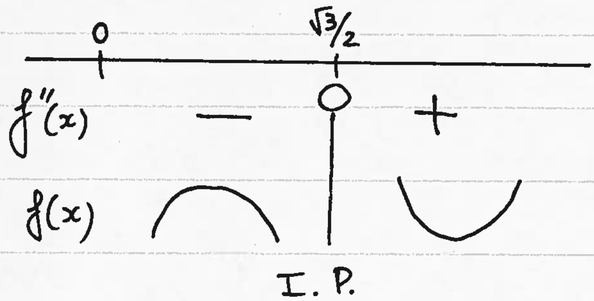
(b) f has a local max $\frac{1}{2}e^{-\frac{1}{2}}$ at $x = 1/2$.

(c) $f''(x) = -4x e^{-2x^2}(1-4x^2) + e^{-2x^2}(-8x)$

$= -4x e^{-2x^2}[1 - 4x^2 + 2]$

$= -4x e^{-2x^2}[3 - 4x^2]$.

$f''(x) = 0 \quad x = 0 \quad x^2 = \frac{3}{4} \Rightarrow x = -\frac{\sqrt{3}}{2}$ (rejected) $\quad x = \frac{\sqrt{3}}{2}$.



f is concave up on $(0, \frac{\sqrt{3}}{2})$

f is concave down on $(\frac{\sqrt{3}}{2}, \infty)$

f has an inflection pt at $x = \frac{\sqrt{3}}{2}$

$y = \frac{\sqrt{3}}{2} e^{-3/2}$

* $\lim_{x\to\infty} x e^{-2x^2} = \lim_{x\to\infty} \frac{x}{e^{2x^2}} \overset{HR}{=} \lim_{x\to\infty} \frac{1}{4x e^{2x^2}} = 0$

$y = 0$ H.A at ∞