15. Method of Characteristics: Two-Dimensional Steady Flow

So far, we have only dealt with flows that are one-dimensional (Chapters 1-11) or are comprised of simple two-dimensional elements such as oblique shocks or Prandtl-Meyer expansion fans (Chapters 12-14). For more complex flows (say, over a curved wing or fuselage), it becomes necessary to develop new methodologies that can solve general two dimensional or multi-dimensional flows. One of the most versatile approaches is called the method of characteristics. It uses the fact that compressible flow is controlled by disturbances that propagate along waves or lines (also called characteristics or characteristic lines). These are not new concepts; we have already encountered the envelope of disturbances called the Mach angle in Section 4.3. As will be discussed in this chapter, the Mach line is a characteristic in steady, two-dimensional supersonic flow. By tracking how lines of influence propagate at the local Mach angle through a flow, it is possible to solve the entire flow field; this is the essence of the method of characteristics.

The method of characteristics is very general and very powerful. It can either be implemented on an ad hoc basis by hand or systemized into a computer-based algorithm. Implementing it for complex flows with shock waves, however, it can become cumbersome, and it has been largely replaced by finite-difference computational fluid dynamics (CFD) in recent decades. The method of characteristics is still worthy of our attention: examining characteristics is often essential to obtain a “feel” for how a flow responds to boundary conditions. In addition, the development of modern CFD codes is often built on an understanding of how characteristics control a compressible flow. Interestingly, some modern developments in CFD (such as space-time finite elements) are remarkably similar to the “classical” method of characteristics we will develop here.

15.1 Governing Equations for Irrotational Flow

In this Chapter, we will limit our attention to two-dimensional, steady supersonic flow. In addition, we will only consider irrotational flow. This is an important assumption, and it is worthwhile to detail what types of flow can, and cannot, be considered irrotational. An irrotational flow is a flow in which the vorticity of the flow is everywhere zero. Vorticity is defined as the curl of the velocity vector: vorticity = \( \nabla \times \hat{\mathbf{v}} \), where \( \hat{\mathbf{v}} \) is the velocity vector and \( \nabla \) is the grad operator for two-dimensional Cartesian coordinates). Swirling flows with eddies are obviously flows with vorticity. Some less obvious examples of rotational flow are flows in boundary layers or flows behind curved shock waves; the version of the method of characteristics developed here will not be applicable to these flows. An example of an irrotational flow (where \( \nabla \times \hat{\mathbf{v}} = 0 \) everywhere in the flow) includes an initially uniform flow passing through an isentropic converging-diverging nozzle.

The fact that the curl of velocity is everywhere zero in an irrotational flow means that the velocity vector \( \hat{\mathbf{v}} \) can be expressed in terms of a potential function \( \phi \):

\[
\hat{\mathbf{v}} = \nabla \phi
\]

Recall that a potential function is a scalar function whose gradient gives a vector field. It is a compact way of expressing the velocity vector field, and since it replaces the components of velocity with a single function, can lead to simplification in solving the flow. For example, the conservation of mass (continuity) for a steady flow:

\[
\nabla \cdot (\rho \hat{\mathbf{v}}) = 0
\]

*It may not be obvious why flows with curved shock waves have vorticity. However, a powerful principle known as Crocco’s theorem, expressed as \( \hat{\mathbf{v}} \times (\nabla \times \hat{\mathbf{v}}) = -T \nabla s \), explicitly links vorticity with a gradient in entropy. Since a curved shock wave in an initially uniform flow produces a gradient in entropy as the strength of the shock changes, the flow downstream of the shock must have non-zero vorticity. Implementing the method of characteristics for rotational flows (such as behind curved shocks) necessitates tracking an additional characteristic to account for the changing in entropy through the flow.
(note that this is simply the vector form of the conservation of mass derived in Section 2.1) can be written using the potential function as

\[
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = \frac{\partial}{\partial x}(\rho \phi_x) + \frac{\partial}{\partial y}(\rho \phi_y) = 0
\]

\[\rho (\phi_x + \phi_y) + \phi_x \frac{\partial \rho}{\partial x} + \phi_y \frac{\partial \rho}{\partial y} = 0\]  \(15.1\)

We can eliminate \(\rho\) (density) completely by invoking the momentum equation (recall Section 3.2):

\[
\frac{dp}{\rho} + VdV = 0
\]

or

\[
dp = -\rho \frac{d(V^2)}{2} = -\rho \frac{d(\phi_x^2 + \phi_y^2)}{2}\]  \(15.2\)

If we further limit ourselves to isentropic flow, we can relate changes in density to changes in pressure by the definition of the speed of sound (recall Section 4.1):

\[
c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s
\]

Thus, at any point in an isentropic flow:

\[
d\rho = \frac{dp}{c^2}
\]

Using (15.2) in this result, we obtain:

\[
dp = -\rho \frac{d(\phi_x^2 + \phi_y^2)}{2c^2}
\]

Thus, we can express the gradient of density in the \(x\)-direction as

\[
\frac{\partial \rho}{\partial x} = \frac{\rho \phi_x}{c^2}\]

and likewise for the \(y\)-direction:

\[
\frac{\partial \rho}{\partial y} = \frac{\rho \phi_y}{c^2}
\]

Thus, the gradient of density can be expressed in terms of the potential function.

We can use these expressions in (15.1) to eliminate density entirely:

\[
\left(1 - \frac{\phi_x^2}{c^2}\right) \phi_{xx} - \frac{2\phi_x \phi_y}{c^2} \phi_{xy} + \left(1 - \frac{\phi_y^2}{c^2}\right) \phi_{yy} = 0
\]

This relation is called the **velocity potential equation** and represents both the continuity and momentum equations for irrotational, isentropic flow. This equation is still coupled to the energy equation via the sound speed \(c\). Since sound speed is a function of temperature, it will be necessary to solve for temperature using the energy equation.

### 15.2 Properties of Hyperbolic Systems: Characteristics

We will re-introduce the velocity components \(u\) and \(v\) for a moment to make a few points:

\[
\left(1 - \frac{u^2}{c^2}\right) \phi_{xx} - \frac{2uv}{c^2} \phi_{xy} + \left(1 - \frac{v^2}{c^2}\right) \phi_{yy} = 0
\]  \(15.3\)

Thus, compressible flow is governed by a PDE of the form:
\[
A \frac{\partial^2 \phi}{\partial x^2} + 2B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} = 0
\]

where \( A = 1 - \frac{u^2}{c^2}, \quad B = -\frac{uv}{c^2}, \quad \text{and} \quad C = 1 - \frac{v^2}{c^2}. \) From the basic theory of partial differential equations, PDE’s of this form can be classified as follows:

- **Elliptic**
  
  \( B^2 - AC < 0 \) \( \Rightarrow \text{elliptic} \)

- **Parabolic**
  
  \( B^2 - AC = 0 \) \( \Rightarrow \text{parabolic} \)

- **Hyperbolic**
  
  \( B^2 - AC > 0 \) \( \Rightarrow \text{hyperbolic} \)

The canonical **elliptic** PDE is the Laplace equation: it is governed by smooth solutions where all boundary conditions have an influence on the solution at every point in the domain (think of the steady state temperature distribution in a plate with temperature prescribed along the edges). The classic example of a **parabolic** equation is the heat (or diffusion) equation: in parabolic systems, initial conditions simply spread or diffuse outward over time, but the influence of a source is felt instantaneously (think of temperature diffusing outward into a rod after being given an initial heat pulse at one end). The most familiar example of a **hyperbolic** equation is the wave equation

\[
\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0
\]

where information propagates via waves that move at a specific speed. With a hyperbolic system, a source or disturbance has a clearly defined zone of influence (return to the picture of throwing rocks into a pond or river, as discussed in Section 4.3).

For steady, irrotational flow, the quantity \( B^2 - AC \) can be shown to equal

\[
B^2 - AC = \frac{u^2 + v^2}{c^2} - 1 = \frac{V^2}{c^2} - 1 = M^2 - 1
\]

Thus, the type of PDE governing a flow changes as the flow goes from subsonic to supersonic. **Subsonic flow** is governed by an elliptic PDE: since the flow is everywhere subsonic, the influence of upstream and downstream conditions can be felt everywhere in the flow, and the solution is able to accommodate smoothly. In fact, in the limit as \( V \ll c \) (incompressible limit), the potential equation reverts to the Laplace equation \( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \). In **supersonic flow**, the effect of a boundary or source is only felt in a domain of influence that is defined by the speed at which information propagates. The fact that every point in the solution cannot communicate to everywhere else in the domain can give rise to discontinuous jumps (shock waves).

Hyperbolic partial differential equations in general have a unique feature. Along certain lines, called characteristics, the partial differentials become undefined (or, more properly, indeterminate). Along these lines the PDE may “collapse” to simpler ordinary differential equations or even algebraic equations. You are encouraged to consult a textbook on partial differential equations for a full development; the development here will only give you a taste of the idea. For example, suppose we solve (15.3) for \( \frac{\partial v}{\partial y} \):

\[
\frac{\partial v}{\partial y} = \phi_{yy} = \frac{\frac{2uv \phi_{yy}}{c^2} - \left( 1 - \frac{u^2}{c^2} \right) \phi_{y}}{\frac{2uv \phi_{yy}}{c^2} - \left( 1 - \frac{v^2}{c^2} \right) \phi_{x}}
\]

(15.4)

Thus, \( \frac{\partial v}{\partial y} \) is known as a function of how \( u \) and \( v \) vary along the \( x \)-direction (\( \frac{\partial u}{\partial x} \)) and \( \frac{\partial v}{\partial x} \)). This partial derivative will be indeterminate, however, when \( v = c \) (denominator goes to zero). Consider a point in the flow where this happens. The angle the flow makes with the horizontal is:
Thus, when the angle of the flow with respect to the horizontal becomes equal to the Mach angle, the partial derivative of the \( v \)-velocity component in the \( y \)-direction becomes undefined. The only way the expression (15.4) can avoid “blowing up” in this case is if the numerator is also zero, and the expression is indeterminate. Setting the denominator of (15.4) equal to zero, we obtain the compatibility equation:

\[
\frac{2uv\partial v}{c^2 \partial x} \left( 1 - \frac{u^2}{c^2} \right) \frac{\partial u}{\partial x} = 0 \text{ along the } x\text{-direction.}
\]

This equation is really an ordinary differential equation, and we should replace \( \frac{\partial}{\partial x} \) with \( \frac{d}{dx} \), since it is only dependent on how properties vary in a single direction (the \( x \)-direction). The horizontal line in this case becomes a characteristic.

Note that the orientation of the \( x \) and \( y \) axes is arbitrary; we can always rotate the axes to find an orientation where one axis will be at the Mach angle with respect to the flow. Thus, any line that is at the Mach angle \( \alpha \) with respect to the flow is a characteristic.

The formal technique to find the characteristics of a system is to express the system as algebraic linear equations:

\[
A \frac{\partial^2 \phi}{\partial x^2} + 2B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} = 0
\]

\[
dx \left( \frac{\partial u}{\partial x} \right) + dy \left( \frac{\partial u}{\partial y} \right) + 0 \left( \frac{\partial v}{\partial y} \right) = du
\]

\[
0 \left( \frac{\partial u}{\partial x} \right) + dx \left( \frac{\partial u}{\partial x} \right) + dy \left( \frac{\partial v}{\partial y} \right) = dv
\]

Solving for \( \frac{\partial u}{\partial x} \) using the techniques of linear algebra (Cramer’s rule):

\[
\frac{\partial u}{\partial x} = \frac{0 \begin{vmatrix} A & B \\ B & C \end{vmatrix} - (-1) \begin{vmatrix} B & C \\ du \ dy \end{vmatrix}}{\begin{vmatrix} A & B \\ dx \ dy \end{vmatrix}} = \frac{-du(2Bdy - Cdx) - dvCd\gamma}{Ady - dx(2Bdy - Cdx)}
\]

(15.5)

And likewise for \( \frac{\partial v}{\partial y} \). Setting the denominator equal to zero gives a quadratic equation for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{-uv \pm \sqrt{u^2 + v^2}}{c^2} \frac{1}{1 - \frac{u^2}{c^2}}
\]

Thus, there are two different lines along which the partial derivative becomes indeterminate, this expression gives the slope of these lines. Introducing \( u = V \cos \theta \) and \( v = V \sin \theta \) and \( \alpha = \sin^{-1} \frac{1}{M} = \sin^{-1} \frac{c}{V} \):

\[
\frac{dy}{dx} = \frac{-\cos \theta \sin \theta \pm \sqrt{\cos^2 \theta + \sin^2 \theta}}{\sin^2 \alpha} \frac{1}{\frac{\cos^2 \theta}{\sin^2 \alpha} - 1}
\]

Note that \( \frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \alpha} - 1 = \frac{1}{\tan \alpha} \), so this expression simplifies to:

\[
\frac{dy}{dx} = \tan(\theta \mp \alpha)
\]
The graphical interpretation of this result is shown here:

Note than the expression \( \frac{dv}{du} = \tan(\theta + \alpha) \) gives the slope of a line that is oriented at the Mach angle \(-\alpha\) (or \(\alpha\)) with respect to the flow direction. We have encountered these lines before: they are Mach lines (see Section 4.3).

We now see that Mach lines are also characteristics. We will denote the two Mach lines as the right-running characteristic \(C_I\) and the left-running characteristic \(C_{II}\) respectively. These names derive from the fact that if we disturb the flow at point \(P\) (imagine, dipping a finger into the flow), we would see one Mach line running off toward the right and another toward the left. In the sketch here, the characteristics are shown to curve, because they may be moving into a region of flow where \(\theta\) and \(\alpha\) are changing. But locally (at the point \(P\)), their slopes are given by \(\tan(\theta + \alpha)\).

We now turn to the numerator of (15.5), its value must also be zero in order to keep the flow field derivatives from blowing up:

\[
\frac{dv}{du} = \frac{dx}{dy} \cdot \frac{2B}{C}
\]

Using the fact that this condition will only apply along the characteristics, where we know \(\frac{dx}{dx} = \frac{dv}{dv}_{\text{char}}\), we can obtain (note: many steps of algebra skipped over):

\[
\frac{dv}{du} = \frac{uv + \sqrt{u^2 + v^2 - 1}}{1 - \frac{v^2}{c^2}}
\]

Using \(u = V\cos\theta\) and \(v = V\sin\theta\) and considerable manipulation, we obtain:

\[
d\theta = \pm \sqrt{M^2 - 1} \frac{dV}{V}
\]

This is our compatibility equation, where the “−” sign applies along \(C_I\) characteristics and the “+” sign applies along \(C_{II}\) characteristics. This is an ordinary differential equation, which we can now integrate. Looking back to Section 13.1, however, we note that we already integrated this relation: when integrated, it yields the Prandtl-Meyer Function: \(v(M)\). Thus, our compatibility conditions become algebraic equations:

\[
C_I = v(M) + \theta = \text{constant along right-running characteristics}
\]
\[
C_{II} = v(M) - \theta = \text{constant along left-running characteristics}
\]

Thus, the quantities \(C_I\) and \(C_{II}\) are constant along right-running and left-running characteristic, respectively. This concept takes some getting used to, but is very powerful: imagine a point in a supersonic flow that generates a tiny disturbance. The disturbance will propagate out from that point along Mach lines (one to the right, one to the left), and they carry a piece of information along with them. That information is that the quantity \(C_I = v(M) + \theta\) is a constant (or \(C_{II} = v(M) - \theta\) is a constant). As this disturbance propagates through the flow, it will encounter regions with different flow angle \(\theta\) and different Mach number \(M\). The sum of \(v(M) + \theta\), however, will always remain a constant (provided the flow remains supersonic and irrotational). This remarkable property permits us to solve an entire supersonic flowfield, provided we know the conditions entering the flow boundary.
15.3 Solution Techniques

15.3.1 Solving for a Point in the Flow

Let us assume that there are two points in a supersonic flow at which we have complete knowledge of the flow: Points 1 and 2. At Point 1, the flow is moving upward and to the right with flow angle $\theta = 20^\circ$ at Mach 2. At Point 2, the flow is less inclined ($\theta = 5^\circ$) but moving a bit faster ($M = 2.1$). We will also assume we know the exact $(x,y)$ coordinates of each point.

We can imagine that there is an infinitesimal disturbance emanating from Point 1, that sends characteristic Mach lines streaming off to the left and right. Since we know $\theta$ and Mach number at Point 1, we know the value of $C_I$ and $C_{II}$ these characteristics carry with them: $(C_I)_1 = 26.38^\circ + 20^\circ = 46.38^\circ$ and $(C_{II})_1 = 26.38^\circ - 20^\circ = 6.38^\circ$. Likewise, Point 2 has two characteristics streaming away from it as well: $(C_I)_2$ and $(C_{II})_2$.

At some point in the flow, the right running characteristic from Point 1 $(C_I)_1$ and the left running characteristic from Point 2 $(C_{II})_2$ will intersect. Note that we can approximately locate this point in the flow by making a geometric construction (i.e., straight edge and protractor): we know the right running and left running characteristics have angles $\theta + \alpha$ and $\theta - \alpha$ with respect to the horizontal. While the characteristic may curve as they move through regions of nonuniform flow, we can locally approximate them as straight lines.

Note that while we do not know a priori what the flow properties are at Point 3, we do know that $(C_I)_1 = 46.38^\circ$ and $(C_{II})_2 = 24.07^\circ$. Both these values must apply at Point 3. Using this information, we can solve for $\theta_3$ and $\nu_3$:

$$\theta = \frac{C_I - C_{II}}{2} \quad \nu = \frac{C_I + C_{II}}{2}$$

$$\theta_3 = \frac{46.38^\circ - 24.07^\circ}{2} = 11.15^\circ \quad \nu_3 = \frac{46.38^\circ + 24.07^\circ}{2} = 35.22^\circ$$

Knowing $\nu_3$, we can find Mach number using a table of Prandtl-Meyer function values: $M_3 = 2.34$. Thus, we can completely solve for the conditions at Point 3 (pressure, temperature, etc. can be found using isentropic relations, assuming we know the stagnation temperature and pressure of the flow).

Now that we know the values of $\theta$ and $\alpha$ at Point 3 ($\theta_3 = 11.15^\circ$, $\alpha_3 = 25.30^\circ$), we could revise the geometric construction we used to find the $(x,y)$ location of Point 3 (i.e., get out protractor and straight edge again) by using the average values of the flow and Mach angles:

$$\theta_{13} = \frac{1}{2}(\theta_1 + \theta_3) = 15.58^\circ, \text{ etc. for } \theta_{23}, \alpha_{13}, \alpha_{23}$$

since these values are more representative of the actual flow the characteristics propagate through. Thus, we can improve the accuracy of our solution. Note that the values of the flow properties at Point 3 do not change as we do this iteration, only the location of Point 3 changes.
Note that the Mach number at Point 3 is greater than at Points 1 and 2. This agrees with the intuition we developed in studying one-dimensional flow: since the flow is supersonic and diverging, it should accelerate and Mach number should increase.

Also note that the characteristics do not terminate at Point 3, they continue on through the flow, carrying their information about $C_I$ and $C_{II}$ with them. Alternatively, we can imagine that new $C_I$ and $C_{II}$ characteristics emanate from Point 3; their values of $(C_I)_3$ and $(C_{II})_3$ will be the same as $(C_I)_1$ and $(C_{II})_2$. When they intersect characteristics of another type (e.g., a right running characteristic intersects a left running characteristic), we can solve for that new point in the flow via the same technique.

15.3.2 Solving for a Point on the Wall

Let us return to the $C_I$ characteristic that emanated from Point 1. Let us assume that this characteristic eventually encounters a wall. Again, we can estimate the point of this intersection with the wall via a geometric construction, assuming the characteristic remains at a constant angle $\theta_1 + \alpha_1$. Where it hits the wall at Point 4, value of $(C_{II})_1$ must still apply. We do not have the $C_I$ characteristic at the point, but since the flow must be parallel to the wall, we do have a value of $\theta$ (say, $\theta_{wall} = \theta_4 = 13^\circ$). Thus, we can solve for $\nu_4$ and Mach number:

$$\nu_4 = (C_{II})_1 + \theta_4 = 19.38^\circ$$

So, $M_4 = 1.75$, from a look-up of Prandtl-Meyer function values. Thus, we now have complete knowledge about this point on the wall, including the value of $(C_{II})_4$ for a new right-running characteristic that is emanated from this point.

*In practice, it is often not worth correcting the position of the new point using the average values. In a numerical solution, it is often easier to just add more characteristics, thus shortening the distance over which they need to travel before solving for the next point and thereby decreasing the error. By continuing to increase the number of characteristics, you can quickly converge to a final solution.

15.4 Solving a Complete Flowfield

By combining the point-wise solution techniques developed above, we can now march through a supersonic flow and solve for each point where characteristics intersect. This requires that we have complete knowledge of the flow along some line (not a characteristic line). Typically, this is an upstream boundary where the flow enters the domain of interest. For a nozzle, for example, we would need to know the flow conditions at some line or plane downstream of the throat. Note that we cannot initialize our characteristic solution with the sonic flow at the throat, since the method of characteristics breaks down for sonic flow. We need to initialize the solution with a supersonic flow (say, Mach 1.1).

Once we have an upstream boundary specified, we can start streaming characteristics downstream from points on the boundary. Where the characteristics intersect, we can solve for the flow conditions at that point. If the characteristics intersect the wall, we will know the wall angle and thus can
solve for the flow there, and stream new characteristics back into the solution domain.

Constructing a table of flow variables and characteristics is often essential to successfully implement the method. This is illustrated in the numerical example below.

### 15.5.1 Numerical Example: Supersonic Flow into a Converging Channel

Consider supersonic flow entering a channel that has converging walls of fixed angle $10^\circ$. The flow enters at Mach 3, and is parallel to the wall at the wall and runs horizontal at the center line of the channel. Using the method of characteristics, solve for the flow as it continues down the channel.

We will initialize our solution by choosing three points on the boundary that was specified (the inlet plane of the channel): Points 1, 2, and 3. Since we have complete knowledge about these points, we can determine the values of $C_I$ and $C_{II}$ and stream those values into the flow. For example, the $C_I$ characteristic from Point 1 intersects the $C_{II}$ characteristic from Point 2 at Point 4. Knowing $C_I$ and $C_{II}$, we solve for Point 4, and then stream characteristics on from Point 4 to Points 6 and 7.

In the table shown here, values that are known \emph{a priori} either from the initial boundary or the wall are denoted with a heavy box. Values that are obtained by streaming characteristics from known values are denoted with a double-lined box. Once any two of the four parameters ($C_I$, $C_{II}$, $\nu$, $\theta$) is known, the other two are automatically determined, and then the Mach number and other flow parameters (Mach angle, etc.) can be solved for. These other values are listed in the table ($M$, $\theta$, $\theta + \alpha$, and $\theta - \alpha$). They are used in performing the graphical construction to determine where the points will lie in physical space. We have stopped at Point 7, but the method can be extended further.

Note that solving for the lower half of the channel was redundant. Since the channel was symmetric, we could have replaced the center line with a horizontal wall and the solution would have remained the same. If we wish to increase the accuracy of our solution, we can increase the number of initial characteristics emitted from the initial datum line. For example, we can introduce a point halfway between Points 1 and 2.

### 15.6 Breakdown of the Method

The solution above can be continued until one of two things happen:

- Characteristics running in the same direction cross.
- The flow becomes subsonic.
“Characteristics running in the same direction” means, for example, two $C_1$ characteristics, streaming from different points in space, intersect, as shown here. When this happens, technically the two waves merge to form a stronger, finite amplitude wave that can no longer be considered a characteristic; it has become a shock wave (albeit a weak one). If you review Chapter 13, this is exactly how we discussed oblique shock waves forming on a smoothly curved corner: Mach lines (or characteristics) intersect and coalesce to form an oblique shock. Once a shock wave appears, our flow is no longer isentropic or irrotational, the key assumptions upon which our method was developed. If the shock wave generated is weak, it may be possible to “get away with” continuing to treat it as a characteristic, although the fact that two characteristics have now become one can wreak havoc on our bookkeeping system (or computerized algorithm).

If characteristics continue to merge, the oblique shock gets stronger, and we really cannot justify continuing the solution. There are ways the method of characteristics can be modified to handle these situations, but finite difference CFD codes are typically much better in these cases.

The other thing that will cause our solution method to break down is the appearance of sonic (or subsonic) flow. Again, there are ways that the method of characteristics can be modified to handle interacting with regions of subsonic flow (although the method itself cannot apply in those regions).

### 15.7 Some Comments

As one is implementing the method of characteristics, the operation using the information propagated along characteristics to solve for the flow at a new point where characteristics interact becomes routine, consisting of simple addition and subtraction and table look-ups of $v(M)$ and $\alpha(M)$. The main challenge becomes careful bookkeeping of the characteristic information. As simple as these operations are, one should not lose sight of the fact that, in implementing the method of characteristics, **you are actually solving a coupled set of nonlinear partial differential equations**! This is no small feat, since in general, solving nonlinear PDE’s is a task that demands considerable mathematical prowess or serious computational power.

Another thought to keep in mind in implementing the method of characteristics is that, to some degree, this is how Nature itself “solves” a compressible flow. In actuality, there are an infinite number of characteristics in a real, physical domain of flow: they form a continuum. In the method of characteristics, these are approximated by a finite number of characteristics. In laboratory experiments, the trajectory of a few characteristics can be observed by, for example, making scratches on the walls of a channel to induce a finite (but still very weak) disturbance that can be visualized by techniques such as Schlieren. This has been done in the photograph shown here, where grooves that have been scratched into the walls of a wind tunnel permit the characteristics to be visualized. **Characteristics are real.** (The zebra-patterns here are not related to characteristics; they are a result of a flow visualization technique called interferometry that permits us to visualize density contours in a flow.)

Finally, if the development of the method of characteristics seems so mathematical as to be opaque, then you might look forward to Chapter 16, where the method is developed for one-dimensional unsteady flow. In one-

dimensional unsteady flow, the characteristics again turn out to be acoustic waves, but the characteristics and compatibility conditions can be derived in a more transparent and intuitive fashion.

15.8 Examples of Applying the Method of Characteristics*

The method of characteristics can be implemented in a computer program such that a large number of characteristics can be tracked. This has been done using Matlab, and the results for an initially slightly supersonic flow ($M_u = 1.000001$) entering a channel with an upper wall that diverges at a constant angle of $20^\circ$ are shown below. The method of characteristics solution was initialized with a Prandtl Meyer expansion fan at the corner: knowing the Prandtl-Meyer flow, a finite number of characteristics were specified. The characteristics then reflected off the lower wall and proceeded to “bounce” back and forth between the two walls, with each interaction between characteristics being solved via the methods outlined above.

It is interesting to see how this two-dimensional solution compares with the one-dimensional solution to isentropic flow in a diverging channel developed in Chapter 5. This comparison is made below, in which the Mach number of the flow at each characteristic intersections is plotted as a function of the area at that cross section (normalized by the inlet area where the flow is sonic). This forms a dense “cloud” of points, since there are a large number of characteristics being used. The solid line is the $\frac{A}{A_i}$ relation for one-dimensional flow from Chapter 5. Note the excellent agreement between the two solutions. The agreement can be made even better if the channel area is modified to vary more slowly. This result is a good validation of the method of characteristics solver (or a good validation of one-dimensional flow, depending on your perspective).

A method of characteristics program such as this can be used to solve more complex problems where one dimensional solutions are not available, such as the supersonic flow over a curved wedge. A solution to the hypersonic flow ($M = 17.7$) over a curved wedge whose surface is described by a power law ($y = 0.2 x^{0.8}$) using the same program is shown below. Note that, due to the

*Calculations in this section performed by Patricia Vu, McGill University.
curved shock wave, the flow is no longer irrotational. Thus, the solver needed to track an additional characteristic (the particle paths) in order to account for the nonuniform entropy of the flow. This required a minor modification to the solution techniques developed in this chapter.