Introduction to 3-dimensional Coordinate Space

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1 Outline

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2 The 3rd Dimension

Objectives

• Become familiar with 3D coordinate system.
• Interpret equations of planes, lines and spheres.
• Generalize the distance formula to points in 3 dimensions

2.1 Distance and Spheres

The Distance Formula

Recall the formula for the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in $\mathbb{R}^2$ (the $xy$-plane):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$ 

What do you think the formula between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in $\mathbb{R}^3$ (3-space) is?

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$ 

Equation of a circle
Similarly, recall that a circle with radius \( r \) and center \( C(x_0, y_0) \) is the set of points \( P(x, y) \) in \( \mathbb{R}^2 \) at a distance \( r \) from \( C \), or, in other words, those which satisfy the equation
\[
r = \sqrt{(x - x_0)^2 + (y - y_0)^2},
\]
more commonly written as
\[
(x - x_0)^2 + (y - y_0)^2 = r^2.
\]

Equation of a Sphere
Define the sphere with radius \( r \) and center \( C(x_0, y_0, z_0) \).

Definition 1. The sphere is the set of points \( P(x, y, z) \) in \( \mathbb{R}^3 \) at a distance \( r \) from \( C \), i.e. the points which satisfy the equation
\[
(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.
\]

Example 2 (Exercise 12.2.52). Find the center and radius of the sphere \( 3x^2 + 3y^2 + 3z^2 + 2y - 2z = 0 \).

Solution 3. We must first complete the square for the variables \( y \) and \( z \)
\[
3y^2 + 2y = 3 \left( y^2 + \frac{2}{3}y + \frac{1}{9} \right) = 3 \left( y + \frac{1}{3} \right)^2 - \frac{1}{3}
\]
Similarly, \( 3z^2 - 2z = 3 \left( z - \frac{1}{3} \right)^2 - \frac{1}{3} \). So the equation becomes
\[
3x^2 + 3 \left( y + \frac{1}{3} \right)^2 + 3 \left( z - \frac{1}{3} \right)^2 = \frac{2}{3}.
\]
Dividing by 3, we see the radius is \( \frac{\sqrt{2}}{3} \) and the center \( (0, -\frac{1}{3}, \frac{1}{3}) \).

2.2 Lines and Planes
Equation of a Plane
Example 4. Consider the equation \( x = 1 \). What set of points does this equation represent?
- In \( \mathbb{R}^2 \), \( x = 1 \) is the set of points of the form \((1, y)\), i.e. the vertical line crossing the x-axis at \((1, 0)\).
- In \( \mathbb{R}^3 \), the \( z \)-coordinate is unrestricted, so \( x = 1 \), which represents the points of the form \((1, y, z)\), describes the plane parallel to the \( yz \)-plane, and containing the point \((1, 0, 0)\).

The same equation may describe either a line or a plane in different contexts.

Equation of a Line
In \( \mathbb{R}^3 \), \( x = 1 \) describes a plane. To describe a line, we make additional restrictions.
Example 5. Write an equation for the line parallel to the \( y \)-axis, passing through the point \((1, 2, 3)\).

Solution 6. All points on this line must have the same \( x \) and \( z \) coordinates, but their \( y \)-value is unrestricted. Rather than a single equation, the line can be described by restricting two coordinates, with the system
\[
\begin{cases}
  x = 1 \\
  z = 3
\end{cases}
\]
Equation of a Circle

Example 7. Describe the circle of radius 5 centered at (3, 2, 1) and lying in a plane parallel to the $xz$-plane.

Solution 8. • The circle lies in the plane parallel to the $xz$-plane, containing the point (3, 2, 1). This gives the restriction $y = 2$.
  • The variables $x$ and $z$ are bound to each other by the equation of the circle $(x - 3)^2 + (z - 1)^2 = 25$.
  • Together, the equations $y = 2$ and $(x - 3)^2 + (z - 1)^2 = 25$ describe this circle.

2.3 Inequalities

Modifying the Equation of a Circle

• If $y = 2$ is omitted from the last example, then $(x - 3)^2 + (z - 1)^2 = 25$ describes an infinite cylinder stretching parallel to the $y$-axis.
  • Restricting $y$ not to a single value, but to a range of values, say,

$$\begin{cases} 2 \leq y \leq 5 \\ (x - 3)^2 + (z - 1)^2 = 25 \end{cases}$$

allows us to describe a cylinder stretching along the $y$-axis from $y = 2$ to $y = 5$.

Equation of a Ring

Example 9. Describe the ring in the $yz$-plane with center at (0, 1, 2), and bounded by the circles with radii 3 and 4.

Solution 10. • The equations of the inner circle are $(y - 1)^2 + (z - 2)^2 = 9$ and $x = 0$.
  • The equations of the outer circle are $(y - 1)^2 + (z - 2)^2 = 16$ and $x = 0$.
  • The equations for the region bounded by the two are

$$\begin{cases} 9 \leq (y - 1)^2 + (z - 2)^2 \leq 16 \\ x = 0 \end{cases}$$

3 Vectors (part 1)

Objectives

• Understanding the properties of a vector
  • Add vectors pictorially
  • Convert vectors between component and polar form
3.1 Introduction

What is a Vector?

- A vector is characterized by
  - a quantity called magnitude, length, or norm
  - a direction
  - a lack of position in space

- A vector is opposed to a scalar, a word used to describe a simple quantity (a real number) and emphasize its lack of a direction.

- A vector can be
  - represented geometrically by a directed line segment or arrow
  - defined by a startpoint and an endpoint
  - defined in rectangular or component form in $\mathbb{R}^2$ and $\mathbb{R}^3$
  - defined in polar form in $\mathbb{R}^2$

Adding Vectors Geometrically

Vectors add head-to-tail. It does not matter how we get there. We only care about where we start and where we end up.

Example 11. $\overrightarrow{AB}$ is the vector with startpoint $A$ and endpoint (arrowhead) $B$.

- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\overrightarrow{AC} + \overrightarrow{DA} = \overrightarrow{DA} + \overrightarrow{AC} = \overrightarrow{DC}$.
- Try this: $\overrightarrow{DB} + \overrightarrow{CO} + \overrightarrow{BC} + \overrightarrow{AD}$ (Make sure to try it first.)
  
  - $\overrightarrow{DB} + \overrightarrow{CO} + \overrightarrow{BC} + \overrightarrow{AD} = \overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC} + \overrightarrow{CO} = \overrightarrow{AO}$.

3.2 Vector Representation

Component Form and Magnitude

Component Form

The component form of a vector is given by the coordinates of the headpoint when the tailpoint is at the origin, i.e., when the vector is in standard position.

Magnitude

To find the magnitude of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ find the distance between the headpoint and tailpoint. When $\vec{v}$ is in standard position, the headpoint is at the origin, and the tailpoint at $(v_1, v_2, v_3)$, so the magnitude of $\vec{v}$ is

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$
Example 12. Let \( A(1,2,3) \) and \( B(0,-1,-2) \). Find the component forms of \( \overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AA} \) and their corresponding magnitudes.

- In general, subtract the coordinates of the first (tail) from those of the second (head). (Think “second minus first” componentwise.)
- \( \overrightarrow{AB} = (0 - 1, -1 - 2, -2 - 3) = (-1, -3, -5) \).
- \( \overrightarrow{BA} = (1 - 0, 2 - (-1), 3 - (-2)) = (1, 3, 5) \).
- \( \overrightarrow{AA} = (0, 0, 0) \).
- The magnitudes can be found from the magnitude formula or directly from the distance formula between the points. \( |\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{35} \) and \( |\overrightarrow{AA}| = 0 \).

The Zero Vector and the Opposite of a Vector

- We saw that \( \overrightarrow{AA} \) has magnitude 0. The vector with magnitude 0
  - is unique and is denoted \( \vec{0} \) or 0.
  - should not be confused with the scalar 0.
  - is the only vector without direction.
  - has all components equal to 0.
- Since \( \overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{0} \), we say that \( \overrightarrow{BA} \) is the opposite of \( \overrightarrow{AB} \) and write \( \overrightarrow{BA} = -\overrightarrow{AB} \). In general, the opposite \( -\vec{v} \) of a vector \( \vec{v} \)
  - has the same magnitude as \( \vec{v} \)
  - has the direction opposite \( \vec{v} \)
  - is defined to be the vector whose sum with \( \vec{v} \) results in the zero vector.

3.3 Vector Conversion

Converting from rectangular to polar form

Example 13. Let \( \vec{v} \) be the vector with magnitude 2, which makes an angle of \( \theta = \frac{2\pi}{3} \) radians with the positive x-axis.

Find the x and y components of the vector.

Solution 14. Let \( \vec{v} = (v_1, v_2) \). Then

\[
v_1 = |\vec{v}| \cos \theta \text{ and } v_2 = |\vec{v}| \sin \theta.
\]

That is, the components are \( v_1 = 2 \cos(\frac{2\pi}{3}) = -1 \) and \( v_2 = 2 \sin(\frac{2\pi}{3}) = \sqrt{3} \).

Converting from polar to rectangular form

Example 15. Let \( \vec{v} = (1, 2) \). Find the magnitude and angle from the positive x-axis.

- The magnitude is \( |\vec{v}| = \sqrt{1^2 + 2^2} = \sqrt{5} \).
- The angle from the positive x-axis is \( \theta = \tan^{-1}(\frac{2}{1}) = \tan^{-1}(2) \).

Note

The angle \( \theta' \) that \( \vec{w} = (-1, -2) \) makes with the positive x-axis is calculated slightly differently. Because the x-component of \( \vec{w} \) is negative, we add \( \pi \) to the final answer.

\[
\theta' = \tan^{-1}(-\frac{2}{-1}) + \pi = \tan^{-1}(2) + \pi.
\]
Vector Notation

A vector can be denoted by

- a single letter with
  - bold face, common in print (e. g. \( \mathbf{v}, \mathbf{F} \)), or
  - an arrow above (e. g. \( \vec{v}, \vec{F} \)), or
  - a tilde below, sometimes used in handwritten documents to indicate bold face (e. g. \( \vec{v}, \vec{F} \))

- two capital letters with an arrow above, when defined by endpoints. (e. g. \( \overrightarrow{AB} \))

- its component form: \( \langle v_1, v_2, v_3 \rangle \) or \( v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \)

Special vectors

- standard unit vectors: \( \hat{i} = \langle 1, 0, 0 \rangle, \hat{j} = \langle 0, 1, 0 \rangle, \hat{k} = \langle 0, 0, 1 \rangle \)

- zero vector: \( 0 \) or \( \vec{0} \)
Scalar product and Dot product

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Objectives

• Add vectors algebraically
• Define and use the scalar product
• Calculate the dot product given the vectors componentwise
• Calculate the dot product given the magnitudes and angle between the vectors
• Interpret the dot product

2 Vectors (part 2)

2.1 Addition of Vectors

Adding Vectors componentwise
Recall that, geometrically, vectors add head to tail.

Rule
Equivalently, vectors add componentwise in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \); this generalizes to an arbitrary number of dimensions.

Example 1. Let \( A(1, 2) \), \( B(-2, -3) \) and \( C(0, 2) \). Then \( \overrightarrow{AB} = (-3, -5) \), \( \overrightarrow{BC} = (2, 5) \) and \( \overrightarrow{AC} = (-1, 0) \). And indeed, we see that if we add these vectors componentwise, that is,

\[
    \text{for } x : -3 + 2 = -1 \text{ and for } y : -5 + 5 = 0,
\]

we have \( \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \).
Adding Vectors in polar form

Example 2. Let \( \mathbf{v} \) and \( \mathbf{w} \) be vectors in \( \mathbb{R}^3 \) such that \( |\mathbf{v}| = 2 \), \( |\mathbf{w}| = 3 \) and with respective angles \( \theta_{\mathbf{v}} = \frac{3\pi}{2} \) and \( \theta_{\mathbf{w}} = \frac{2\pi}{3} \).

Solution 3. Clearly we cannot add the angles and magnitudes together, as the resulting vector would not be the desired head-to-tail vector.

- convert the vectors to component form:
  - \( \mathbf{v} = \langle 3\cos(3\pi/2), 3\sin(3\pi/2) \rangle = \langle 0, -3 \rangle \)
  - \( \mathbf{w} = \langle 2\cos(2\pi/3), 2\sin(2\pi/3) \rangle = \langle -1, \sqrt{3} \rangle \)
- add them componentwise: \( \mathbf{v} + \mathbf{w} = \langle -1, \sqrt{3} - 3 \rangle \)
- convert the resulting vector back to polar form:
  - \( |\mathbf{v} + \mathbf{w}| = \sqrt{1 + (\sqrt{3} - 3)^2} = \sqrt{13 - 6\sqrt{3}} \)
  - \( \theta_{\mathbf{v}+\mathbf{w}} = \tan^{-1}(3 - \sqrt{3}) + \pi \) (add \( \pi \) since \( (\mathbf{v} + \mathbf{w})_x < 0 \).)

2.2 Scalar Multiplication

Scalar or Vector?

Examples

Are the following expressions vector or scalar quantities? If it is a vector, find its length.

- \( |\mathbf{\vec{u}}| \): scalar
- \( |\mathbf{\vec{u}}| \mathbf{\vec{u}} \): vector because it is the result of a scalar product. \( |\mathbf{\vec{u}}| \mathbf{\vec{u}} |\mathbf{\vec{u}}| = |\mathbf{\vec{u}}|^2 \) by the rule \( a\mathbf{\vec{u}} = |a||\mathbf{\vec{u}}| \).
- \( \frac{\mathbf{\vec{u}}}{|\mathbf{\vec{u}}|} \): vector, this is the scalar product \( \frac{1}{|\mathbf{\vec{u}}|} \mathbf{\vec{u}} = \frac{1}{|\mathbf{\vec{u}}|} \mathbf{\vec{u}} = |\mathbf{\vec{u}}| \frac{\mathbf{\vec{u}}}{|\mathbf{\vec{u}}|} = 1 \). \( \frac{\mathbf{\vec{u}}}{|\mathbf{\vec{u}}|} \) is called a unit vector, that is, it has length 1.
- \( \frac{|\mathbf{\vec{u}}|}{\mathbf{\vec{u}}} \): neither, division by a vector is undefined.

Vector Expressions

Examples

Write an expression for the following expressions:

- a unit vector in the direction of the sum of \( \mathbf{a} \) and \( \mathbf{b} \).
  - to obtain a unit vector, divide a vector in the correct direction by its length: \( \frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|} \).
- a vector with the same magnitude as \( \mathbf{a} \) in the same direction as \( \mathbf{b} \).
  - to get a vector of the desired magnitude, first obtain a unit vector in the desired direction (\( \mathbf{b}/|\mathbf{b}| \)) and multiply by the right magnitude: \( |\mathbf{a}| \frac{\mathbf{b}}{|\mathbf{b}|} \).
3 The Dot Product

3.1 Computation

The Dot Product is a Scalar

The dot product, denoted by the symbol “·”, is an operation on two vectors whose result is a scalar.

Examples

Which of the following expressions are defined?

- \((\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}\): no, since \(\mathbf{u} \cdot \mathbf{v}\) is a scalar, it can’t be dotted with \(\mathbf{w}\).
- \((a\mathbf{u}) \cdot \mathbf{v}\) where \(a \in \mathbb{R}\) (a is real): yes, since \(a\mathbf{u}\) is scalar product, hence a vector, it can be dotted with \(\mathbf{v}\).
- \(a(\mathbf{u} \cdot \mathbf{v})\): yes, the dot product is between vectors, and their result is multiplied by \(a\) in the real sense of product.
- \(|\mathbf{u}| \cdot \mathbf{v}\): no, \(|\mathbf{u}|\) is a scalar.
- \(|\mathbf{u} \cdot \mathbf{v}|\): yes. Note that the absolute value sign represents real absolute value (as in \(|-1| = 1\)) and not magnitude because its argument is a scalar.

Calculation in Component Form

Definition 4. The dot product of two vectors \(\mathbf{u} = \langle u_1, u_2, u_3 \rangle\) and \(\mathbf{v} = \langle v_1, v_2, v_3 \rangle\) in \(\mathbb{R}^3\) is defined to be

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.
\]

Because this rule generalizes to \(\mathbb{R}^n\) (\(n\)-dimension) and in particular to \(\mathbb{R}^2\), we take this to be the definition of dot product. Notice that, by this definition, the dot product is commutative: \(\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}\). Why? Because

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \quad \text{and} \quad \mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3,
\]

and these two are equal.

Do not confuse the different products.

Dot Product

- takes two vectors
- computes a scalar

Scalar Product

- takes a scalar and a vector
- computes a vector

Real Product (regular product)

- takes two scalars, i.e., reals
- computes a scalar, i.e., a real.
Calculation with Magnitudes and Angle Between

Intuitively, the dot product \( \vec{u} \cdot \vec{v} \) represents the (real) product of the magnitude of \( \vec{u} \) and the length of the projection of \( \vec{v} \) onto \( \vec{u} \). The length of this projection can be computed as \( |\vec{v}| \cos \theta \), where \( \theta \) is the angle between the two vectors. In other words, we could simply define the dot product as

\[
\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta.
\]

However, this calculation does not easily generalize to \( \mathbb{R}^n \). How would you define an angle in, say, \( \mathbb{R}^4 \)? (see linear algebra class)

**Magnitude**
In particular, for any vector, \( \vec{v} \cdot \vec{v} = |\vec{v}| |\vec{v}| \cos 0 \), so

\[
|\vec{v}|^2 = \vec{v} \cdot \vec{v}.
\]

**Remarks on the Dot Product**

- In defining the dot product geometrically, it is equally fair to say that \( \vec{u} \cdot \vec{v} \) represents
  - the length of \( \vec{u} \) times the length of the projection of \( \vec{v} \) onto \( \vec{u} \)
  - the length of \( \vec{v} \) times the length of the projection of \( \vec{u} \) onto \( \vec{v} \)

because the first is given by \( |\vec{u}| (|\vec{v}| \cos \theta) \) and the second by \( |\vec{v}| (|\vec{u}| \cos \theta) \).

- Given \( \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \), we notice that
  - \( \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \) when the vectors point in the same direction,
  - \( \vec{u} \cdot \vec{v} = -|\vec{u}| |\vec{v}| \) when the vectors point in opposite directions,
  - \( \vec{u} \cdot \vec{v} = 0 \) exactly when the vectors are perpendicular.

- To verify that the two forms of the dot product agree for \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), use the Law of Cosines, or see the proof of Theorem 1 (p 847) in the textbook.

### 3.2 Interpretation

**Projection**

- Find the length of the projection of \( \vec{u} \) onto \( \vec{v} \), which is also called the *scalar component of \( \vec{u} \) in the direction of \( \vec{v} \).*
  - As mentioned before, it is given by \( |\vec{u}| \cos \theta \) where \( \theta \) is the angle between the two vectors. Also, since \( \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \), we have that this scalar component is equal to which, when written as \( \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} \), can be interpreted as its dot product with the unit vector in the direction of \( \vec{v} \).

- Find the projection of \( \vec{u} \) onto \( \vec{v} \).
  - The projection vector’s magnitude is given by the previous question, and its direction is that of \( \vec{v} \), so it is equal to \( \text{proj}_\vec{v} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \).
Example (12.3.10)

Let $u = 2i - 2j + k$, $v = 3i + 4k$, $\theta$ be the angle between $u$ and $v$.

- $u \cdot v = 2 \cdot 3 - 2 \cdot 0 + 1 \cdot 4 = 10$, $|u| = 3$ and $|v| = 5$
- $\cos \theta = \frac{u \cdot v}{|u||v|} = \frac{2}{3}$ and $\theta = \cos^{-1} \left( \frac{2}{3} \right)$
- scalar projection: $|\text{proj}_u v| = v \cdot \frac{u}{|u|} = \frac{10}{3}$
- component of $v$ parallel to $u$: $\text{proj}_u v = \frac{v \cdot u}{|u|^2} u = \frac{10}{3} u = \frac{20}{3} i - \frac{20}{3} j + \frac{10}{3} k$
- component of $v$ perpendicular to $u$: $v - \text{proj}_u v = (3i + 4k) - \left( \frac{20}{3} i - \frac{20}{3} j + \frac{10}{3} k \right) = \frac{7}{3} i - \frac{20}{3} j + \frac{26}{3} k$
- decomposition of $v$ in terms of $u$: $v = \left( \frac{20}{9} i - \frac{20}{9} j + \frac{10}{9} k \right) + \left( \frac{7}{9} i - \frac{20}{9} j + \frac{26}{9} k \right)$
Cross product and Equations of Lines

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Objectives

• Calculate the cross product of 2 vectors
• Compute the area of a triangle given 3 points
• Compute the volume of a parallelepiped
• Parametrize a line
• Compute the distance between a point and a line

2 Cross Product

The Cross Product is a Vector

Unlike the dot product, the result of a cross product is a vector.

Examples

Are the following expressions defined? Explain.

• $x \cdot (y \times z)$: Yes, the result of the cross product is a vector, which can be dotted with $x$.
• $(x \cdot y) \times z$: No, the result of the dot product is a scalar, which cannot be crossed with $z$.
• $(x \cdot y) (z \times w)$: Yes, this is the scalar product of the scalar $x \cdot y$ and the vector $z \times w$.
• $(x \times y) (z \times w)$: No, the two vectors $x \times y$ and $z \times w$ are neither crossed nor dotted.
• $(x \cdot y) (z \cdot w)$: Yes, this is the real product of the scalars $x \cdot y$ and $z \cdot w$.

2.1 Magnitude of the Cross Product

Magnitudes and Angle between

The magnitude of the cross product $u \times v$ can be computed by

$$|u \times v| = |u||v| \sin \theta$$

where $\theta$ is the angle between the two vectors.
• Recall, dotting a vector \( u \) with a unit vector \( \hat{v} \) yields the length of the component of \( u \) parallel to \( \hat{v} \).

• Similarly, crossing a vector \( u \) with a unit vector \( \hat{v} \) and taking its magnitude yields the length of the component of \( u \) perpendicular to \( \hat{v} \). (Convince yourself of this fact by drawing a picture.)

• The magnitude \( |u \times v| \) represents the area of the parallelogram formed by \( u \) and \( v \).

With Components

The cross product can be directly computed from the components.

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix}
\]

where \( u = \langle u_1, u_2, u_3 \rangle \) and \( v = \langle v_1, v_2, v_3 \rangle \). The result is a vector.

Example 1. Let \( u = 2\mathbf{i} + \mathbf{j} \) and \( v = -3\mathbf{i} + 2\mathbf{k} \). Find the dot, cross and its magnitude.

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 1 & 0 \\
-3 & 0 & 2
\end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}
\]

Hence \( |\mathbf{u} \times \mathbf{v}| = \sqrt{29} \). Also \( \mathbf{u} \cdot \mathbf{v} = -6 \) (not a vector).

2.2 Direction of the Cross Product

The Cross Product Defined Only in \( \mathbb{R}^3 \)

The cross product is an operation defined uniquely for vectors in \( \mathbb{R}^3 \), unlike the dot product which generalizes to vectors in \( \mathbb{R}^n \).

Example 2. Find the dot, cross and magnitude for \( \vec{u} = \langle 1, 2 \rangle \) and \( \vec{v} = \langle 1, -1 \rangle \).

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 2 & 0 \\
1 & -1 & 0
\end{vmatrix} = -3\vec{k}
\]

\( |\vec{u} \times \vec{v}| = 3 \) and \( \vec{u} \cdot \vec{v} = -1 \). To compute the cross, the vectors must be considered as vectors in \( \mathbb{R}^3 \). The resulting vector is purely in the \( \vec{k} \) direction. In other words, the cross product is perpendicular to both the original vectors.

The Right-Hand Rule

We know that \( (\mathbf{u} \times \mathbf{v}) \perp \mathbf{u} \) and \( (\mathbf{u} \times \mathbf{v}) \perp \mathbf{v} \). This leaves two possible directions for the cross product, the correct one being determined by the so-called right-hand rule:

• sweep the fingers of your right hand across the area between \( \mathbf{u} \) to \( \mathbf{v} \), and specifically from \( \mathbf{u} \) to \( \mathbf{v} \).

• Curl the fingers, the thumb will point in the direction of the cross product \( \mathbf{u} \times \mathbf{v} \), call it \( \mathbf{w} \).

Now sweep the fingers across from \( \mathbf{v} \) to \( \mathbf{u} \) (you have to twist your wrist to do that); the cross product points in the opposite direction. In other words, if \( \mathbf{u} \times \mathbf{v} = \mathbf{w} \), then \( \mathbf{v} \times \mathbf{u} = -\mathbf{w} \).

\[
\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}
\]

Remember: the cross product is not commutative, but the scalar product is.
The Right-Hand Rule and Standard Unit Vectors

Finally, if you apply the right-hand rule to $i$ to $j$, then $i \times j = k$. This is why $\mathbb{R}^3$ is called a right-hand coordinate system. Also, you can rotate the vectors in the previous equation, and the result will still hold:

$$i \times j = k \quad k \times i = j \quad j \times k = i$$

**Question**
What is $j \times i$ equal to? Answer: $j \times i = -i \times j = -k$

**Example**
Find the cross product of $u = 2i + j$ and $v = -3i + 2k$ using the right-hand rule for the standard unit vectors.

$$u \times j = (2i + j) \times (-3i + 2k)$$

$$= -6i \times i + 4i \times k - 3j \times i + 2j \times k$$

$$= 0 - 4j + 3k + 2i = 2i - 4j + 3k.$$  

**Remarks**
- $i \times i = 0$. In general $u \times v = 0$ if the two vectors are parallel, since the parallelogram formed by them has area 0.
- This cross product was computed earlier with determinants. You should be reassured that the two do agree indeed.

### 2.3 The Triple Scalar Product

**Triple Scalar Product**

- Calculating the triple scalar product is straightforward, simply apply the determinant of the coordinates.

$$(u \times v) \cdot w = \left| \begin{array}{ccc} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{array} \right|$$

- Note that unlike the cross product alone, the result is a scalar.
- This scalar quantity represents the volume of the parallelepiped defined by the three vectors.

### 3 Equations of Lines

#### 3.1 Parametrization

**Parametrization of a line**

A line in $\mathbb{R}^2$ or $\mathbb{R}^3$ is determined by a point and a direction. Hence it can be described by the vector function

$$r(t) = r_0 + vt$$

where
- $r_0$ represents the position of a point on the line
- $v$ represents the direction of the line
- $t$ is any real number; there is no restriction on $t$ because the line is infinite in both directions.
Equation of a Line

Example 3. Let \( A(0, 1, 2) \) and \( B(1, -1, 0) \).

- Since \( \vec{AB} = (1, -2, -2) \), a parametrization for the line through \( A \) and \( B \) is
  \[
  r(t) = (0, 1, 2) + (1, -2, -2) t, \quad t \in \mathbb{R}.
  \]

- The half line starting at \( A \) in the direction of \( B \) is parametrized by the same equation, but with the restriction \( t \geq 0 \).

- The segment from \( A \) to \( B \) can be parametrized by the same equation, with \( 0 \leq t \leq 1 \) since \( r(0) = r_A \) and \( r(1) = r_B \).

Equation of a Line (cont.)

Example 4. The segment with \( A \) as an endpoint and \( B \) as a midpoint can be parametrized by

\[
\begin{align*}
  r(t) &= (0, 1, 2) + (1, -2, -2) t, \quad 0 \leq t \leq 2 \\
  &\text{or, moving twice as fast,} \\
  r(t) &= (0, 1, 2) + (2, -4, -4) t, \quad 0 \leq t \leq 1.
\end{align*}
\]

Parametrization as Position of Traveling Particle

The parametrization can be thought of as the position of a point-particle at time \( t \) traveling along the line. The parametrization of the line is not unique and can vary according to

- \( r_0 \): any point on the line can be used as the initial position.
- \( v \): any vector parallel to \( v \) can be used. \( v \) can be thought of as the velocity of the traveling particle and \( |v| \) as its speed.

3.2 Distance from a Point to a Line

Distance from a Point \( P_0 \) to a Line \( \ell \)

Key Idea

- Find a point \( P \) on the line
- Find the direction vector \( \overrightarrow{P_0P} \) from the point on the line to \( P_0 \)
- Cross this vector \( \overrightarrow{P_0P} \) with a unit vector \( \hat{v} \) in the direction of the line and take its magnitude to find the length of the component of \( \overrightarrow{P_0P} \) perpendicular to \( \hat{v} \).

In short,

\[
d = |\overrightarrow{P_0P} \times \hat{v}| = \frac{|\overrightarrow{P_0P} \times v|}{|v|}.
\]

where \( P \) is a point on \( \ell \) and \( v \) is a vector in the direction of the line.

Final Comments on Vectors

We have seen that vectors, such as those in the parametrization \( r(t) = r_0 + vt \), come in two kinds:

- position vectors: such as \( r_0 \) and \( r(t) \), they represent the coordinates of a point in \( \mathbb{R}^3 \), and are drawn in standard position
- direction vectors: such as \( \vec{v} \) or \( \overrightarrow{AB} \), they represent a direction and are drawn emanating from a particular point, not necessarily the origin.

Though this difference in vectors may be confusing, it should not be stressed because it is only a superficial one.
Equations of Planes,  
Cylinders and Quadratic Surfaces

Jenny Lam  
January 27, 2009

1 Outline

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3 Cylinders and Quadratic Surfaces 4

Objectives
  • Find the equation of a plane
  • Find the distance between a point and a plane
  • Find the angle between two intersecting planes
  • Sketch cylinders
  • Sketch and name distinguished quadratic surfaces

2 Planes

2.1 Description of a Plane

Parametrization of a Plane
  A plane in $\mathbb{R}^3$ is determined by a point and two directions. Hence it can be described by the vector function
  \[ \mathbf{r}(t, u) = \mathbf{r}_0 + v_1 t + v_2 u \]

where
  • $\mathbf{r}_0$ represents the position of a point in the plane
  • $v_1$ and $v_2$ represent two non-parallel directions in the plane
  • $t$ and $u$ are any real numbers; there is no restriction on them because the plane extends infinitely in all four directions.

The parametrization of planes and surfaces in general will be discussed later in the textbook (in section 16.6).
Derivation of the Equation of a Plane

Let \( \mathbf{r}(t, u) = \mathbf{r}_0 + \mathbf{v}_1 t + \mathbf{v}_2 u \) be the parametrization of a plane. Then the vector \( \mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 \) is perpendicular to both \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) and hence to the plane defined by these two vectors. Also

\[
\mathbf{n} \cdot (\mathbf{r}(t, u) - \mathbf{r}_0) = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot (\mathbf{v}_1 t + \mathbf{v}_2 u) = t (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_1 + u (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_2 = 0
\]

(Why?) Thus the plane containing the point \( \mathbf{r}_0 \) and perpendicular to the normal vector \( \mathbf{n} \) is the set of points \( \mathbf{r} \) such that

\[
\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.
\]

Geometrically, this states that the normal vector must be perpendicular to the vector joining \( \mathbf{r}_0 \) to any other point \( \mathbf{r} \) on the plane.

Example

Find the equation of the plane containing the points \( A(1, 2, 3), B(1, -1, 0) \) and \( C(0, 0, 1) \).

Solution 1. Since \( \overrightarrow{AB} = (0, -3, -3) \) and \( \overrightarrow{AC} = (-1, -2, -2) \), the plane can be parametrized by

\[
\mathbf{r}(t, u) = (1, 2, 3) + (0, -3, -3) t + (-1, -2, -2) u
\]

where \( t, u \in \mathbb{R} \). This vector equation is an abbreviated form of the collection of the three coordinate equations

\[
\begin{align*}
x & = 1 - u \\
y & = 2 - 3t - 2u \\
z & = 3 - 3t - 2u
\end{align*}
\]

Example (cont.)

Find the equation of the plane containing the points \( A(1, 2, 3), B(1, -1, 0) \) and \( C(0, 0, 1) \).

Solution 2. Alternatively, let \( \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (0, -3, -3) \times (-1, -2, -2) \)

\[
\mathbf{n} = \begin{vmatrix}
i & j & k \\
0 & -3 & -3 \\
-1 & -2 & -2
\end{vmatrix} = (0, 3, -3)
\]

Hence an equation for the plane is \( \langle 0, 3, -3 \rangle \cdot (\mathbf{r} - (0, 1, 2)) = 0 \). We can carry out the dot product by letting \( \mathbf{r} \) have the arbitrary coordinates \( \langle x, y, z \rangle \).

\[
3(y - 1) - 3(z - 2) = 0 \quad \text{or} \quad 3y - 3z = -3.
\]

Remarks

- Carrying out the dot product in the last example results in not 3, but one single equation because of the dot product sums up the coordinates.
- The equations for a plane of the form
  - \( A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \) and
  - \( Ax + By + Cz = D \)

have a normal vector that is easily found as \( \mathbf{n} = \langle A, B, C \rangle \).
- To find a point in the plane \( Ax + By + Cz = D \), choose values for two of the coordinates and solve for the third.
  - For example, let \( x = y = 0 \), then \( z = D/C \), which means that the point \( (0, 0, D/C) \) is a point in the plane.
2.2 Applications

Parallel Planes

Example 3. Which of the following planes are parallel? Find a point in each.

- $\Pi_1 : x - 2y + 3z = 1$
- $\Pi_2 : 2x - 4y + 6z = -1$
- $\Pi_3 : x - 2y - 3z = 0$

- $\mathbf{n}_1 = (1, -2, 3)$, $\mathbf{n}_2 = (2, -4, 6)$ and $\mathbf{n}_3 = (1, -2, -3)$ are normal vectors for the planes $\Pi_1$, $\Pi_2$ and $\Pi_3$ respectively.

- Since $\mathbf{n}_2 = 2\mathbf{n}_1$ are parallel, $\Pi_1$ and $\Pi_2$ are parallel.
- Since $\mathbf{n}_1 \times \mathbf{n}_2 = (12, 6, 0)$ is non-zero, these vectors are not parallel, and so $\Pi_1$ and $\Pi_3$ are not parallel.
- $\Pi_2$ and $\Pi_3$ are not parallel, for, if they were, the fact that $\Pi_1$ and $\Pi_2$ are parallel would imply that $\Pi_1$ and $\Pi_3$ are also parallel, contradicting the previous point.

- Let $y = z = 0$ for ease of computation. Then $(1, 0, 0) \in \Pi_1$, $(-1/2, 0, 0) \in \Pi_2$ and $(0, 0, 0) \in \Pi_3$.

Intersecting Planes

Example 4. $\Pi_1 : x - 2y + 3z = 1$ and $\Pi_3 : x - 2y - 3z = 0$ are not parallel. Find the line along which they intersect.

Solution 5. We find two points on the line of intersection.

- Let $x = 0$ to solve $-2y + 3z = 1$ and $-2y - 3z = 0$. Subtracting the two equations yields $z = 1/6$. Plugging this value into the first equation yields $y = -1/4$. So $A(0, -1/4, 1/6)$ is on the line.

- Let $y = 0$ to find that $B(1/2, 0, 1/6)$ is on the line.

Now $\overrightarrow{AB} = (1/2, 1/4, 0)$, so the line is parametrized by $r(t) = (0, -1/4, 1/6) + (1/2, 1/4, 0) t$.

Intersecting Planes

Example 6. $\Pi_1 : x - 2y + 3z = 1$ and $\Pi_3 : x - 2y - 3z = 0$ are not parallel. Find the angle between the planes.

Solution 7. It suffices to find the angle between the normal vectors $\mathbf{n}_1 = (1, -2, 3)$ and $\mathbf{n}_3 = (1, -2, -3)$ for these planes.

$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_3}{|\mathbf{n}_1| |\mathbf{n}_3|} \right) = \cos^{-1} \left( -\frac{2}{7} \right)$$

Distance between a Point $P_0$ and a Plane $\Pi$

Key Idea

- Find the direction vector $\overrightarrow{PP_0}$ from a point $P$ in the plane to $P_0$.
- Find a unit normal vector $\mathbf{n}$ for plane $\Pi$ by dividing by its norm if necessary.
- Dot the vectors $\overrightarrow{PP_0}$ with a unit normal vector $\mathbf{n}$ to find the length of the component of $\overrightarrow{PP_0}$ parallel to $\mathbf{n}$.

In short,

$$d = \overrightarrow{PP_0} \cdot \mathbf{n} = \frac{\overrightarrow{PP_0} \times \mathbf{n}}{|\mathbf{n}|}$$

where $P$ is a point in $\Pi$ and $\mathbf{n}$ is a vector normal to $\Pi$. 

3
3 Cylinders and Quadratic Surfaces

Cylinders and Quadratic Surfaces
Refer to section 12.6 of your textbook.

Things to Remember

• Know the graphs of the basic quadratic curves.
  – ellipses (and circles)
  – hyperbolas
  – parabola

• To draw surfaces
  – set one of the coordinates to zero and draw the quadratic curve
  – recognize axis symmetries. e. g. $x^2 + z^2 - y^2 = 1$ has a symmetry along the $y$-axis because $x$ and $z$ can be interchanged in the equation.
Vector-Valued Functions,  
Motion in Space and Projectile Motion  

Jenny Lam  
February 2, 2009

1 Outline

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1 Outline  
2 Vector-Valued Function  
2.1 Notion of a Parametric Curve and Functions  
2.2 Calculus of Parametric Curves  
3 Motion in Space  
3.1 General Motion of a Particle  
3.2 Projectile Motion  

Objectives

• find the domain, codomain and range of a function  
• eliminate the parameter in the parametrization of a curve  
• parametrize a curve given its equation  
• find limits, derivatives, definite and indefinite integrals of vector-valued functions  
• find the velocity, acceleration and speed of a curve  
• show that motion on a sphere implies the perpendicularity of position and velocity  
• solve projectile motion problems

2 Vector-Valued Function

2.1 Notion of a Parametric Curve and Functions  

Parametrization of a Curve  
A parametrization \( r(t) = (f(t), g(t), h(t)) \) for a curve is a function which takes a single input, the parameter \( t \), and outputs an ordered triple of real numbers.

• Write \( r : \mathbb{R} \to \mathbb{R}^3 \), and read “\( r \) is a function from \( \mathbb{R} \) to \( \mathbb{R}^3 \)”.
  – the set of inputs (in this case \( \mathbb{R} \)) is the domain.
the set of outputs is the range
the set in which the outputs lie (in this case $\mathbb{R}^3$) is the codomain

- When defining a function, we must specify the domain and codomain. The range can take work to find and is not always easy to describe.

Scalar vs vector input and output
Let $f: \mathbb{R}^n \to \mathbb{R}^m$.

- If $f$ is a vector-valued function is, its output is a vector, that is, $m > 1$
- If $f$ is a scalar- or real-valued function, its output is a scalar, that is, $m = 1$.
- If $f$ is a single-variable function, its input is a scalar, that is, $n = 1$.
- If $f$ is a multi-variable function, its input is a vector, that is, $n > 1$.

Domain, Range and Codomain for a real-valued function
Example 1. Let $f(x) = \frac{1}{\sqrt{x-1}}$.

- The domain is a subset of $\mathbb{R}$. More specifically, it is the set of $x \in \mathbb{R}$ such that $x > 1$.
- $f$ is a real-valued function, so $\mathbb{R}$ is the codomain.
- Since the range of the function $g(x) = \sqrt{x-1}$ is $[0, \infty)$, the range of $f$ is $(0, \infty)$.

We write $f: (1, \infty) \to \mathbb{R}$.

Domain, Range and Codomain for a parametrized curve
The parametrization of a curve is a single-variable, vector-valued function $r: \mathbb{R} \to \mathbb{R}^n$, where $n = 2$ or $n = 3$. The range of $r$ can be described by a curve in $\mathbb{R}^n$.

Example 2. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be parametrized by the curve

$$x(t) = a \cos t \quad \text{and} \quad y(t) = b \sin t, \quad \text{or} \quad r(t) = (a \cos t)i + (b \sin t)j$$

because these $x$ and $y$ values satisfy the equation of the ellipse for all $t \in \mathbb{R}$.

From a Parametrized Curve to an Equation
Example 3 (13.1.2). Conversely, given the curve $r(t) = (t^2 + 1)i + (2t - 1)j$, or

$$x = t^2 + 1 \quad \text{and} \quad y = 2t - 1$$

we try to eliminate the parameter $t$ to rewrite $r$ into a single equation in $x$ and $y$ only. We do so by solving the second equation for $t$ and substitute it in the first equation.

$$x = \left( \frac{y + 1}{2} \right)^2 + 1.$$

Question: do you recognize the curve? It is that of a parabola opening sideways.
From a Parametrized Curve to an Equation

Remarks

• The parametrization \( \mathbf{r}(t) \) and the equation in \( x \) and \( y \) are not equivalent. The equation only describes the range of the function \( \mathbf{r} \). It does not describe how fast the curve is traced out.

• If the parametrization contains trigonometric functions, one can sometimes take advantage of trigonometric identities to eliminate the parameter. See the previous ellipse example.

• The process of eliminating the parameter to write an equation in terms of the coordinates can be difficult and not always possible.

2.2 Calculus of Parametric Curves

Componentwise calculations

Limits, continuity and derivatives are determined componentwise: if \( \mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle \) and \( \mathbf{L} = \langle L_1, L_2, L_3 \rangle \)

\( \lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L} \iff \lim_{t \to t_0} r_i(t) = L_i \) for \( i = 1, 2 \) and 3

\( \mathbf{r} \) is continuous at \( t \) iff \( r_i \) is continuous at \( t \) for \( i = 1, 2 \) and 3

\( \mathbf{r} \) is differentiable at \( t \) iff \( r_i \) is differentiable at \( t \) for \( i = 1, 2 \) and 3

\( \mathbf{r} \) is integrable on \( [a, b] \) iff \( r_i \) is integrable on \( [a, b] \) for \( i = 1, 2 \) and 3

\( \frac{d}{dt} \mathbf{r} = \langle \frac{dr_1}{dt}, \frac{dr_2}{dt}, \frac{dr_3}{dt} \rangle \)

\( \int_a^b \mathbf{r}(t) \, dt = \langle \int_a^b r_1(t) \, dt, \int_a^b r_2(t) \, dt, \int_a^b r_3(t) \, dt \rangle \)

Componentwise calculations

These generalizations are valid because continuity, differentiability and integrability are defined with limits, which can be evaluated componentwise.

However, it takes some work to show that this latter fact is true. See exercises for more on the topic.

Differentiation

For \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^n \), the derivative \( \mathbf{r}' : \mathbb{R} \to \mathbb{R}^n \) is also a vector. The rules for the differentiation of vector-valued functions are similar to those of real functions.

• constant rule: \( \frac{d}{dt} \mathbf{C} = \mathbf{0} \)

• addition: \( \frac{d}{dt} [\mathbf{r} + \mathbf{s}] = \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{s}}{dt} \)

• product rule—the interesting one:
  - scalar product: \( \frac{d}{dt} [\mathbf{a} \cdot \mathbf{r}] = \frac{d\mathbf{a}}{dt} \cdot \mathbf{r}(t) + \mathbf{a}(t) \frac{d\mathbf{r}}{dt} \)
  - dot product: \( \frac{d}{dt} [\mathbf{r} \cdot \mathbf{s}] = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \frac{d\mathbf{s}}{dt} \)
  - cross product (order matters): \( \frac{d}{dt} [\mathbf{r} \times \mathbf{s}] = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s}(t) + \mathbf{r}(t) \times \frac{d\mathbf{s}}{dt} \)

• chain rule—also interesting: let \( y = f(t) \), \( \frac{d}{dt} [f(\mathbf{r}(t))] = f'(t) \mathbf{r}'(f(t)) \) or \( \frac{d}{dt} [f(\mathbf{r}(t))] = \frac{dy}{dt} \frac{dr}{dt} \frac{dg}{dt} \)
3 Motion in Space

3.1 General Motion of a Particle

Velocity and Acceleration
If \( r(t) \) is the parametrization of a curve,
- \( v(t) = r'(t) \) is the velocity of the curve
- \( a(t) = r''(t) \) is the acceleration of the curve
- \( v = |v|(t) \) is the speed of the curve.

Important Fact
If \( |r| \) is constant, then \( r \cdot r' = 0 \). This is clear geometrically: if a curve lies on a sphere, its tangent is always perpendicular to the radius, hence position vector. Know the proof (page 897).

3.2 Projectile Motion

The Projectile Motion Problem
Projectile motion is described by
\[
    r(t) = r_0 + v_0 t + \frac{1}{2} g t^2,
\]
where
- \( t \) is the time
- \( r(t) \) is the position of the projectile at time \( t \)
- \( r_0 \) is the initial position of the projectile
- \( v_0 \) is the initial velocity of the projectile
- \( g = -g j \) is the acceleration due to gravity

The coordinate equations are convenient when solving problems. Let \( v_0 = |v_0| \) be the initial speed.
\[
    x = x_0 + (v_0 \cos \theta) t \quad \text{and} \quad y = y_0 + (v_0 \sin \theta) t - \frac{1}{2} gt^2.
\]

Strategies for Solving
- Variables: Identify the known and unknown quantities to be found. This step should give you an idea on how to go about solving the problem.
- Origin: choose a convenient origin, usually the launch point even if it is above the ground. This way, \( x_0 = y_0 = 0 \).
- Symbolic solution: solve the problem with variables, and plug the numerical values at the end. Advantages include
  - Ease of computation
  - Ease when fixing mistakes
  - Solving the problem once and for all
  - Understanding the reason behind patterns.
- Divide: to solving a system of 2 equations with 2 unknowns, one can use substitution or elimination. A little discussed but useful technique is to divide one equation by the other to eliminate variables.
Example (13.2.4)

A baseball is thrown from the stands 32 ft above the field at an angle of 30° up from the horizontal. When and how far away will the ball strike the ground if its initial speed is 32 ft/sec?

Solution 4. If the origin is on the ground, the launchpoint will be at \( y_0 = 32 \) ft. Let the origin be at the launchpoint, so the ground is at \( y_f = -32 \) ft. We know \( v_0 \) and \( \theta \) and must find \( x_f \) and \( t_f \). The equations are

\[
x_f = (v_0 \cos \theta)t_f \quad \text{and} \quad y_f = (v_0 \sin \theta)t_f - \frac{1}{2}gt_f^2.
\]

Solve the \( y \)-equation by quadratic formula for \( t_f \). Knowing \( t_f \), use the \( x \)-equation to find \( x_f \). Finding the numerical solutions is left to the reader.

\[
t_f = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta - 2gy_f}}{g}.
\]

Example (13.2.4 reversed)

Suppose now a baseball is thrown from the stands 32 ft above the field and hits the ground \( x_f \) feet away \( t_f \) seconds later. Find the initial speed and launching angle from the horizontal.

Solution 5. Again let the origin be the launchpoint, so \( y_f = -32 \) ft, and

\[
x_f = (v_0 \cos \theta)t_f \quad \text{and} \quad y_f = (v_0 \sin \theta)t_f - \frac{1}{2}gt_f^2.
\]

We know \( x_f \) and \( t_f \), solve for \( \theta \) by dividing \( x_f \) into \( y_f + \frac{1}{2}gt_f^2 \),

\[
\frac{y_f}{x_f} + \frac{gt_f^2}{2x_f} = \tan \theta, \quad \text{so} \quad \theta = \tan^{-1} \left( \frac{y_f}{x_f} + \frac{gt_f^2}{2x_f} \right)
\]

and, knowing \( \theta \), solve the first equation for \( v_0 = \frac{x_f}{\cos \theta t_f} \).
Arc Length and the Unit Tangent Vector

Jenny Lam

February 2, 2009

1 Outline

Contents

1 Outline 1
2 Arc Length 1

Objectives

• find the length of a curve
• find the arc length parameter of a curve
• reparametrize a curve into one with arc length parameter
• find the unit tangent vector $T$

2 Arc Length

Arc Length Formula

Definition 1 (Smoothness). A curve $r : \mathbb{R} \to \mathbb{R}^n \ (n = 2, 3)$ is smooth if derivatives of all orders exist.

Definition 2 (Arc Length Formula). Let $r(t) = (x(t), y(t), z(t))$ be a smooth curve. Its arc length on $[a, b]$ is the length of the curve $r(t)$ from the point $r(a)$ to the point $r(b)$.

$$ L = \int_a^b v(t) \, dt $$

where $v(t)$ is the speed of the curve.

$$ v(t) = |r'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$ 

Arc Length Parameter

In the arc length formula on $[a, b]$, suppose that $a$ is fixed, but that $b$ is a variable. Then the arc length is a function of $b$:

$$ L(b) = \int_a^b v(t) \, dt $$
In the more conventional notation, we let the basepoint \( a \) be \( t_0 \), and we replace the variable \( b \) by \( t \). The variable of integration must be different from any limit of integration, so we let that be \( \tau \) (“tau”), and we call the arc length function \( s \):

\[
    s(t) = \int_{t_0}^{t} v(\tau) \, d\tau.
\]

**Reparametrizing a Curve with Arc Length**

The function \( s : \mathbb{R} \to \mathbb{R} \) (see exercises) defined by \( s(t) = \int_{t_0}^{t} v(\tau) \, d\tau \) is invertible. So it has an inverse function \( t(s) = t \). The composition \( r(t(s)) \) rewrites the curve \( r(t) \) into one with a new parameter, \( s \):

\[
    r(t(s)) = r(s).
\]

We call \( s \) the arc length parameter.

**Example 3.** one in which we reparametrize a line.

**Why arc length parameter?**

Recall that a curve can be parametrized many different ways. Arc length parameter distinguishes itself in the following way:

- **Speed of \( r(t) \):** by definition \( s(t) = \int_{t_0}^{t} v(\tau) \, d\tau \). Its derivative, by the Second Fundamental Theorem of Calculus is

  \[
  \frac{ds}{dt} = \frac{d}{dt} \int_{t_0}^{t} v(\tau) \, d\tau = v(t).
  \]

  This property will become very useful in this chapter. Question: rewrite this property in terms of the curve. Answer: \( \frac{ds}{dt} = \left| \frac{dr}{dt} \right| \).

- **Speed of \( r(s) \):** use Chain Rule.

  \[
  \left| \frac{dr}{ds} \right| = \left| \frac{dr}{dt} \frac{dt}{ds} \right| = \frac{dr}{ds} = \frac{v}{v} = 1.
  \]

  A curve parametrized by arc length always has unit speed.

**New notation**

Let \( r(t) \) be a curve, and \( r(s) \) be its arc length parametrization. We use “prime” notation for derivatives with respect to \( t \)

\[
    \frac{dr}{dt} = r'(t).
\]

We use “dot” notation for derivatives with respect to \( s \)

\[
    \frac{dr}{ds} = \dot{r}(s).
\]

**3 Unit Tangent Vector**

**The Tangent Direction**

Let \( r(t) \) be the parametrization of a curve. Then

\[
    r' = \frac{dr}{dt} = \frac{ds}{dt} \frac{dr}{ds} = v\dot{r}.
\]
In other words, the derivatives of \( r \) with respect to any parameter always point in the same direction, the tangent direction. In particular, \( \dot{r} \) points in the tangent direction, and we have seen that \( |\dot{r}| = 1 \). Thus we define the unit tangent vector to be a function of \( s \) by

\[
T = \dot{r}.
\]

Dividing the first equation by \( v \), this means that, the unit tangent vector is a function of \( t \),

\[
T = \frac{r'}{v}.
\]

Summary

Let \( r(t) \) be the parametrization of a curve.

- **Arc length parameter:** \( s(t) = \int_{t_0}^{t} v(\tau)d\tau \) where \( v = |\frac{dr}{dt}| \).
- **Derivative notation**
  - parameter \( t \): \( r' = \frac{dr}{dt} \)
  - parameter \( s \): \( \dot{r} = \frac{dr}{ds} \)
- **Speed**
  - parameter \( t \): \( |r'| = v = \frac{ds}{dt} \)
  - parameter \( s \): \( |\dot{r}| = 1 \)
- **Unit tangent**
  - parameter \( t \): \( T = \frac{r'}{|r'|} \)
  - parameter \( s \): \( T = \dot{r} \)
Curvature and the Unit Normal Vector

Jenny Lam

January 27, 2009

Objectives

- find the unit normal vector $N$ of a curve
- find the curvature $\kappa$ of a curve
- derive properties of curves using $T$, $N$ and $\kappa$
- find the osculating circle of a curve

Motivation

Intuitively, the normal vector at a point $P$ on the curve should

- be perpendicular to the unit tangent vector
- lie in the plane of the curve near $P$
  - Specifically, we require that, for a planar curve, the normal vector lie in the plane of the curve
- point in the direction in which the curve bends
- have unit length

A candidate for the normal vector

Property 1

- Recall that if $|r| = c$ is constant, then $r \perp r'$.
- Since $|T| = 1$, then $T \perp \dot{T}$.
- Therefore, $\dot{T}$ is a candidate to be a normal vector. Our task is now to show that it satisfies the three other properties.

A candidate for the normal vector

Property 2

Let $r$ be a planar curve. This means that for it satisfies the equation of some plane

$$ n \cdot (r - r_0) = 0. $$

(Caution: $n$ is a normal to the plane in which the curve lies, not the normal to the curve that we are currently trying to find.) We must show that $\dot{T}$ also satisfies this equation. Take the derivative by applying dot product rule and keeping in mind that $n$ and $r_0$ are constant.

$$ n \cdot \dot{r} = 0, \quad \text{hence} \quad n \cdot T = 0 $$

Take a second derivative $n \cdot \ddot{T} = 0$. Therefore, $\ddot{T}$ lies in the same plane as a planar curve.
A candidate for the normal vector

Property 3
Properties 1 and 2 restrict the direction of the normal to two possible choices: the direction towards which the curve bends, and the opposite direction. To show that $\dot{T}$ points towards the curve, we appeal to the fact that

$$\dot{T} \approx \Delta T,$$

that is, the derivative of the tangent is approximately equal to a small change or difference in the tangent at two very close points on the curve. To complete the argument, we draw a picture. (picture here)

A candidate for the normal vector

Property 4
Nothing guarantees that $\dot{T}$ has unit length. In fact, the more the curve bends, the greater the length of $\dot{T}$. Since $|\dot{T}|$ measures how much the curve bends, we call this quantity curvature

$$\kappa = |\dot{T}|.$$

For the normal vector to have unit length, we define

$$N = \frac{T}{|T|} = \frac{\dot{T}}{\kappa}.$$

Shortcut: from $\frac{d}{ds}$ to $\frac{d}{dt}$

We often need to take the derivative with respect to $t$, knowing the derivative with respect to $s$. Suppose $F$ is a vector function of $t$. Then

$$\dot{F} = \frac{dF}{ds} = \frac{dt}{ds} \frac{dF}{dt} = \frac{dF}{ds} = \frac{dF}{dt}.$$

In short,

$$\dot{F} = \frac{F'}{v}.$$

Note that we have used this calculation before to show that $T = \dot{r} = \frac{r'}{v}$.

$N$ and $\kappa$ in terms of parameter $t$

By the shortcut,

$$\dot{T} = \frac{T'}{v}.$$

- $N = \frac{T}{|T|} = \frac{T'}{v} |v| = \frac{\dot{T}}{|T|}.$$

  - Question: why is $|v| = v$?

  - Answer: Recall that $v = |v|$, so not only is $v$ a scalar, $v = \sqrt{x'^2 + y'^2 + z'^2}$ is positive.

- $\kappa = |\dot{T}| = \frac{|T'|}{v}.$
Calculations: $\frac{d}{ds}$ versus $\frac{d}{dt}$

Although definitions of $T$, $N$ and $\kappa$ in terms of $s$ are nicer, in practice, it is simpler to compute them directly in terms of $t$. This way, we don’t have to reparametrize the curve first.

**Example 1.** Find $T$, $N$ and $\kappa$ for $r(t) = \langle \ln \sec t, t \rangle$ for $-\pi/2 < t < \pi/2$.

- $r'(t) = \langle \tan t, 1 \rangle$
- $v = \sqrt{\tan^2 t + 1} = \sec t = |\sec t|$. Why can we drop the absolute values in this case?
- $T = \frac{r'(t)}{v} = \frac{(\tan t, 1)}{\sec t} = (\sin t, \cos t)$
- $T' = (\cos t, -\sin t)$ hence $|T'| = 1$
- $N = \frac{T'}{|T'|} = \langle \cos t, -\sin t \rangle$
- $\kappa = \frac{|T'|}{v} \frac{1}{\sec t} = \cos t$.

**Osculating Circle**

An *osculating* (i.e. kissing) circle to a curve at a point $P$ is the circle of best fit. That is, at $P$, the osculating circle and the curve have

- the same tangent vector
- the same normal vector
- the same curvature

The radius of the osculating circle called the radius of curvature and is equal to

$$\rho = \frac{1}{\kappa}.$$

**Summary**

- **Normal Unit Vector**
  - parameter $s$: $N = \frac{T}{|T|}$ so $N = \frac{T}{|\kappa|}$
  - parameter $t$: $N = \frac{T'}{|T'|}$

- **Curvature**
  - parameter $s$: $\kappa = \frac{|T'|}{|T|}$
  - parameter $t$: $\kappa = \frac{|T'|}{v}$
Torsion and the Unit Binormal Vector

Jenny Lam

February 2, 2009

Objectives

• compute the torsion of a curve
• compute the binormal of a curve
• compute the tangential and normal components of acceleration
• understand the meaning of the components of acceleration

Completing the moving frame

So far, we have seen that \( T \) and \( N \) were two perpendicular unit vectors that accompany a curve. To complete this frame, we define the binormal vector to be

\[
B = T \times N
\]

Check that \( |B| \) is a unit vector. Answer: since \( T \) and \( N \) are perpendicular,

\[
|B| = |T \times N| = |T| |N| \sin \frac{\pi}{2} = 1.
\]

Understanding torsion

We would like to measure how much a curve twists out of its plane of motion (also called osculating plane) which is defined by \( T \) and \( N \). Since a vector normal to this plane is \( B \), a curve twists out if \( B \) changes direction, that is, if \( B \neq 0 \). Therefore, we want to define torsion \( \tau \) so that

\[
|\tau| = |\dot{B}|
\]

Unlike curvature, we will be able to define torsion with a sign, which gives additional information about the twisting.

Defining torsion

We show that \( \dot{B} \) is parallel to \( N \).

• \( \dot{B} \) is perpendicular to \( B \) because \( B \) is a unit vector.
• \( \dot{B} \) is perpendicular to \( T \) because \( B = T \times N \) implies that

\[
\dot{B} = \dot{T} \times N + T \times \dot{N} = \kappa N \times N + T \times \dot{N} = T \times \dot{N}
\]

• Another vector that is both perpendicular to \( B \) and \( T \) is \( N \). Therefore, \( \dot{B} \) and \( N \) are parallel.
\( \dot{\mathbf{B}} \) and \( \mathbf{N} \) are parallel but do not necessarily point in the same direction. Define torsion to be the real number such that
\[
\dot{\mathbf{B}} = -\tau \mathbf{N}
\]
This definition yields the desired property that \( |\tau| = |\dot{\mathbf{B}}| \). Moreover, if \( \tau > 0 \), then \( \dot{\mathbf{B}} \) and \( \mathbf{N} \) point in opposite directions and if \( \tau < 0 \), then \( \dot{\mathbf{B}} \) and \( \mathbf{N} \) point in the same direction.

**The Frenet frame**

We think of the vectors \( \mathbf{T}, \mathbf{N} \), and \( \mathbf{B} \) as a frame moving along the curve. We call this moving frame the *Frenet frame*. We can define several vectors in terms of this frame. By definition,
\[
\dot{\mathbf{T}} = \kappa \mathbf{N} \quad \text{and} \quad \dot{\mathbf{B}} = -\tau \mathbf{N}.
\]
What about \( \dot{\mathbf{N}} \)? Since \( \mathbf{N} = -\mathbf{T} \times \mathbf{B} \),
\[
\dot{\mathbf{N}} = -\dot{\mathbf{T}} \times \mathbf{B} - \mathbf{T} \times \dot{\mathbf{B}} = -\kappa \mathbf{N} \times \mathbf{B} + \mathbf{T} \times \tau \mathbf{N} = -\kappa \mathbf{T} + \tau \mathbf{B}.
\]

**The Frenet equations**

These three equations,
\[
\dot{\mathbf{T}} = \kappa \mathbf{N} \\
\dot{\mathbf{N}} = -\kappa \mathbf{T} + \tau \mathbf{B} \\
\dot{\mathbf{B}} = -\tau \mathbf{N}
\]
are called the Frenet equations. The symmetry in these equations is important for solving this system of differential equations easily, and it justifies the choice of defining \( \tau \) with a negative sign rather than a positive sign: \( \dot{\mathbf{B}} = -\tau \mathbf{B} \) (See linear algebra and differential equations class for more on this topic.)

**Components of acceleration**

Another vector that can be described in terms of the Frenet frame is acceleration.
\[
\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}
\]
are the tangential and normal components.

- \( \mathbf{a} \) does not have any binormal component and always lies in the osculating plane (plane of motion).
- \( a_T \) measures linear acceleration, that is, the change in speed.
- \( a_N \) measures radial acceleration, the change in direction.
\[
a_T = v' \quad a_N = \kappa v^2.
\]
Curvature and Torsion revisited

An easier way to find $a_N$, $\kappa$ and $\tau$ is given by the formulas

\begin{align*}
    a_N &= \sqrt{|a|^2 - a_T^2}, \\
    \kappa &= \frac{|x' \times x''|}{y^3}, \\
    \tau &= \frac{x' \times x'' \cdot x'''}{|x' \times x''|^2}
\end{align*}

These formulas can be derived all at once, and guidance through these derivations can be found in the Discussion Board questions.
Functions of Several Variables and their Limits

Jenny Lam

February 15, 2009

1 Outline

Contents

1 Outline 1
2 Domains 1
3 Limits 3

Objectives

• Determine the domain and range of a multivariable function
• Sketch level curves of a function \( f : \mathbb{R}^2 \to \mathbb{R} \)
• Recognize level surfaces of a function \( f : \mathbb{R}^3 \to \mathbb{R} \)
• Evaluate limits of \( f : \mathbb{R}^2 \to \mathbb{R} \) at a point
• Show the non-existence of a limit
• Use polar coordinates to prove the existence of a limit

2 Domains

Open and Closed sets in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)

For this class, it suffices to understand open and closed sets intuitively. For example,

• an open disk (in \( \mathbb{R}^2 \)) or ball (in \( \mathbb{R}^3 \)) does not have a boundary.
• a closed disk or ball has a boundary.

Points on a boundary are called boundary points. Those that are not are called interior points. The formal definition of interior and boundary points (p 951 and 954) can be skipped. They are part of a branch of mathematics called topology, the study of open and closed sets.
Functions of Several Variables
We begin the study of functions of several variables of the form
\[ f : \mathbb{R}^n \to \mathbb{R}, \quad n = 2 \text{ or } 3 \]

**Example 1.** Let \( f(x, y) = e^{x+y} \).

- \( f \) has two variables, so its domain lies in \( \mathbb{R}^2 \). Moreover, there is no restriction on either variable, so its domain is all of \( \mathbb{R}^2 \).
- The output is a scalar (not a vector), so the codomain is \( \mathbb{R} \).
- Fix \( y = 0 \) and see that \( f(x, 0) = e^x \), which has range \((0, \infty)\). So the range of \( f \) includes \((0, \infty)\). Varying the value of \( y \) does not add any more values to \((0, \infty)\) because \( f(x, y) = f(x+y, 0) \). So the range of \( f \) is exactly \((0, \infty)\).

**Example:** find the domain and range
Let \( g \) be defined by \( g(x, y) = e^{\sqrt{x-1}} \).

- \( y \) is unrestricted, but \( x \) is bound by the condition \( x - 1 \geq 0 \). These conditions can be described by a graph in \( \mathbb{R}^2 \).
- The exponent contains a radical and can only have non-negative values. So based on the graph of the exponential function, the range of \( g \) is \([1, \infty)\).

**Graphs of functions**
Recall, the graph of an equation is the set of solutions to that equation. The graph is a subset of \( \mathbb{R}^n \) where \( n \) is the number of equations. In other words, we need one spatial dimension to represent each variable in the equation.

- The graph of \( f(x) = \sin x \) is the set of solutions of the equation \( y = \sin x \).
- The graph of \( g(x, y) = x \sin y \) is the set of solutions of the equation \( z = x \sin y \).

More generally, the graph of a function \( f \) is the set of solutions to the equation \( f(x) = y \) if \( f \) has 1 variable and \( f(x, y) = z \) if it has 2 variables. To graph a function \( f : \mathbb{R}^n \to \mathbb{R} \), we need \( n + 1 \) dimensions.

**Level Surfaces**
To graph \( f(x, y, z) \), or equivalently to graph \( w = f(x, y, z) \) would require 4 spatial dimensions. To circumvent this problem, we fix the dependent variable \( w \) at several values and graph the resulting 3-variable equations.

**Example 2.** Let \( T : \mathbb{R}^3 \to \mathbb{R} \), defined by \( T(x, y, z) = x^2 + y^2 + z^2 \) be the temperature in a room. The points \((x, y, z)\) in the room which have the same temperature \( T \) satisfy the equation \( x^2 + y^2 + z^2 = T \) and therefore all must lie on a sphere centered at the origin and with radius \( \sqrt{T} \). These spheres of points at equal temperature are called level surfaces.

**Level Curves**
Although a function \( f : \mathbb{R}^2 \to \mathbb{R} \) can be represented in \( \mathbb{R}^3 \), it is often more convenient (and accurate) to represent it in \( \mathbb{R}^2 \).

**Example 3.** Consider the function \( H(x, y) = x^2 - y^2 \) which assigns to each point on a map its height. The points \((x, y)\) which have the same height \( H \) satisfy the equation \( x^2 - y^2 = H \) form parabolas opening sideways if \( H > 0 \), up and down if \( H < 0 \) and pairs of lines (i.e. degenerate parabolas) if \( H = 0 \). These parabolas represent points at equal height and are called level curves.
Graphing Summary

Curves
• parametric: \( f : \mathbb{R} \rightarrow \mathbb{R}^n, \ n = 2 \) or 3 (Ch 13)
• graph of a function: \( f : \mathbb{R} \rightarrow \mathbb{R} \)
• graph of a level curve: \( f(x, y) = a \) where \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)

Surfaces
• parametric: \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) (Ch 16)
• graph of a function: \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)
• graph of a level surface: \( f(x, y, z) = a \) where \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \)

Remember
Only the dependent (range) variables are represented by parametrizations, while both the independent (domain) and dependent (range) variables are represented by graphs of functions.

3 Limits

Limits in \( \mathbb{R} \)
Recall that the limit of \( f : \mathbb{R} \rightarrow \mathbb{R} \) at a point \( a \) exist only if the one-sided limits must agree.

Example 4. Let
\[
f(x) = |x| = \begin{cases} -1 & -1 \leq x < 0, \\ 0 & 0 \leq x < 1, \\ \vdots & \end{cases} \quad g(x) = |x| = \begin{cases} -x & x < 0, \\ x & x \geq 0. \end{cases}
\]

- \( \lim_{x \to 0^+} f(x) = 0 \) and \( \lim_{x \to 0^-} f(x) = -1 \), so \( \lim_{x \to 0} f(x) \) does not exist.
- \( \lim_{x \to 0^+} g(x) = 0 \) and \( \lim_{x \to 0^-} g(x) = 0 \), so \( \lim_{x \to 0} f(x) \) exists and is 0.

Limits for functions of 2 variables
The limit of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) at a point \((a, b)\) exists if the limits coming from all directions are equal. Many of the techniques used to evaluate limits of single-variable functions still work.

Evaluation of limit by evaluating the function
This technique works if the function is continuous at \((a, b)\)
\[
\lim_{(x,y) \to (1,1)} \cos \frac{\sqrt{|xy|} - 1}{x+y} = \cos 0 = 1.
\]

Limits for functions of 2 variables

Evaluation of limit by reducing the fraction or conjugation
If evaluating the function leads to a division by 0, we can sometimes avoid this problem by factoring and reducing the fraction or by conjugating:
\[
\lim_{(x,y) \to (1,1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \to (1,1)} \frac{(x + y)(x - y)}{x - y} = 2.
\]

By conjugation
\[
\lim_{(x,y) \to (2,2)} \frac{x + y - 4}{\sqrt{x + y} - 2} = \lim_{(x,y) \to (2,2)} \frac{(x + y - 4)(\sqrt{x + y} + 2)}{x + y - 4} = 4.
\]

Warning: do not ever write 0 in the denominator; use limits instead.
Proving the existence or non-existence of a limit
To show that a limit of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) at a point

- exists
  - show that the limit is the same coming from all sides.
- does not exist
  - show that the limit coming from two different directions are different (or Two-Path test).

Example: non-existence of a limit
Show that \( f(x, y) = \frac{x^4}{x^4 + y^2} \) does not have a limit as \((x, y) \to (0, 0)\).

Solution
Choose 2 different paths in \( \mathbb{R}^2 \) that go through the limit point \((0, 0)\) and evaluate the limit of \( f \) on that path.

- **Path 1:** \( y = 0 \) (\( x \) is unrestricted). Then
  \[
  \lim_{(x,0) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^4}{x^4} = 1.
  \]

- **Path 2:** \( y = x \) (\( x \) is unrestricted). Then
  \[
  \lim_{(x,x) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^4}{x^4 + x^2} = 1
  \]
  by taking the ratio of leading coefficients. (See review on p 115, dominant terms).

Example: non-existence of a limit

Remarks
Paths 1 and 2 have limits that agree. This is not enough to conclude that the limit exists, because as long as one other path has a different limit, the limit will not exist.

- **Path 3:** \( y = x^2 \) (\( x \) is unrestricted). Then
  \[
  \lim_{(x,x^2) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}.
  \]

The limit of \( f \) as \((x, y) \to (0, 0)\) on path 1 is 1 and the limit on path 3 is 1/2. So the limit of \( f \) at the origin does not exist.

Existence of a limit
Intuitively, we are trying to show that the limit coming from any direction will yield the same value. One strategy is to use polar coordinates and show that the limit as \( r \to 0 \) is fixed regardless of the value of \( \theta \).

*Example 5.* We want to find the limit of \( f(x, y) = \frac{2x}{x^2 + x + y^2} \) as \((x, y) \to (0, 0)\). Since

\[
  f(x, y) = \frac{2x}{x^2 + x + y^2} = \frac{2r \cos \theta}{r^2 + r \cos \theta},
\]

As \( r \to 0, \theta \) remains constant and the dominant terms are \( r \) and \( r^2 \). Therefore \( \lim_{r \to 0} f(r, \theta) = 0 \).
Summary of limit techniques

• To evaluate a limit
  – evaluate the function if it is continuous at the limit point
  – reduce the rational expression by factoring if necessary
  – multiply top and bottom of the fraction by the conjugate
  – as \((x, y) \to (0, 0)\): convert to polar form and evaluate the limit as \(r \to 0\), fixing \(\theta\).

• To show non-existence of a limit as \((x, y) \to (0, 0)\), find paths on which the limits are different. Paths to try:
  – \(x = 0\) or \(y = 0\)
  – \(x = y\) or \(y = x\)
  – \(x = y^n\) or \(y = x^n\) where \(n\) is chosen as to simplify the limit.
Continuity of Functions of Several Variables 
and Partial Derivatives

Jenny Lam

February 22, 2009

Objectives

• determine the set of points at which a function is continuous
• extend the domain of a function to make it continuous at a point
• know the definition and geometric interpretation of partial derivatives
• compute partial derivatives: notation, implicit, 2nd order, mixed (Fubini) and higher order

Definition of continuity

Recall that a function $f : D \rightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}$ ($D$ is a subset of $\mathbb{R}$) is continuous at a point $a$ if

• $a \in D$, i. e., $f$ is defined at $a$,
• $\lim_{x \to a} f(x)$, i. e., the limit of $f$ as $x \to a$ exists,
• $\lim_{x \to a} f(x) = f(a)$, i. e., $f$ and its limit agree at $a$.

We can generalize this definition to a function $f : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, in any number of variables. Note that continuity is a pointwise property. A function is said to be continuous (without specifying any point) if it is continuous everywhere on its domain.

Questions: is the function $1/x$ continuous? Answer: yes.

Example

• Let $h_1(x, y) = 3x^2 - 2y + 1$ for $(x, y) \neq (0, 0)$. Since $h_1$ is undefined at the origin, it is not continuous there.

• Let $h_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as $h_1$ with $h_2((0,0)) = 0$. Then $\lim_{(x,y)\rightarrow(0,0)} h_2(x,y) = 1$ is not equal to $h_2(0,0) = 0$. Therefore, $h_2$ is not continuous at $(0,0)$. Why do we evaluate the first expression to find the limit? Because the limit is concerned with the behavior near the point $(0,0)$, not at the point.

• Let $h_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $h_1$ with $h_3((0,0)) = 1$. Then $\lim_{(x,y)\rightarrow(0,0)} h_3(x,y) = 1 = h_3(0,0)$. Therefore, $h_3$ is continuous at $(0,0)$. In fact, we can simply define $h_3(x,y) = 3x^2 - 2y + 1$ for all $(x,y) \in \mathbb{R}^2$.

When we calculate limits by evaluating functions, we appeal to the fact that the functions are continuous at the point, so that limit and value at the point are equal.
Finding points at which a function is continuous

To determine the points at which a function is continuous, one must, by definition, find the points in the domain, whose limit exists and agrees with the value at that point. Consider the following.

- Elementary functions (absolute, power, trigonometric, inverse trigonometric, exponential, logarithmic functions) are continuous on their domains.

- Combining continuous functions (by adding, subtracting, multiplying, dividing and composing) results in a continuous function.

So in practice, the task of finding the points at which a function is continuous amounts to finding the domain of the function. This is true of most of the textbook problems, but not true when the function is defined piecewise such as in the previous example.

Partial derivatives: definition

For a function \( f : \mathbb{R}^2 \to \mathbb{R} \), the partial derivative with respect to \( x \) at \((a, b)\) is defined to be

\[
\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(x + h, b) - f(a, b)}{h}.
\]

If \( F(x) = f(x, b) \) is the function obtained from \( f \) by fixing the \( y \) component at \( b \), then \( \frac{\partial f}{\partial y}(a, b) = F'(x) \) is the derivative of \( f \). In other words, the partial derivative of \( f \) with respect to \( x \) is simply the derivative of \( f \) as a function of \( x \) only, where \( y \) is taken to be a constant. Similarly, the partial derivative with respect to \( y \) at \((a, b)\) is

\[
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, y + h) - f(a, b)}{h}.
\]

Partial derivatives: geometric interpretation

Geometrically, the partial derivatives at \((a, b)\) are the slopes of the curves obtained by slicing the graph of \( f \) with the planes \( x = a \) and \( y = b \).

This lesson is computationally heavy, although not difficult. Refer to the textbook for examples on computing partial derivatives

- of the first order

- of the second order (and equality of mixed partials \( f_{xy} = f_{yx} \))

- implicit partial differentiation

Remarks on notation

- we use the \( \partial \) ("del") notation to take derivatives of multivariable functions. For single-variable functions (even if it is vector-valued such as curves in Ch 13), we use the \( d \) notation (or "prime" or "dot" notation).

- Note the position of the exponents on the second partial derivative \( \frac{\partial^2 f}{\partial x^2} \). This is actually short for \( \frac{\partial}{\partial x} \frac{\partial}{\partial x} f \). Think of \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) as differential operators applied to \( f \), which can be denoted either at the top of the fraction or behind it.
1 Outline

Contents

Objectives

• apply chain rule to functions of 1 variable
• apply chain rule to functions of several variables
• use tree diagrams to apply chain rule
• state the definition of a directional derivative
• compute directional derivatives using the limit definition
• compute directional derivatives using the gradient vector

2 Chain Rule

Chain Rule for functions of 1 variable
Let \( z = f(x, y) \).

• if \( x = x(t) \) and \( y = y(t) \), then \( f \) is a function of \( t \) and its derivative is

\[
\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

Remark
Note the use of \( d \) for derivatives of functions of single variables, versus \( \partial \) for derivatives of several variables. For example, in the first case, \( z, x \) and \( y \) are functions of a single variable \( t \). But \( f \) is a function of two variables \( x \) and \( y \).

Chain Rule for functions of 2 and 3 variables
Let \( z = f(x, y) \).

• if \( x = x(u, v) \) and \( y = y(u, v) \), then \( f \) is a function of \( u \) and \( v \), and it has two partial derivatives

\[
\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.
\]
• if \( x = x(u,v,w) \) and \( y = y(u,v,w) \), then \( f \) is a function of \( u, v \) and \( w \), and it has three partial derivatives

\[
\begin{align*}
\frac{\partial z}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \\
\frac{\partial z}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial w} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w}
\end{align*}
\]

Chain Rule: Example 1

Suppose that \( w = f(x,y,z,t) \), \( x = x(u,v,t) \), \( y = y(u,t) \) and \( z = z(t) \). Then \( f \) is a function of \( u, v \) and \( t \) and has three partial derivatives

\[
\begin{align*}
\frac{\partial w}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \\
\frac{\partial w}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} \\
\frac{\partial w}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial f}{\partial t}
\end{align*}
\]

Chain Rule: Example with 3 levels of composition

Suppose that \( w = f(x,y) \), \( x = x(u,v) \), \( y = y(u) \), \( u = u(s,t) \) and \( v = v(t) \). Then \( f \) is a function of \( s \) and \( t \) and has 2 partial derivatives. We can compute these partial derivatives in two steps.

\[
\begin{align*}
\frac{\partial w}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \\
\frac{\partial w}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial f}{\partial t}
\end{align*}
\]

3 Directional Derivatives

Computing directional derivative from gradient vector

• Motivation: geometrically the directional derivative of \( z = f(x,y) \) at \((a,b)\) in the direction of \( \mathbf{u} \) is the slope of the tangent line to the function \( z = f(x,y) \) at \((a,b)\), in the direction of \( \mathbf{u} \) which lies in the \( xy \)-plane.

• Problem: We would like to find this directional derivative using our geometric intuition.

• Idea: slice the graph of the function \( z = f(x,y) \) along the vertical plane parallel to \( \mathbf{u} \). Then we only consider the graph of \( f \) on that plane as a function of a new variable \( t \). Then the derivative \( \frac{df}{dt} \) is the slope of the tangent.
Computing directional derivative from gradient vector

- Solution: If \( u = \langle u_1, u_2 \rangle \), then we can parametrize the line passing through \((a, b)\) along \( u \) by
  \[
  x = a + u_1 t \quad y = b + u_2 t.
  \]
  So along \( u \), \( f \) is really a function of the single-variable \( t \). By chain rule, we have that the directional derivative is
  \[
  \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot u.
  \]
  The directional derivative along \( u \) is denoted \( \left( \frac{\partial f}{\partial s} \right)_u \) and \( \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \nabla f \) is called the gradient of \( f \).
  So we have
  \[
  \left( \frac{\partial f}{\partial s} \right)_u = \nabla f \cdot u.
  \]

Definition + calculation from definition

The actual definition of directional derivative is, like derivatives in single-variable calculus and for partial derivatives, in terms of the limit of a difference quotient. More precisely, the derivative of \( f \) in the direction of \( u = \langle u_1, u_2 \rangle \) at the point \( r = (a, b) \) is

\[
\left( \frac{\partial f}{\partial s} \right)_{u,(a,b)} = \lim_{h \to 0} \left( \frac{f(r + hu) - f(r)}{h} \right) = \lim_{h \to 0} \left( \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \right)
\]
Gradients, Tangent lines, Tangent Planes
and the Linearization of functions of several variables

Jenny Lam

February 24, 2009

Objectives

- interpret the gradient as the direction of greatest change
- compute the maximal rate of change at a point
- find equations of tangent lines to curves and tangent planes to surfaces
- find the linearization of a function at a point
- find the total differential of a function at a point.

Notation: in this presentation, we write \( \nabla f(a, b) \) to mean \( \nabla f \bigg|_{a,b} \), the gradient of \( f \) evaluated at \((a, b)\).

Gradient Vector

Several consequences follow from the fact that the directional derivative is given by a dot product

\[
\left( \frac{df}{ds} \right)_{a, \langle a, b \rangle} = \nabla f(\langle a, b \rangle) \cdot u = |\nabla f(\langle a, b \rangle)| \cos \theta
\]

- The directional derivative is maximal when \( \theta = 0 \), i.e. the gradient points in the direction of steepest increase.
- the directional derivative is minimal when \( \theta = \pi \), i.e. the direction of steepest decrease is opposite the gradient.
- the maximal rate of increase is equal to the magnitude of the gradient.
- the directional derivative is 0 when \( \theta = \pm \pi/2 \), i.e. the direction of zero increase is perpendicular to the gradient.
- on a topographic map, the gradient is perpendicular to the level curves (which indicate the direction of zero increase).

Tangent Line

Suppose that \( f(x, y) = c \) is the equation of a curve in \( \mathbb{R}^2 \) satisfied by \( \mathbf{a} = \langle a, b \rangle \). Since \( f(x, y) = c \) is a level curve of \( f \), then \( \nabla f(a, b) \) is perpendicular to this curve. Moreover, all the points \( \mathbf{x} \) on the tangent line are such that \( \mathbf{x} - \mathbf{a} \) are parallel to the tangent line. So an equation for the tangent to the curve \( f(x, y) = c \) at \( \mathbf{a} \) is

\[
\nabla f(a, b) \cdot (x - a) = 0.
\]
Tangent Plane

The procedure is analogous to that of the tangent line problem. If \( \mathbf{a} = \langle a, b, c \rangle \) is a solution to the equation \( f(x, y, z) = c \), whose graph is a level surface of \( f \), then \( \nabla f(a, b, c) \) is perpendicular to the level surface. If \( \mathbf{x} \) is on the tangent plane to the surface, then \( \mathbf{x} - \mathbf{a} \) is parallel to the tangent plane, and so perpendicular to the gradient vector. An equation for the tangent plane is

\[
\nabla f(a, b, c) \cdot (\mathbf{x} - \mathbf{a}) = 0.
\]

In coordinate form, this equation translates to

\[
f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.
\]

Tangent Plane to a graph

Now we apply the previous result to the special case of a surface that is the graph of a function \( g(x, y) = z \). Equivalently, this is the level curve \( g(x, y) - z = 0 \) of the function \( f(x, y, z) = g(x, y) - z \), and

\[
\nabla f(a, b, g(a, b)) = \langle g_x(a, b), g_y(a, b), -1 \rangle.
\]

(Why? some details must be filled in). So an equation for the tangent plane at \( (a, b, g(a, b)) \) is

\[
g_x(a, b)(x - a) + g_y(a, b)(y - b) = z - g(a, b).
\]

Linearization

Recall that, if \( f(x) = y \) is differentiable at \( a \), then it can be approximated by the function

\[
f(x) \approx L(x) = f(a) + f'(a)(x - a) \quad \text{for all } x \text{ near } a
\]

whose graph is the tangent line to \( f \) at \( a \). Similarly \( g(x, y) \) can be approximated near \( \mathbf{a} = \langle a, b \rangle \) by the function whose graph is the tangent plane to \( g \) at \( \mathbf{a} \).

\[
L(x, y) = g(a, b) + g_x(a, b)(x - a) + g_y(a, b)
\]

And \( h(x, y, z) \) can be approximated near \( \mathbf{a} = \langle a, b, c \rangle \) by

\[
L(x, y, z) = h(a, b, c) + h_x(a, b, c)(x - a) + h_y(a, b, c)(y - b) + h_z(a, b, c)(z - c).
\]

In general, the linearization of a function \( f(\mathbf{x}) \) at \( \mathbf{a} \) is

\[
L(\mathbf{x}) = L(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}).
\]

Total Differential

The linearization of a function \( f(\mathbf{x}) \) at \( \mathbf{a} \) is given by

\[
L(\mathbf{x}) - L(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}),
\]

where the left-hand side represent the approximate change in \( z \) (the output variable) as we go from \( \mathbf{a} \) to \( \mathbf{x} \). Let \( \mathbf{x} \to \mathbf{a} \), and we obtain the total differential

\[
dz = \nabla f(\mathbf{a}) \cdot d\mathbf{x},
\]

where \( d\mathbf{x} \) is short for \( \langle dx, dy \rangle \) or \( \langle dx, dy, dz \rangle \).

- The equation for a function \( f(x, y) = z \) translates to

\[
dz = f_x(a, b)dx + f_y(a, b)dy,
\]

- and for a function \( f(x, y, z) = w \) (notice the different output variable)

\[
dw = f_x(a, b, c)dx + f_y(a, b, c)dy + f_z(a, b, c)dz.
\]
Summary
Let \( f(x) = z \) be a function of 1, 2, or 3 variables and \( a \) be in its domain. Then

- the tangent line/plane/"space" to \( f \) at \( a \) is given by
  \[
  z - f(a) = \nabla f(a) \cdot (x - a).
  \]

- the linearization of \( f \) near \( a \) is the function \( L(x) \) given by
  \[
  L(x) - L(a) = \nabla f(a) \cdot (x - a).
  \]
  \( L(x) \) approximates \( f(x) \) for values \( x \) near \( a \).

- the total differential of \( f \) at \( a \) is
  \[
  dz = \nabla f(a) \cdot dx.
  \]
Extreme Values of a Function of Several Variables

Jenny Lam
February 24, 2009

Objectives

• Find local extrema of a function of 2 variables on an open region of $\mathbb{R}^2$
• Apply the Second Derivative Test to determine the nature of critical points
• Find absolute extrema of a function of 2 variables on a closed region of $\mathbb{R}^2$

Local extrema
In this section, we restrict our attention to functions of 2 variables only. To find local extrema (i.e., maxima or minima) on an open region of $\mathbb{R}^2$

• find the critical points, i.e., the points where both partial derivatives vanish
• to determine the type of critical point, apply the 2nd derivative test (see Theorem 11, p 1014)

Absolute extrema
To find absolute extrema of a function $f$ of 2 variables on a closed region of $\mathbb{R}^2$

• find the critical points on the interior of the region, i.e. on the region without its boundary
  – use the techniques for finding local extrema from the previous slide
  – the 2nd Derivative Test is not needed
• find the critical points on each piece of the boundary
  – parametrize the boundary piece so that $f$ is a function of 1 variable $t$ only
  – set $f'(t) = 0$ and solve (see Calculus I)
• Evaluate $f$ at the critical points in the interior, the critical points on the boundary and the endpoints of the boundary and compare.
  – The largest value of $f$ is the absolute maximum.
  – The least value of $f$ is the absolute minimum.
Lagrangian Multipliers

Jenny Lam
March 8, 2009

Objectives

• Solve an extrema problem with a function of 2 variables and 1 constraint
• Solve an extrema problem with a function of 3 variables and 1 constraint
• Solve an extrema problem with a function of 2 variables and 2 constraints

Overview of Lagrangian Multipliers

• Lagrange’s method is used to find extrema of a function when the domain of the function is restricted by constraint equations.
• The method introduces new variables, the Lagrange multipliers, but also a sufficient number of new equations to make the new system of equations solvable.

A comparison of the 3 cases

• To find extrema for \( f(x, y) = z \) constrained by \( g(x, y) = c \), we have
  
  – 3 equations: \( g(x, y) = c \) and \( \nabla f = \lambda \nabla g(x, y) \) (consisting of 2 equations when written in coordinate form)
  
  – and 3 variables: \( x, y \) and \( \lambda \).

• To find extrema for \( f(x, y, z) = w \) constrained by \( g(x, y, z) = c \), we have
  
  – 4 equations: \( g(x, y, z) = c \) and \( \nabla f = \lambda \nabla g \) (consisting of 3 equations when written in coordinate form)
  
  – and 4 variables: \( x, y, z \) and \( \lambda \).

• To find extrema for \( f(x, y, z) = w \) constrained by \( g(x, y, z) = c \) and \( h(x, y, z) = d \), we have
  
  – 5 equations: \( g(x, y, z) = c, h(x, y, z) = d \) and \( \nabla f = \lambda \nabla g + \mu h \) (consisting of 3 equations when written in coordinate form)
  
  – and 5 variables: \( x, y, z, \lambda \) and \( \mu \).
Partial Derivatives with Constraints and
Double integrals

Jenny Lam
March 23, 2009

1 Outline

Contents

Objectives
• Find partial derivatives given constraints on the variables
• Evaluate double integrals using geometric formulas
• Evaluate double integrals using iterated integrals
• Change the order of integration of a double integral

2 Partial derivatives with constraints

Motivation
Suppose that \( w = f(x, y), \ x = x(u, v), \ y = y(u, v) \) and \( u = u(v) \). Then
\[
\frac{df}{dv} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.
\]
We must be careful when finding \( \frac{\partial x}{\partial v} \) and \( \frac{\partial y}{\partial v} \). Because \( x \) depends on \( v \) directly and via \( u \),
\[
\frac{\partial x}{\partial v} = \frac{\partial x}{\partial u} \frac{du}{dv} + \frac{\partial x}{\partial v}
\]
and similarly for \( \frac{\partial y}{\partial v} \). Note that \( \frac{\partial x}{\partial v} \) has different meanings. On the left, \( u \) is treated as a variable dependent on \( v \). On the right, \( u \) is treated as a variable independent from \( v \).

Motivation: new notation
The notation for partial derivatives becomes ambiguous when the variables are constrained by an equation. In the previous example, we use
\[
\left( \frac{\partial x}{\partial v} \right)_u
\]
to mean the partial derivative where \( u \) and \( v \) are treated as independent variables. This is the partial derivative we already know how to compute, by treating \( u \) as a constant. On the other hand,
\[
\frac{\partial x}{\partial v}
\]
refers to the partial derivative where \( u \) is dependent on \( v \). This is the partial derivative we are usually asked to find.
Example
Let \( x = u^2 + u \cos v + t \) and \( u = e^{2v} + t \). Then

- \( \frac{\partial x}{\partial v} = u \sin v \)
- \( \frac{\partial x}{\partial u} = \cos v \)
- \( \frac{\partial u}{\partial v} = 2e^{2v} \)

Therefore

\[
\frac{\partial x}{\partial v} = \left( \frac{\partial x}{\partial u} \right)_v \frac{\partial u}{\partial v} + \left( \frac{\partial x}{\partial v} \right)_u = 2(\cos v)e^{2v} + u \sin v.
\]

The General Case
If \( f(x, y, z) \) is a function whose variables are constrained by the equation \( g(x, y, z) = c \) where \( c \) is a constant, then

\[
\left( \frac{\partial f}{\partial x} \right)_y \quad \text{and} \quad \left( \frac{\partial f}{\partial y} \right)_x
\]

represent partial derivatives where \( x \) and \( y \) are treated as independent variables.

- The variable with respect to which the derivative is taken is always treated as an independent variable.
- The other variables that are to be treated as independent variables are listed as subscripts.
- To find these partials above, we try to eliminate the dependent variable \( z \).

Example
Let \( w = x^2 + 2y^2 + \sin z \), subject to \( x^3z = x^2 + y^2 \).

- \( \frac{\partial w}{\partial z} \bigg|_{x, y} = \cos z \): \( x \) and \( y \) are independent. This is equivalent to ignoring the existence of the constraint equation and taking the partial as usual, treating \( x \) and \( y \) as constants.

- \( \frac{\partial w}{\partial z} \bigg|_x \): eliminate the dependent variable \( y \), then take the partial as usual, treating \( x \) as a constant.

\[
w = x^2 + 2(x^3z - x^2) + \sin z = 2zx^3 - x^2 + \sin z.
\]

So \( \frac{\partial w}{\partial z} \bigg|_x = 2x^3 + \cos z \).

Example, continued
Let \( w = x^2 + 2y^2 + \sin z \), subject to \( x^3z = x^2 + y^2 \). \( \frac{\partial w}{\partial z} \bigg|_y \): it is not clear that we can eliminate the dependent variable \( x \) by solving for it in the constraint equation. Instead, take the partials with respect to \( z \) implicitly in both equations, treating \( x \) as a variable dependent on \( z \). Note that since \( y \) is independent, \( \frac{\partial y}{\partial z} = 0 \):

\[
\left( \frac{\partial w}{\partial z} \right)_y = 2x \frac{\partial x}{\partial z} + \cos z, \quad 3x^2 \frac{\partial x}{\partial z} + x^3 = 2x \frac{\partial x}{\partial z}.
\]

We have found \( \frac{\partial w}{\partial z} \bigg|_y \) but need to rewrite it in terms of the original variables. We eliminate \( \frac{\partial x}{\partial z} \) by solving for it in the second equation:

\[
\frac{\partial x}{\partial z} = \frac{x^3}{2x - 3x^2z}, \quad \text{so} \quad \left( \frac{\partial w}{\partial z} \right)_y = \frac{2x^4}{2x - 3x^2z} + \cos z
\]
3 Double integrals

Definition
The double integral
\[ \int \int_R f(x,y)dA \]
represents the volume under the surface \( z = f(x,y) \) over the region \( R \subseteq \mathbb{R}^2 \). (Assume that the surface is above the \( xy \)-plane, that is, \( f(x,y) \geq 0 \)).

The formal definition of a double integral, which uses a limit of Riemann sums \( \lim_{n \to \infty} S_n \) where \( S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A \), is motivated by this geometric interpretation.

Caution
When accompanied by \( dA \), the two integral signs \( \int \int \) should be treated as a single symbol, that may not be broken apart. In fact, some textbooks prefer writing
\[ \int_R f(x,y)dA \]

Example
Evaluate
\[ I_1 = \int \int_{\text{unit disk}} \sqrt{x^2 + y^2}dA \quad \text{and} \quad I_2 = \int \int_{-1 \leq x \leq 1, 2 \leq y \leq 5} 4dA. \]

- The integrand represents the upper hemisphere of the unit sphere; so \( I_1 = \frac{1}{2} \frac{4}{3} \pi 1^3 = \frac{2}{3} \pi \).
- \( I_2 \) describes the volume of the rectangular solid under the horizontal plane \( z = 4 \) and above a \( 2 \times 3 \) rectangle, so \( I_2 = 4 \cdot 2 \cdot 3 = 24 \).

Iterated integrals: Fubini’s Theorem
For more complicated shapes, it becomes no longer practical to evaluate double integrals by using geometric formulas for volumes. Instead, Fubini’s Theorem allows us to compute a double integral by rewriting it as two nested one-dimensional, regular integrals, one with respect to \( x \), the other with respect to \( y \). We first integrate the innermost integral, holding the other variable constant. Then we integrate the outer integral. The nested integrals are often referred to as an iterated integral.

Example
Integrate \( f(x,y) = \frac{1}{xy} \) over the region \( R = [1, 2] \times [3, 4] \). \( R \) represents the rectangle in the \( xy \)-plane where \( 1 \leq x \leq 2 \) and \( 3 \leq y \leq 4 \).

**Solution 1.** The order of integration is important. Like the layers of an onion, if we integrate with respect
to x first, the bounds of the inner integral are going to be 1 and 2.

\[ \int_R f(x, y) \, dA = \int_3^4 \int_1^2 \frac{1}{xy} \, dx \, dy \]

\[ = \int_3^4 \frac{1}{y} \int_1^2 \frac{1}{x} \, dx \, dy \]

\[ = \int_3^4 \frac{1}{y} \ln x \bigg|_1^2 \, dy \]

\[ = \int_3^4 (\ln 2) \frac{1}{y} \, dy \]

\[ = \ln 2 \ln y \bigg|_3^4 \]

\[ = \ln 2 \ln \frac{4}{3} \]

Volume: integrating over y first

Find the volume bounded above by the cylinder \( z = x^2 \) and over the region between \( y = x \) and \( y = 2 - x^2 \).

We can choose to integrate with respect to either variable first. Sometimes, one option is easier. To integrate with respect to \( y \) first, note that to fall in the region, we need \( x \leq y \leq 2 - x^2 \). The bounds on \( x \) are constant and determined by the points of intersection of the two curves: \((-2, -2)\) and \((1, 1)\).

\[ \text{Volume} = \int_{-2}^1 \int_{\sqrt{2-y}}^{\sqrt{2-y}} x^2 \, dx \, dy + \int_1^2 \int_{\sqrt{2-y}}^{\sqrt{2-y}} x^2 \, dx \, dy \]

These integrals are doable and left as an exercise to the reader.
Volume
Remarks regarding the previous example.

• If we integrate with respect to $x$ first, the bounds on $x$ cannot depend on $x$. They may only depend on $y$. The reverse is true as well.

• To figure out the bounds on the inner integral (say it is with respect to $x$), rewrite the equations so that they are solved for $x$. This may mean that the curve is split into two equations, such as when $y = 2 - x^2$ becomes $x = \pm \sqrt{2 - y}$.

• The outer integral may only have constant bounds when computing volumes. To figure these constant bounds, it can helpful to think of the curves as collapsing onto the axes. For example, when integrating with respect to $y$ first, the curves $y = x$ and $y = 2 - x^2$ collapse onto the $x$-axis after the integration with respect to $y$. The region now corresponds to the interval $[-2, 1]$ on the $x$-axis.
Areas, Moments and Center of Mass

Jenny Lam

March 24, 2009

1 Outline

Contents

1 Outline 1

Objectives

To calculate the following quantities.

• area
• average value of a function
• mass given density \( \delta(x, y) \)
• first moments and center of mass
• moments of inertia

Area

Recall that \( \int_R f(x, y) \, dA \) (where \( f(x, y) \geq 0 \) on \( R \)) is the volume of the solid region bounded above by the graph of \( f \) and below by the \( xy \)-plane on region \( R \). In particular, if we let \( f(x, y) = 1 \), then the integral

\[
\int_{[a,b] \times [c,d]} 1 \, dA
\]

represents the solid region above \( R \) with thickness 1. This is, of course, also the area of \( R \).

Average value

Recall the average of \( n \) numbers \( f(1), f(2), f(3), \ldots, f(n) \) is

\[
\frac{f(1) + f(2) + f(3) + \ldots + f(n)}{n} = \frac{1}{n} \sum_{i=1}^{n} f(n).
\]

Similarly, the average of all values \( f(x) \) over the interval \([a, b]\) is

\[
\frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
\]

We generalize that to the average of all values \( f(x, y) \) over the rectangular region \([a, b] \times [c, d]\) is

\[
\frac{1}{(b - a)(d - c)} \int_{[a,b] \times [c,d]} f(x, y) \, dA.
\]

Recall that the symbol \( \int \) is a German “S” and stands for sum. An integral is a sum of infinitesimal quantities, be it areas, values of a function, or, as we will now see, masses.
Mass
The mass of a flat object (lying on the $xy$-plane) with a variable density $\delta(x,y)$ can be determined by integrating mass elements $dm$ over $R$. Assume that density is constant over volume elements $dV$, then

$$dm = \delta(x,y) \, dV.$$ 

Therefore,

$$\text{mass} = \int_R dm = \int\int_R \delta(x,y) \, dA.$$ 

First Moment: a system of 2 particles
Suppose a massless stick on the $x$-axis has a point mass at each end: $m_a$ at $(a,0)$ and $m_b$ at $(b,0)$ and is balanced on the $y$-axis. The stick’s tendency to lean towards one side of the $x$-axis or the other depends on the distances of the masses to the $y$-axis, and how massive (for the lack of a better word) the masses are. In short, in particular if the quantities $m_a |a|$ and $m_b |b|$ are equal and $a$ and $b$ are on opposite sides of the $y$-axis, then the stick is balanced and does not lean. The quantity which summarizes the leaning tendency of this system of two particles $m_a$ and $m_b$ is the first moment of inertia about the $y$-axis:

$$I_y = m_a a + m_b b.$$ 

Note that $a$ and $b$ are coordinates rather than distances, reflecting the fact that one must be negative for the moment to be 0.

First Moments and Center of Mass
Suppose a thin plate covering a region $R \subseteq \mathbb{R}^2$ with density function $\delta(x,y)$ is balanced on the top of the $y$-axis. The plate is made up of infinitely many mass elements $dm = \delta(x,y)dA$ located at a point $(x,y)$, each of which has coordinate $x$. Therefore, the first moment of this plate about the $y$-axis is

$$I_y = \int_R xd\!m = \int\int_R x\delta(x,y) \, dA.$$ 

Similarly, the first moment about the $x$-axis is

$$I_x = \int_R yd\!m = \int\int_R y\delta(x,y) \, dA.$$ 

The center of mass is the point at which the plate could be supported at by, say, a pencil tip, and not tilt over. Its coordinates are given by

$$x_m = \frac{I_x}{m}, \quad y_m = \frac{I_y}{m}.$$ 

Moments of Inertia
Moments of inertia are also known in physics as rotational inertia about a given axis. Given by the equation

$$I_L = \int_R r^2d\!m = \int\int_R r^2(x,y)\delta(x,y)dA,$$ 

the rotational inertia measures an object’s tendency to resist change of its rotational motion about an axis $L$. It is the rotational analog of mass in many equations of motion. Moments of inertia also measure resistance to bending, as described in the textbook.
Integration in Polar Coordinates

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In rectangular coordinates, we use the fact that

\[ \int \int_R dA = \int \int_R dx \, dy \]

to compute the area of a region \( R \). Now, what if the curves bounding \( R \) are expressed in polar form \( r = r(\theta) \)? Then a more suitable formula to use is

\[ \int \int_R dA = \int \int_R r \, dr \, d\theta. \]

But why is it that \( dx \, dy = r \, dr \, d\theta \)? We begin by expressing the rectangular coordinates in polar form:

\[ x = r \cos \theta \quad r = \sin \theta. \]

Then we take their differentials.

\[ dx = dr \cos \theta - r \sin \theta \, d\theta \]
\[ dy = dr \sin \theta + r \cos \theta \, d\theta \]

so

\[ dA = dx \, dy = \cos \theta \sin \theta (dr)^2 + r \cos^2 \theta \, dr \, d\theta - r \sin^2 \theta \, d\theta \, dr - r^2 \sin \theta \cos \theta (d\theta)^2. \]

Now the product of two differentials is anticommutative, that is, \( drd\theta = -d\theta dr \). This also implies that \( drdr = d\theta d\theta = 0 \). (To emphasize the fact that product of differentials does not behave like product of reals, we often write \( dr \wedge d\theta \), where \( \wedge \) is called the wedge product.) Using these facts, we conclude the result which was to be proven.

The area element \( dA \) is a small rectangle in rectangular coordinates, and a small wedge with the point removed in polar coordinates.
Remark. This technique for computing the area element $dA$ and the volume element $dV$ generalizes, as we shall see at the end of this chapter.

Example. To evaluate the integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \, dy \, dx,$$

we notice that the expressions $\sqrt{1-x^2}$ and $x^2+y^2$ make conversion into polar form tempting. We begin by sketching the region $R$ over which the integration takes place.

The region of integration $R$ is bound by the curve $y = \sqrt{1-x^2}$, the $x$ and the $y$-axes. In polar form, the distance $r$ from the origin to any point in $R$ can be anywhere from 0 to the boundary curve

$$y = \sqrt{1-x^2} \implies x^2 + y^2 = 1 \implies r^2 = 1 \implies r = 1.$$  

Therefore the limits of integration on $r$ are $0 \leq r \leq 1$. The rays emanating from the origin indicate the limits of integration of $\theta$, namely $0 \leq \theta \leq \pi/2$. Therefore,

$$I = \int_0^{\pi/2} \int_0^1 e^{-r^2} r \, dr \, d\theta = \frac{\pi}{4} \left(1 - \frac{1}{e}\right).$$

Remark. It is interesting to note that Gauss introduced the change of variables to polar coordinates to integrate the normal distribution curve $e^{-x^2}$.

Example. To find the area of the region $R_1$ that lies outside the curve $r = 1 + \cos \theta$ and inside the curve $r = 1$, we must find the limits of integration.
Clearly, we must have $1 + \cos \theta \leq r \leq 1$ to satisfy the conditions set by the problem. To find the limits on $\theta$, picture a ray emanating from the origin, sweeping the plane as its angle with the positive $x$-axis increases. The ray enters the region $R_1$ when $\theta = \pi/2$ and exits it when $\theta = 3\pi/2$. For an algebraic solution, set the curves equal to each other:

$$1 + \cos \theta = 1, \quad \text{or} \quad \theta = \frac{\pi}{2} + n\pi.$$ 

We use the sketch of $R_1$ to choose the correct limits on $\theta$ out of the solutions we found. The area of $R$ is given by

$$I_1 = \int_{R_1} dA = \int_{\pi/2}^{3\pi/2} \int_{1+\cos \theta}^{1} r \, dr \, d\theta.$$ 

On the other hand, to find the area of the region $R_2$ that lies inside the curve $r = 1 + \cos \theta$ and outside the curve $r = 1$, we let $-\pi/2 \leq \theta \leq \pi/2$. This means that the area of $R_2$ is given by

$$I_2 = \int_{R_2} dA = \int_{-\pi/2}^{\pi/2} \int_{1+\cos \theta}^{1} r \, dr \, d\theta.$$ 

Finally, the area of the region $R_3$ inside both curves is found by breaking the region into two parts, one below each curve:

$$I_3 = \int_{R_3} dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{1} r \, dr \, d\theta + \int_{\pi/2}^{3\pi/2} \int_{0}^{1+\cos \theta} r \, dr \, d\theta.$$ 

In particular, the area of one lobe $R_4$ of the cardioid is given by

$$I_4 = \int_{R_4} dA = \int_{\pi/2}^{\pi} \int_{0}^{1+\cos \theta} r \, dr \, d\theta.$$ 

This is because $r$ is bounded above by the curve $r = 1 + \cos \theta$, and the bounds of $\theta$, which can be found by letting a ray emanating from the origin sweep the plane, are $\pi/2$ and $\pi$. The evaluation of these integrals is left as an exercise to the reader.

**Example.** The first moments of $R_1$ in the previous example can be found by the formulas of the previous lesson, with the polar coordinates transformation. Immediately the symmetry with respect to the $x$-axis implies that $M_x = 0$. The first moment with respect to the $y$-axis is given by

$$M_y = \int_{R_1} x \, dm = \int_{R_1} (r \cos \theta) \delta \, dA = \delta \int_{\pi/2}^{3\pi/2} \int_{0}^{1} (r \cos \theta) r \, dr \, d\theta.$$ 

In general, the density is assumed to be constant unless otherwise specified. The evaluation of the integral is also again left as an exercise to the reader.
The techniques and formulas for integrating in the plane readily generalize to integration in 3-space. The challenge lies in visualization, which is generally necessary to determine limits of integration. We begin by a few tips on graphing in 3-space.

- To plot a point, say \((a, b, c)\), picture traveling along the edges of the box formed by the points \(\vec{a} = (a, 0, 0)\), \(\vec{b} = (0, b, 0)\) and \(\vec{c} = (0, 0, c)\). Lines parallel to the axes should remain parallel in the sketch.

- To plot a line, plot two points as mentioned above.

- To plot a plane, plot the intercepts if possible. For example, the intercepts of \(2x + 3y + 4z = 12\) are \((6, 0, 0)\), \((0, 4, 0)\) and \((0, 0, 3)\).

**Example.** Let \(R\) be the solid region in the first quadrant, bounded above by the cylinder \(y^2 + 4z^2 = 16\), below by the \(xy\)-plane, and in the front by the plane \(x + y = 4\).
The limits on $z$ are given by the cylinder and the $xy$-plane, whose equations may be solved for $z$. The cylinder’s equation yields

$$z = \pm \sqrt{16 - y^2}$$

and we choose the positive sheet, which is the part which lies above the $xy$-plane, whose equation is $z = 0$. So the constraints on $z$ are $0 \leq z \leq \sqrt{16 - y^2}/4$. To find the bounds on $x$ and $y$, project the solid onto the $xy$-plane. We obtain the planar region $R'$, whose $y$ values are bounded below by $x = 0$ and above by the line $x + y = 4$. So the constraints on $y$ are $0 \leq y \leq 4 - x$. Finally, $0 \leq x \leq 4$. Therefore, the volume of $R$ is given by

$$\int \int_R dV = \int_0^4 \int_0^{4-x} \int_0^{\sqrt{16-y^2}/4} dz \, dy \, dx.$$ 

However, this integral is difficult to evaluate. If we choose to integrate with respect to $x$ instead, we use the fact that $R$ is bounded in the back by the $yz$-plane, and in the front by the $x + y = 4$ plane. So the bounds on $x$ are $0 \leq x \leq y - 4$. Next, $z$ is bound by $0 \leq z \leq \sqrt{16 - y^2}/4$, and finally $0 \leq y \leq 4$. So the volume of $R$ is given by

$$I = \int \int_R dV = \int_0^4 \int_0^{\sqrt{16-y^2}/4} \int_0^{4-y} dx \, dz \, dy.$$ 

After the first two integrals, we find that

$$I = \int_0^4 (4 - y) \frac{16 - y^2}{4} dy = \int_0^4 \sqrt{16 - y^2} \, dy - \frac{1}{4} \int_0^4 y \sqrt{16 - y^2} \, dy = 4\pi - \frac{16}{3},$$

where the first integral was found geometrically, while the second was integrated.

**Example.** Let $R$ be the region bounded in the back by $x = 0$, on the front and on the sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the $xy$-plane.
The vertical direction $z$ is constrained by $0 \leq z \leq x^2 + y^2$. Projecting the solid region onto the $xy$-plane, we see that $0 \leq x \leq 1 - y^2$ and $0 \leq y \leq 1$. So

$$\iiint_R dV = \int_0^1 \int_{1-y^2}^0 \int_0^{x^2+y^2} dz \, dx \, dy = \frac{2}{7}.$$

The evaluation of the integral is straightforward.
Masses and Moments in three Dimensions

Jenny Lam

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The formulas from mass and moments easily generalize from two dimensions to three: simply replace the area element $dA$ by the volume element $dV$. The density function $\delta(x, y, z)$ represents mass per volume instead of mass per area. So mass is given by

$$M = \iiint dm = \iiint R \delta \, dV,$$

and the first moment with respect to the plane $\Pi$ is

$$M_\Pi = \iint r \, dm = \iint R r \delta \, dV.$$

where $r$ is the distance to the plane $\Pi$. For example, the first moment with respect to the $yz$-plane is

$$M_{yz} = \iint R x \delta(x, y, z) \, dx \, dy \, dz.$$

Similarly, the second moment with respect to a line $\ell$ is

$$I_\ell = \iint r^2 \, dm = \iiint R r^2 \delta \, dV,$$

where $r$ is the distance to the line $\ell$. For example, the second moment with respect to the $x$-axis is

$$I_x = \iiint R (y^2 + z^2) \delta(x, y, z) \, dx \, dy \, dz.$$

Example. Consider the following solid.

![Diagram of a solid with centroids labeled](image)
We would like to find its first moments. We begin by finding the limits of integration. The bounds on $x$ are constant, that is, $-a/2 \leq x \leq a/2$. We also look at a cross section of the solid that is parallel to the $yz$-plane, because the $z$-values are bounded above by a plane which depends on $y$.

To find the equation of the slanted line in the $yz$-plane (hence the plane in $\mathbb{R}^3$), we can use the points $(-b/3, 2c/3)$ and $(2b/3, -c/3)$ or set up a proportionality between similar triangles. We end up with the line

$$z = \frac{c}{3} - \frac{cy}{b}.$$  

Therefore the bounds on $z$ are $-c/3 \leq c/3 - cy/b$ and the bounds on $y$ are $-b/3 \leq y \leq 2b/3$. This allows us to set up the integrals to find the first moments.

$$M_x = \iiint \delta x \, dV = \int_{-a/2}^{a/2} \int_{-b/3}^{2b/3} \int_{-c/3}^{c/3 - cy/b} x \, dz \, dy \, dx = 0.$$  

From our understanding of first moments, we could have also seen that the symmetry of the solid with respect to the $x$-axis implies that the moment with respect to it is 0. Similarly we set up the integrals for the first moments with respect to the other two axes, and leave the details of the computation to the reader.

$$M_y = \int_{-a/2}^{a/2} \int_{-b/3}^{2b/3} \int_{-c/3}^{c/3 - cy/b} y \, dz \, dy \, dx = 0. M_z = \int_{-a/2}^{a/2} \int_{-b/3}^{2b/3} \int_{-c/3}^{c/3 - cy/b} z \, dz \, dy \, dx = 0.$$  

Amazingly, we also find that the first moments are 0 with respect to the $y$ and $z$ axis. In fact, this means that the center of mass of this solid lies at the intersection of all three axes.

**Example.** Consider the solid with density $\delta(x, y, z) = \sqrt{x^2 + y^2}$ bounded above by the paraboloid $z = 16 - 2x^2 - 2y^2$ and below by the paraboloid $z = 2x^2 + 2y^2$. The curve of intersection of the two surfaces can be found by setting the $z$ variables equal. We find that $16 = 4x^2 + 4y^2$ or $x^2 + y^2 = 4$. This curve does not lie in the $xy$-plane. To find its height, plug $x^2 + y^2 = 4$ back into one of the surface equations. Then $z = 8$. The bounds on $z$ are simply the two surfaces: $16 - 2x^2 - 2y^2 \leq z \leq 2x^2 + 2y^2$. To find the bounds on $x$ and $y$, we project the solid onto the $xy$ plane, and obtain the area contained inside the curve of intersection we found earlier $x^2 + y^2 = 4$. If we decide to integrate first, solve this equation for $y$, so $y = \pm \sqrt{4 - x^2}$ so that $y$ is bounded by two semicircles $-\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}$. 

2
Finally the bounds on the last variable must be constant: $-2 \leq x \leq 2$. Therefore the mass is given by

$$M = \iiint \delta \, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2x^2+2y^2}^{16-2x^2-2y^2} \sqrt{x^2 + y^2} \, dz \, dy \, dx.$$

The integration process is a bit difficult in cartesian coordinates. However, if we convert $x$ and $y$ into polar coordinates and leave $z$ as is, we obtain a much more tractable integral. Note that we must apply the conversion $dy \, dx = r \, dr \, d\theta$.

$$M = \int_{0}^{2\pi} \int_{0}^{2} \int_{2r^2}^{16-2r^2} r \, dz \, r \, dr \, d\theta = 32\pi.$$
Cylindrical Coordinates

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Last week, we learned that some integrals in two dimensions were more easily evaluated in polar form than in rectangular form. Similarly, three dimensional integrals can be rewritten from rectangular form to cylindrical or spherical form. First, cylindrical coordinates are a direct extension of polar coordinates, that is, we apply the polar coordinate transformation to two of the variables, and leave the third one alone. The most common cylindrical coordinate transformation is

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \]

The corresponding volume element is

\[ dV = r \, dr \, d\theta \, dz. \]

Note that cylindrical coordinates should be used with intelligence; when integrating over a cylinder that is symmetric about the y-axis, we should apply the polar coordinate transformation to \( x \) and \( z \), not \( x \) and \( y \).

**Example.** (Problem 15.6.19) Consider the prism in the first octant, that is bounded above by the plane \( z = 2 - y \), in the front by the plane \( x = 1 \) and in the back by the plane \( y = x \).

The volume is given in rectangular coordinates by

\[ V = \int_0^1 \int_y^1 \int_0^{2-y} \, dz \, dx \, dy = \int_0^1 (2 - y)(1 - y) \, dy = \frac{5}{6}. \]
We can also evaluate this integral in polar coordinates by letting $x = r \cos \theta$ and $y = r \sin \theta$. The bounds on $\theta$ are determined by the angles the $x$-axis and the line $y = x$ make with the positive $x$-axis. Hence $0 \leq \theta \leq \pi/4$. Next, $r$ is bounded inward by the origin and outward by the line $x = 1$ or $r \cos \theta = 1$. Solving for $r$, we have $r = \sec \theta$. Therefore, $0 \leq r \leq \sec \theta$ and the volume is given in cylindrical coordinates by

$$V = \int_0^{\pi/4} \int_0^{\sec \theta} \int_0^{2-r \sin \theta} r \, dz \, dr \, d\theta = \cdots = \int_0^{\pi/4} \sec^2 \theta - \frac{\sin \theta}{3 \cos^3 \theta} = \tan \theta - \frac{\cos^{-2} \theta}{6} \bigg|_0^{\pi/4} = \frac{5}{6}.$$

The other substitution is a change to polar coordinates $\rho$, $\phi$, $\theta$, which is useful for measuring position on objects related to spheres. $\rho$ (pronounced “rho”) is used to measure distance to the origin, unlike $r$ in cylindrical coordinates, which is used to measure the distance to the axis of symmetry. Just like in polar coordinates, $\theta$ measures the angle from the positive $x$-axis to the point (once it is projected onto the $xy$-plane). $\phi$ measures the angle from the positive $z$-axis to the point (without projection onto any plane). On the Earth, $\theta$ would be measuring longitude, i.e. how far East or West one is. And $\phi$ would be measuring latitude (north-south), except that we measure from the North Pole rather than from the equator. The substitution is given by

$$x = r \cos \theta = (\rho \sin \phi) \cos \theta$$
$$y = r \sin \theta = (\rho \sin \phi) \sin \theta$$
$$z = \rho \cos \phi$$
$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

One key remark, if integrating over an entire sphere, $\theta$ is evaluated over $[0, 2\pi]$, hence $\phi$ only needs to be evaluated on $[0, \pi]$. This is because, $\theta$ already takes care of the evaluation over the “back” of the sphere.

**Example.** (15.6.35) The solid enclosed by the cardioid of revolution $\rho = 1 - \cos \phi$

![Cardioid of Revolution](image)

has the following limits of integration:

$$0 \leq \rho \leq 1 - \cos \phi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$
The integral is
\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^{1-cos \phi} \rho^2 \sin \phi d\rho \ d\phi \ d\theta = \frac{8\pi}{3}.
\]

Example. (15.5.61) Let \( R \) be the solid cut from the solid cylinder \( x^2 + y^2 \leq 1 \) by the sphere \( x^2 + y^2 + z^2 = 4 \). We will evaluate the volume for the top half of the solid and double the result. The solid is divided into two parts by the cone whose edge is the intersection between the cylinder and the sphere. The angle \( \phi \) of the points on this circle is found by setting \( x^2 + y^2 = 1 \) on the sphere.

\[
1 + z^2 = 4, \quad \implies \quad \rho = 2, \quad z^2 = 3 = 4 \sin^2 \phi, \quad \implies \quad \sin \phi = \frac{\sqrt{3}}{2}, \quad \implies \quad \phi = \frac{\pi}{6}.
\]

We sum up the two spherical integrals. For the region inside the cone, \( \rho \) is integrated over \([0, 2]\) and \( \phi \) over \([0, \pi/6]\). For the region outside the cone, we convert the equation of the cylinder to spherical coordinates:

\[
1 = x^2 + y^2 = (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \cos \theta)^2 = \rho^2 \sin^2 \phi \quad \implies \quad \rho = \csc \phi.
\]

So \( \rho \) is integrated over \([0, \csc \phi]\) and \( \phi \) over \([\pi/6, \pi/2]\). The volume of \( R \) is given by

\[
V = 2 \left( \int_0^{2\pi} \int_0^{\pi/6} \int_0^{2} \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta \right)
\]

\[
= 2 \left( \frac{16\pi}{3} - 2\sqrt{3}\pi \right) = \frac{32\pi}{3} - 4\sqrt{3}\pi.
\]
Substitution in Multiple Integrals

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In the lesson on integration in polar coordinates, we performed the change of variables \((x, y) \rightarrow (r, \theta)\). Using multiplication of differentials, we derived a formula for the area element \(dA\), namely

\[dA = dx \ dy \ dz = r \ dr \ d\theta.\]

This method can be generalized and formulated in terms of the determinant of partial derivatives, called the Jacobian: since the transformation from rectangular to polar coordinates is given by

\[x = r \cos \theta, \quad y = r \sin \theta,\]

the Jacobian of this transformation is

\[
\left| \begin{array}{cc}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{array} \right| = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.
\]

The absolute value of the Jacobian is the scaling factor of the area element resulting from the transformation. More generally, a coordinate transformation in two dimensions \((x, y) \rightarrow (u, v)\) yields an area element given by

\[dA = |J(u, v)| \ du \ dv \quad \text{where} \quad J(u, v) = \left| \begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{array} \right|\]

and a coordinate transformation in three dimensions \((x, y, z) \rightarrow (u, v, w)\) will yield the volume element

\[dV = |J(u, v, w)| \ du \ dv \ dw \quad \text{where} \quad J(u, v, w) = \left| \begin{array}{ccc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{array} \right|\]

Note that the Jacobian may be negative, so we surround it with absolute value signs to get the scaling factor. As we will see in an example coming up, the absolute value can be dropped if the Jacobian is positive over the domain of integration.

Example. \((15.7.9)\) To evaluate

\[I = \iint_{R} \sqrt{x^2 + xy} \ dx \ dy\]

where \(R\) is bounded above by the curve \(xy = 9\), below by the curve \(xy = 1\), to the left \(y = 4x\) and to the right \(y = x\),
we perform the transformation of variables

\[ x = \frac{u}{v}, \quad y = uv \]

which has Jacobian

\[ J(u, v) = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{array} \right| = \frac{2u}{v}. \]

The new boundary equations can be found by applying the transformations to the equations of the boundary curves.

\[
\begin{align*}
xy = 9 & \quad \rightarrow \quad u^2 = 9 & \quad \rightarrow & \quad u = \pm 3 \\
xy = 1 & \quad \rightarrow \quad u^2 = 1 & \quad \rightarrow & \quad u = \pm 1 \\
y = 4x & \quad \rightarrow \quad v^2 = 4 & \quad \rightarrow & \quad v = \pm 2 \\
y = x & \quad \rightarrow \quad v^2 = 1 & \quad \rightarrow & \quad v = \pm 1
\end{align*}
\]

The new boundary equations yield four possible regions over which to integrate, but we note that the transformation is not one-to-one and that it will take any one of these regions will take us back to \( R \) in \( xy \)-space. So integrating over any of these regions will yield the correct result. We choose the one in the first quadrant.

After performing the transformation on the integrand gives

\[
\sqrt{\frac{x}{y}} + \sqrt{xy} = \sqrt{\frac{1}{u^2}} + \sqrt{u^2} = \frac{1}{|v|} + |u|. 
\]
It is important to keep the absolute value until we are able to handle its sign correctly. Since we are integrating over a region in the first quadrant, both \( u \) and \( v \) are positive and we may simply drop the absolute value. However, if we were to integrate over the third quadrant, we would need to negate both \( u \) and \( v \).

\[
I = \int_1^2 \int_1^3 \left( \frac{1}{v} + u \right) \frac{2u}{v} \, du \, dv = 2 \left( 2 + \frac{26}{3} \ln 2 \right).
\]

**Example.** (15.7.21) To integrate

\[
I = \iiint_{R} |xyz| \, dx \, dy \, dz \quad \text{over} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1,
\]

where we assume that \( a, b \) and \( c \) are positive, we perform the substitution

\[
x = au, \quad y = bv \quad z = cw
\]

with Jacobian

\[
J(u, v, w) = \begin{vmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{vmatrix} = abc,
\]

so that

\[
I = \int \int \int_R |(au)(bv)(cw)| \, |abc| \, du \, dv \, dw = a^2 b^2 c^2 \iiint |uvw| \, du \, dv \, dw \quad \text{over} \quad u^2 + v^2 + w^2 \leq 1.
\]

Note that this substitution allows us to go from an integral over a solid ellipsoid to an integral over a ball. Now we perform a second substitution, to spherical coordinates.

\[
I = a^2 b^2 c^2 \int_0^{2\pi} \int_0^{\pi} \int_0^1 |(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi)| \, |\rho^2 \sin \phi| \, d\rho \, d\phi \, d\theta
\]

\[
= a^2 b^2 c^2 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^5 |\sin^3 \phi \cos \phi| \, |\cos \theta \sin \theta| \, d\rho \, d\phi \, d\theta.
\]

Since \( \rho \) is integrated over a positive numbers only, we may drop the absolute values and integrate:

\[
\int_0^1 \rho^5 \, d\rho = \frac{\rho^6}{6} \bigg|_0^1 = \frac{1}{6}.
\]

Hence

\[
I = \frac{1}{6} a^2 b^2 c^2 \int_0^{2\pi} \int_0^{\pi} |\sin^3 \phi \cos \phi| \, |\cos \theta \sin \theta| \, d\phi \, d\theta.
\]

We note that \( \sin \phi \) is positive over \([0, \pi]\), so that \( \sin^3 \phi \) is too. However, \( \cos \phi \) is only positive over the first half, \([0, \pi/2]\). The values of cosine function on the second half \([\pi/2, \pi]\) is equal and opposite to its values on \([0, \pi/2]\). Therefore, to integrate with respect to \( \phi \), we
can simply integrate over $[0, \pi/2]$, where the integrand is positive, and then double its value to get the integral of the absolute value over $[0, 2\pi]$:

$$\int_0^\pi |\sin^3 \phi \cos \phi| \, d\phi = 2 \int_0^{\pi/2} \sin^3 \phi \cos \phi \, d\phi = -2 \frac{\sin^4 \phi}{4} \bigg|_0^{\pi/2} = -\frac{1}{2}$$

Hence

$$I = \frac{1}{6} a^2 b^2 c^2 \left(-\frac{1}{2}\right) \int_0^{2\pi} |\cos \theta \sin \theta| \, d\theta.$$ 

Finally, $\cos \theta \sin \theta$ is positive on $[0, \pi/2]$ and takes on the same but negative values on the second and fourth quadrants $[\pi/2, \pi] \text{ and } [3\pi/2, 2\pi]$. The values are equal in the third quadrant $[\pi, 3\pi/2]$. Therefore, we integrate over the first quadrant and multiply the result by 4,

$$\int_0^{2\pi} |\sin \phi \cos \phi| = 4 \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi = -2.$$ 

So

$$I = \frac{1}{6} a^2 b^2 c^2 \left(-\frac{1}{2}\right) (-2) = \frac{1}{6} a^2 b^2 c^2.$$
Recall the different forms of arc length of a curve $x(t)$ you should have encountered throughout your calculus education:

$$s = \int ds = \int |v(t)| dt = \int \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int \sqrt{f'(x)^2 + 1} dx.$$

The one we would like to focus on now is the form $\int ds$ where $ds$ is the arc length element, that is, a little piece of a curve. A line integral is simply an integral over a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, whose general form is

$$\int_C f(s) \, ds$$

where $C$ is a curve in $\mathbb{R}^n$ where $n$ is 2 or 3, where $f$ depends indirectly through $s$ on the variables of $\mathbb{R}^n$ (e. g. $x$ and $y$, or $r$ and $\theta$, or $\rho$, $\phi$ and $\theta$, etc.) and $ds$ is the arc length element just discussed.

When integrating line integrals, you should keep in mind the meaning of this symbol, and that there are several ways to translate it, depending on the specific variables used. It is also worth noting that very few arc length integrals can be integrated in closed form. In other words, there are a dozen such forms that can be integrated and all other integrals must be evaluated numerically. By contrast, the focus of this lesson is the process of setting up line integrals, rather than their evaluation.

**Example.** Let us review the most basic of line integrals, the arc length integral $\int_C ds$, along the already parametrized curve $r(t) = 2ti + tj + (2 - 2t)k$ where $0 \leq t \leq 1$. (We will focus on parametrization next.) We must set up the integral in terms of $t$ by rewriting $ds$ in terms of the parameter variable $t$.

$$ds = |v(t)| \, dt = \sqrt{2^2 + 1^2 + (-2)^2} \, dt = 3 \, dt.$$

One should not assume from this example that $ds$ is always a constant multiple of $dt$. The arc length is therefore

$$I = \int_C ds = \int_0^1 3 \, dt = 3.$$
The first step in setting up a line integral is to parametrize the curve, a skill which requires that we devote some attention to. Even though the curve depends on the variables of the space in which it lies, it is a one-dimensional object, and should therefore be described using one variable only for the purpose of integration.

**Example.** To parametrize the line joining the points $A(0,1)$ and $B(1,0)$, we can start by writing the $y = mx + b$ form of the equation of the line. In this case, it is $y = -x + 1$. Recall that there are two variants for representing the parametrization:

- the coordinate equations:
  
  \[
  \begin{align*}
  x &= t \\
  y &= -t + 1,
  \end{align*}
  \]
  or, since $t$ is really just a dummy variable
  
  \[
  \begin{align*}
  x &= x \\
  y &= -x + 1,
  \end{align*}
  \]

- the more compact, vector form: $\mathbf{r}(t) = \langle -x, -x + 1 \rangle$

For a method which generalizes to three dimensions, we use a point on the line and a direction vector. So the parametrization is

\[
\mathbf{r}(t) = \mathbf{r}_A + \overrightarrow{AB}(t) = \langle 0, 1 \rangle + \langle 1, -1 \rangle t = \langle t, 1 - t \rangle, \quad \text{or} \quad \begin{cases} \ x = t, \\ y = 1 - t. \end{cases}
\]

To parametrize a unit circle centered at the origin, we begin with its equation $x^2 + y^2 = 1$. We can parametrize it by solving for $y$ and actually using two separate parametrizations:

\[
\begin{align*}
  \begin{cases}
    x = x, \\
    y = \sqrt{1 - x^2},
  \end{cases}
  \quad \text{and} \quad 
  \begin{cases}
    x = x, \\
    y = -\sqrt{1 - x^2}.
  \end{cases}
\]

Breaking the curve into several pieces and parametrizing each piece separately can be useful, as in the case of the four sides of a square, but in this case, it is unnecessary, not to mention ugly. Use the following polar coordinate parametrization instead:

\[
\begin{cases}
  x = \cos \theta, \\
  y = \sin \theta.
\end{cases}
\]

Notice that $r$ is omitted because the radius is fixed when describing the circle. Moreover, using $r$ and $\theta$ together to describe a curve would be problematic since it violates the rule of 1 variable per dimension when parametrizing. To verify that this parametrization works, check that these expressions satisfy the equation of the circle $x^2 + y^2 = 1$. Thus we see that many other parametrizations work, among which are

\[
\begin{align*}
  \begin{cases}
    x = \sin \theta, \\
    y = \cos \theta,
  \end{cases} \quad \begin{cases}
    x = -\cos \theta, \\
    y = -\sin \theta,
  \end{cases} \quad \text{and} \quad 
  \begin{cases}
    x = \cos(2\theta), \\
    y = \sin(2\theta),
  \end{cases}
\]
\]
but not
\[
\begin{align*}
  x &= 2 \cos \theta \\
y &= 2 \sin \theta.
\end{align*}
\]

**Example.** (15.7.11) Let \( C \) be the curve parametrized by \( x = 2t, y = t \) and \( z = 2 - 2t \) where \( 0 \leq t \leq 1 \). Let us evaluate
\[
I = \int_C (xy + y + z) \, ds.
\]
We know that \( ds = 3 \, dt \) from the first example. We must also convert the integrand using the parametrization:
\[
xy + y + z = 2t^2 + t + (2 - 2t) = 2t^2 - t + 2.
\]
Therefore, in terms of \( t \)
\[
I = \int_0^1 3(2t^2 - t + 2) \, dt = 3 \left( \frac{2}{3}t^3 - \frac{t^2}{2} + 2t \right) \bigg|_0^1 = \frac{13}{2}.
\]

In line with the physics applications of this course, we can integrate the density \( \delta \) of a linear object \( C \), say a wire, to get its mass:
\[
M = \int_C dm = \int_C \delta(t) \, ds
\]
where \( \delta(t) \) is the linear density (density per unit length) of the object \( C \), which is a curve parametrized with respect to the same parameter \( t \) used to describe the density. Similarly, we can use line integrals to describe moments. The first moment about the \( yz \) plane is given by
\[
M_{yz} = \int_C x \, dm = \int_C x \delta \, ds = \int_C x(t)\delta(t) |v(t)| \, dt,
\]
and the second moment about the \( x \) axis is given by
\[
I_x = \int_C r^2 \, dm = \int_C (x^2 + y^2) \delta \, ds = \int_C ((x(t))^2 + (y(t))^2) \delta(t) |v(t)| \, dt.
\]

**Example.** (15.7.27) Suppose that the curve \( C \) given by the \( x^2 + y^2 = a^2 \) has constant density \( \delta \). To find the moment of inertia (or second moment) about the \( z \)-axis, we must evaluate
\[
I_z = \int_C (x^2 + y^2) \delta \, ds.
\]
If we parametrize the curve with
\[
\begin{align*}
x &= a \cos t \\
y &= a \sin t
\end{align*}
\]
where \( 0 \leq t \leq 2\pi \),
then the integrand \( x^2 + y^2 = a^2 \) and \( ds = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = a \). So
\[
I_z = \int_0^{2\pi} a^2 \cdot \delta \cdot a \, dt = 2\pi a^3 \delta.
\]
As a summary, a line integral has the form
\[ I = \int_C f(x, y, z) \, ds \]
where \( ds \) is the line element. To evaluate it,

1. Parametrize \( C \): \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) where \( a \leq t \leq b \), so that \( x, y \) and \( z \) are functions of \( t \):
\[
\begin{cases}
  x = x(t) \\
  y = y(t) \\
  z = z(t)
\end{cases}
\]

2. Use this parametrization to rewrite \( f(x, y, z) \) in terms of \( t \): \( f(x, y, z) = f(x(t), y(t), z(t)) \).

3. Use the parametrization to rewrite \( ds \) in terms of \( dt \): \( ds = |\mathbf{v}(t)| \, dt \) or
\[
ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.
\]

Putting this all together, we get
\[
I = \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.
\]
This formula does not need to be memorized if one understands the idea behind a line integral.
Work, Flow, Circulation and Work

Jenny Lam

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Last time, we were introduced to line integrals, and we integrated scalar functions with respect to the arclength element:

$$\int_C f(s) \, ds.$$ 

Now, we explore additional applications which use vector fields. Intuitively, a vector field is obtained by associating each point in space with a vector. Formally, a vector field in the plane is a function from $\mathbb{R}^2 \to \mathbb{R}^2$ and a vector field in space is a function $\mathbb{R}^3 \to \mathbb{R}^3$.

Example. 

• The gravitational force exerted by an object at point A onto an object at point B is given by

$$F_{AB} = -\frac{k}{|r|^2} \frac{r}{|r|}$$

where $r = \overrightarrow{AB}$. We assume $k$ to a constant depending on the physical configuration of the system. The negative sign signifies attraction, namely the force vector $F_{AB}$ is parallel and opposite the distance vector $\overrightarrow{AB}$. If we let point B move in space, then the gravitational force will vary as well: each point B in space is associated with a force. Hence the gravitational force field is a vector field.

• Let $f(x,y,z)$ be scalar function, also called scalar field as opposed to a vector field. To find the gradient vector at point $(a,b,c)$, we first take the partials with respect to each variable, $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ and then evaluate it at the point $(a,b,c)$. Since each point $(a,b,c)$ is associated with the gradient vector $\nabla f(a,b,c)$, we say that $\nabla f(x,y,z)$ is a vector field, called the gradient field of $f$.

• A gravitational field is the opposite of the gradient field of the gravitational potential function.

• A fluid’s motion can be described by a velocity field. For example, the speed at which water is falling from the faucet increases as we move down from the faucet. The velocity field of water going down the sink could indicate a churning motion.

Let $F(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$ be a vector field and $C$ be a curve in $\mathbb{R}^3$. Then the integral

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$
is a line integral which goes by several different names depending on its interpretation.

- If $\mathbf{F}$ is a force, then $I$ is work.
- If $\mathbf{F}$ is a velocity field, then $I$ is flow, that is, it describes how fast fluid is flowing along the curve. So if the velocity is perpendicular to the curve at every point, then the flow is 0. A negative value for $I$ indicates a flow in the direction opposite to the direction of parametrization of the curve.
- In particular, if $\mathbf{F}$ is a velocity field and if the curve $C$ is closed, then flow is called circulation, a quantity which describes the amount of fluid motion along the closed curve.

These quantities have equivalents in $\mathbb{R}^2$.

If $\mathbf{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ is a vector field and $C$ be a closed curve in $\mathbb{R}^2$. Then the integral

$$ I = \oint_C (\mathbf{F} \times d\mathbf{r}) \cdot \mathbf{k} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds, $$

where $\mathbf{n}$ is a vector perpendicular to the curve, pointing outward, is called flux. The circle through the integral symbol is used to denote integration along a closed curve. Flux describes the net amount of “flow” of a vector field through a closed curve. Note that the vector field does not have to be a velocity field. Thus, we talk about the flux of a magnetic field. Flux has a three dimensional counterpart, which describes the flow through a surface, and will be discussed later in this chapter.

The reason for dotting the result of the cross product $\mathbf{F} \times d\mathbf{r}$, a vector always parallel to $\mathbf{k}$, with $\mathbf{k}$ is to obtain a scalar quantity while preserving its sign. The computation is actually rather simple. The sign of the flux indicates whether the net flow is outward (positive flux) or inward (negative flux).

**Example.** Let $\mathbf{F} = \langle -y, x \rangle$ and $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. So along the curve,

$$ \mathbf{F} = \langle -\sin t, \cos t \rangle. $$

Since

$$ d\mathbf{r} = \frac{d\mathbf{r}}{dt} \, dt = \langle -\sin t, \cos t \rangle \, dt $$

(this is how we find $d\mathbf{r}$ in practice), then

$$ \mathbf{F} \cdot d\mathbf{r} = \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = (\sin^2 t + \cos^2 t) \, dt = \, dt, $$

and

$$ \mathbf{F} \times d\mathbf{r} = \langle -\sin t, \cos t, 0 \rangle \times \langle -\sin t, \cos t, 0 \rangle \, dt = -2(\sin t)(\cos t) \mathbf{k} \, dt. $$

So $\mathbf{F} \times d\mathbf{r} \cdot \mathbf{k} = -2\sin t \cos t \, dt$. Remember that to compute a scalar product, you should compute the cross product before the dot product, or compute the determinant of the components of all three vectors at once. The circulation is thus

$$ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi, $$

2
and the flux is

$$\int_C \mathbf{F} \times d\mathbf{r} \cdot \mathbf{k} = \int_0^{2\pi} -2(\sin t)(\cos t) \, dt = \cos^2 t \bigg|_0^{2\pi} = 0.$$  

This means that along the unit circle, $\mathbf{F}$ flows along the curve counterclockwise (consistent with the parametrization of the curve which traced out the circle counterclockwise). Its net flux is 0, a situation which can occur if all the vectors of $\mathbf{F}$ are tangent to the curve. Indeed, there are no vector pointing inward or outward. To verify our conclusion, we can sketch the vector field. One approach to sketching vector fields in $\mathbb{R}^2$ by hand so is to start by plotting the vectors along the $x$-axis (let $y = 0$ and the $y$-axis. Then do those on the $y = \pm x$ lines. Then fill in at other points.

As a final thought, you may have noticed that the textbook gives a different formula to use to compute the flux of a vector field, namely

$$\text{circulation} = \int_C N \, dx - M \, dy.$$  

While you may find this formula easier to use, you should check for yourself that it is actually identical to the formula presented in these notes. The rational for presenting the formulas this way is to exhibit the similarities and differences between circulation and flux: the former computes the net flow parallel to the curve, while the latter computes the net flow perpendicular (and outward) to the curve. The computations are similar too.
In the previous lesson, we learned to integrate a vector field $\mathbf{F}$ over a curve $C$ given by the parametrization $\mathbf{r}(t)$:

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$  

We would like to know if it is possible to simplify the integration process in a way that is reminiscent of the Fundamental Theorem of Calculus. Recall that if $f$ is an integrable function, and if it has an antiderivative $F$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Notice that the theorem only applies when $f$ has an antiderivative, that is, when there is a function $F$ such that $F'(x) = f(x)$. But not all integrable functions $f$ are guaranteed to have this magical function. For example, $e^{-x^2}$ does not have an antiderivative in closed form.

Likewise, we would like to know if the vector field $\mathbf{F}$ has a corresponding magical function $f$ such that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(b) - f(a),$$

where $a$ and $b$ are the endpoints of the curve we are integrating over. It turns out that under the right conditions, the statement holds and that, just like $F$ was the antiderivative of $f$ in the Fundamental Theorem of Calculus, $f$ in this case is a potential function of $\mathbf{F}$. In other words $\mathbf{F}$ is the gradient field of $f$:

$$\nabla \mathbf{F} = f.$$  

A vector field which has a potential function is commonly called conservative. This property is equivalent to the path-independence of all its integrals in a connected open set $D$, which means that integration over any two paths in $D$ with the same endpoints should result in the same value. Path-independence is in turn equivalent to every integral on a closed loop in $D$, that is every circulation integral, equaling 0. A useful criterion to determine if any of these equivalent conditions hold true is the exactness of the differential form $\mathbf{F} \cdot d\mathbf{r} = M \, dx + N \, dy + P \, dz$:

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. $$
The compact form of this test is

\[ \nabla \times \mathbf{F} = 0, \quad \text{that is,} \quad \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = 0. \]

The following example illustrates the three types of problems you are expected to be able to solve in this section:

- confirming that a vector field is conservative,
- finding a potential function of a conservative vector field, and
- using the Fundamental Theorem of line integrals.

**Example.** Let

\[ \mathbf{F}(x, y, z) = (\ln x + \sec^2(x + y)) \mathbf{i} + \left( \sec^2(x + y) + \frac{y}{y^2 + z^2} \right) \mathbf{j} + \frac{z}{y^2 + z^2} \mathbf{k}. \]

Let \( M, N \) and \( P \) refer to the components of \( \mathbf{F} \). Then

\[
\begin{align*}
\frac{\partial M}{\partial y} &= 2 \sec^2(x + y) \tan(x + y), & \frac{\partial N}{\partial x} &= 2 \sec^2(x + y) \tan(x + y), & \text{so} & \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \\
\frac{\partial M}{\partial z} &= 0, & \frac{\partial P}{\partial z} &= 0, & \text{so} & \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \\
\frac{\partial N}{\partial z} &= -\frac{2yz}{(y^2 + z^2)^2}, & \frac{\partial P}{\partial y} &= -\frac{2yz}{(y^2 + z^2)^2}, & \text{so} & \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.
\end{align*}
\]

Therefore, \( M \, dx + N \, dy + P \, dz \) is exact, which means that \( \mathbf{F} \) is conservative. Note that this criterion only serves to prove that a vector field is conservative, and that failing to meet this condition in no way shows that \( \mathbf{F} \) is non-conservative. Let us find its potential function \( f \). In particular \( \frac{\partial f}{\partial x} = M \), so

\[
f(x, y, z) = \int M \, dx = \int \ln x + \sec^2(x + y) \, dx = \frac{1}{x} + \tan(x + y) + g(y, z). \quad (1)
\]

Taking the partial derivative of the above with respect to \( y \)

\[
\frac{\partial f}{\partial y} = \sec^2(x + y) + \frac{\partial g}{\partial y},
\]

and since \( \frac{\partial f}{\partial y} = N = \sec^2(x + y) + y/(y^2 + z^2) \), then

\[
\frac{\partial g}{\partial y} = \frac{y}{y^2 + z^2}.
\]

Therefore

\[
g(y, z) = \int \frac{y}{y^2 + z^2} \, dy = \frac{1}{2} \ln(y^2 + z^2) + h(z),
\]

2
which, combined with equation (1) implies that
\[ f(x, y, z) = \frac{1}{x} + \tan(x + y) + \frac{1}{2} \ln(y^2 + z^2) + h(z). \] (2)

Finally, taking the practical derivative of the above with respect to \( z \)
\[ \frac{\partial f}{\partial z} = \frac{z}{y^2 + z^2} + \frac{\partial h}{\partial z} \]
and setting it equal to \( \frac{\partial f}{\partial z} = P = z/(y^2 + z^2) \) yields that \( \frac{\partial h}{\partial z} = 0 \), and hence that \( h(z) \) is a constant. From equation (2), this means that the potential functions of \( \mathbf{F} \) have the form
\[ f(x, y, z) = \frac{1}{x} + \tan(x + y) + \frac{1}{2} \ln(y^2 + z^2) + C. \]

Now by the Fundamental Theorem of line integrals, if \( C \) is any curve beginning at \((1, 0, 1)\) and ending at \((1, 2, 3)\), then
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 3) - f(0, 0, 0) = (1 + \tan 3 + \frac{1}{2} \ln 13) - (1 + \tan 1). \]
Green’s Theorem

Jenny Lam

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In the Fundamental Theorem of Calculus, and in the Fundamental Theorem of Line integrals, we compute integrals over intervals or curves by evaluating special functions (antiderivatives and potential functions) at the endpoints. We can think of the endpoints of the curves as the boundary of the curve, or symbolically,

\[ \{a, b\} = \partial[a, b] \]

where \( \partial \) is an informal symbol for “boundary.” It turns out the generalization of the Fundamental Theorem of Calculus (FTC for short) which we are after relates an integral over a certain geometric object to the object’s boundary. In this section, we explore Green’s Theorem, a 2-dimensional version of the FTC which allows us to transform line integrals over a closed curve in the plane into integrals over the area enclosed by the curve:

\[
\oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dy \, dx,
\]

where \( C \) is a simple (non-intersecting) closed curve such that \( C = \partial R \). Note that this theorem is only valid if the path is integrated counterclockwise.

**Example.** Our goal is to illustrate the value of Green’s theorem by evaluating

\[ I = \oint_C y^2 \, dx + x^2 \, dy \]

where \( C \) is the triangle bounded by the \( x \)- and \( y \)-axes and the line \( x + y = 1 \), first by evaluating the line integral directly, as we have learned before, and second by applying Green’s Theorem.

First, we perform a direct integration. We evaluate the integral on each of the three pieces, \( C_1 \), \( C_2 \) and \( C_3 \), which make up the boundary \( C \), so that

\[
I = \int_{C_1} y^2 \, dx + x^2 \, dy + \int_{C_2} y^2 \, dx + x^2 \, dy + \int_{C_3} y^2 \, dx + x^2 \, dy = I_1 + I_2 + I_3
\]

Recall that the process of evaluating a line integral begins with the parametrization of the curve. Making sure that the parametrizations are chosen as to trace out the curve
counterclockwise, we have

\[ C_1 : r_1(t) = ti, \quad 0 \leq t \leq 1, \]
\[ C_2 : r_2(t) = (1 - t)i + tj, \quad 0 \leq t \leq 1, \]
\[ C_3 : r_3(t) = -tj, \quad 0 \leq t \leq 1. \]

On \( C_1 \), the \( y \)-component of \( y \) is 0, so \( y^2 \) and \( dy = 0 \), which imply that \( I_1 = 0 \). Similarly, on \( C_3 \), \( x \) is 0, so \( I_3 = 0 \). On \( C_2 \), we have

\[ I_2 = \int_0^1 t^3(- dt) + (1 - t)^2 dt = -\frac{t^3}{3} - \frac{(1 - t)^3}{3}\bigg|_0^1 = -\frac{1}{3} + \frac{1}{3} = 0. \]

Therefore, \( I = I_1 + I_2 + I_3 = 0 \).

Let us now evaluate \( I \) by Green’s Theorem.

\[ I = \oint_C y^2 \, dx + x^2 \, dy = \int_0^1 \int_0^{1-x} (2x - 2y) \, dy \, dx \\
= \int_0^1 2xy - y^2 \bigg|_0^{1-x} \, dx = \int_0^1 (-3x^2 + 4x - 1) \, dx \\
= -x^3 + 2x^2 - x \bigg|_0^1 = 0. \]

By applying Green’s Theorem, we have reduced the problem of integrating over three separate curves into the problem of integrating over one solid region in the plane.

We now apply Green’s theorem to the problem of evaluating a circulation integral over a plane. Recall that the circulation of a vector field \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} \) over a closed curve \( C \) is given by the integral

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_M dx + N \, dy \]

where \( C = \partial R \)

\[ = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dy \, dx \quad \text{by Green’s Theorem, with } P = M \text{ and } Q = N. \]

Now the integrand that is given by Green’s Theorem

\[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \]

has meaning of its own. It is the component of a vector known as curl in the direction of \( \mathbf{k} \), and describes the amount of whirling around an axis in the \( \mathbf{k} \) direction at a given point. This quantity is also known as the circulation density in the \( k \)-direction. While circulation describes the amount of whirling along a curve, circulation density describes the amount of whirling at a point.
Example. Let \( \mathbf{F} = \langle xy, y^2 \rangle \). By Green’s Theorem the counterclockwise circulation around the closed loop defined below by the line \( y = x \) and above by the curve \( y = x^2 \) is given by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_{x^2}^x (0 - x) \ dy \ dx = \int_0^1 (x^3 - x) \ dx = \frac{x^4}{4} - \frac{x^3}{3} \bigg|_0^1 = -\frac{1}{12}.
\]

Green’s Theorem can also be applied to simplify the problem of finding outward flux. Recall that the flux of a vector field \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} \) across a closed planar curve \( C \) parametrized counterclockwise is given by the integral

\[
\oint_C \mathbf{F} \times d\mathbf{r} \cdot \mathbf{k} = \iint_R \left( \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) \ dy \ dx \quad \text{where } C = \partial R
\]

by Green’s Theorem, with \( P = -N \) and \( Q = M \),

\[
= \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \ dy \ dx \quad \text{the negative sign comes out.}
\]

As expected the integrand that is given by Green’s Theorem

\[
\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}
\]

also has meaning. This scalar quantity, which is known as divergence of \( \mathbf{F} \), describes how much of the vector field flows out at a given point. In other words, if the divergence at a point is positive, then \( \mathbf{F} \) has a spring at that point; and if the divergence is negative, \( \mathbf{F} \) has a sink. While flux describes the amount of outward flow across a curve, divergence or flux density describes the amount of outward flow at a point.

Example. Let us revisit the vector field \( \mathbf{F} = \langle xy, y^2 \rangle \) of the previous example. By Green’s Theorem the counterclockwise flux across the closed loop defined below by the line \( y = x \) and above by the curve \( y = x^2 \) is given by

\[
\int_C \mathbf{F} \times d\mathbf{r} \cdot \mathbf{k} = \int_0^1 \int_{x^2}^x (y + 2y) \ dy \ dx = \int_0^1 \frac{3}{2} y^2 \bigg|_{x^2}^x \ dx
\]

\[
= \int_0^1 \frac{3}{2} (x^2 - x^4) \ dx = \frac{3}{2} \left( x^3 - \frac{x^5}{5} \right) \bigg|_0^1 = \frac{1}{5}.
\]
Surface Integrals

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Surface Integrals are to double integrals as line integrals are to regular one-dimensional integrals. Recall that the most basic line integral, \( \int_C ds \) computes the arc length of a curve. In truth, we did not really know how to compute \( ds \) directly, so we converted it into a one-dimensional integral. In calculus 1, the curve was described by an equation of the form \( y = f(x) \), and so we let \( ds = \sqrt{(f'(x))^2 + 1} \, dx \). Of course, if the curve was given in term of \( y \), say \( g(y) = x \), then we could have let \( ds = \sqrt{(g'(y))^2 + 1} \) instead. Then in calculus 3, we learned to described curves with parametrizations \( r(t) = (x(t), y(t), z(t)) \). So we let \( ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt \).

Similarly, the most basic surface integral is the one which does not have any integrands, and simply computes the area of a surface \( S \). It is written in the form

\[
\iint_S d\sigma,
\]

where \( d\sigma \) is an area element on the actual surface. In this section, we discuss the computation of this integral when the surface is described in the familiar way of using \( x, y \) and \( z \) coordinates for an equation of the form \( g(x, y, z) = 0 \). To turn this integral into a double integral, we have the choice of integrating over one of the standard planes in \( \mathbb{R}^3 \). Let \( \mathbf{p} \) be the standard unit vector perpendicular to the plane of integration. Then

\[
d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA.
\]

This is really three formulas in one, one for each plane over which to integrate. If we decide to integrate over the \(xz\)-plane, then \( \mathbf{p} \) is \( \mathbf{j} \) and \( dA = dx \, dz = dz \, dx \).

Example. (16.5.11) To find the surface area of the portion of the surface given by \( x^2 - 2 \ln x + \sqrt{15} - z = 0 \) above the square \( R = [1, 2] \times [0, 1] \) in the \( xy \)-plane, we let \( g(x, y, z) = x^2 - 2 \ln x + \sqrt{15} - z \). So

\[
\nabla g(x, y, z) = \left(2x - \frac{2}{x}\right) \mathbf{i} + \sqrt{15} \mathbf{j} - \mathbf{k},
\]
and therefore,
\[ |\nabla g| = \sqrt{4(x^2 - 2 + \frac{1}{x^2}) + 15} = 2 \sqrt{x^2 + 2 + \frac{1}{x^2}} = 2 \sqrt{(x + \frac{1}{x})^2} = 2 \left| x + \frac{1}{x} \right| = 2 \left( x + \frac{1}{x} \right), \]
with the last equality holding because we are integrating over \( R \). Also,
\[ |\nabla \cdot k| = 1.\]
Therefore,
\[ \iint_S d\sigma = \int_1^2 \int_0^1 2 \left( x + \frac{1}{x} \right) dy \, dx = 2 \left( \frac{x^2}{2} + \ln x \right) \bigg|_1^2 = 3 + 2 \ln 2. \]

The surface may be given in the form \( z = f(x, y) \), which fits the more general form, say \( g(x, y, z) = 0 \), if we move \( z \) across, so that \( g(x, y, z) = f(x, y) - z \). Note that a treatment of this special case can be found on p 1175 of the textbook, under the section “special formulas.” Of course, it is unnecessary to memorize the formula, because the more general method also applies to this case. It is interesting to see that the formula for \( d\sigma \) becomes very similar to the Calculus 1 version of \( ds \) in the case where the surface is a graph (that is, given by the equation \( z = f(x, y) \)). Notice that the top part of the fraction \( |\nabla f| \) is the magnitude of a vector, so it becomes a square root of elements squared, while the bottom part is really just the absolute value of a scalar, and in our case it is simply 1.

A word of caution. In the previous example, the surface could just as well have been described by the function \( f(x, y) = x^2 - 2 \ln x + \sqrt{15} y \). To use the formula for \( d\sigma \), we do not find the gradient of \( f \). Instead, rewrite the equation as a level surface: \( x^2 - 2 \ln x + \sqrt{15} - z = 0 \). This is a level surface of the function \( g(x, y, z) = x^2 - 2 \ln x + \sqrt{15} - z \) and that is what we find the gradient of.

We know from earlier lessons that line integrals are also used to compute masses, momenta, etc, and we expect the same out of surface integrals. The general form of a surface integral is

\[ \iint_S h(x, y, z) \, d\sigma \]

where \( h \) is a function not to be confused with the one used to describe the surface \( S \).

**Example.** (16.5.17) We aim at evaluating

\[ I = \iint_S g(x, y, z) \, d\sigma \]

where \( g(x, y, z) = x + y + z \) and \( S \) is the portion of the plane given by \( 2x + 2y + z = 2 \) that lies in the first octant.
We begin by computing \( d\sigma \). Let \( f(x, y, z) = 2x + 2y + z \). Then
\[ \nabla f = \langle 2, 2, 1 \rangle, \quad |\nabla f| = 3 \quad \text{and} \quad \nabla f \cdot k = 1. \]
To find the limits of integration in the $xy$-plane, we let $z = 0$ in the equation of the plane $S$, and see that $y = 1 - x$. Therefore the limits are $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$. Finally, we rewrite the integrand in terms of $x$ and $y$ only, using the fact that $z = 2 - 2x - 2y$ on $S$, so $g(x, y, z) = 2 - x - y$. So

$$I = \int_0^1 \int_0^{1-x} (2 - x - y) \, dy \, dx = 2.$$ 

In addition to computing surface integrals with scalar integrands, we can also compute ones with vector fields. The flux across a surface is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{\sigma} \quad \text{where} \quad d\mathbf{\sigma} = \frac{\nabla g}{|\nabla g|} \cdot \mathbf{p} \, dA.$$ 

Notice that now $d\mathbf{\sigma}$ is a vector element, which encodes two pieces of information: its magnitude still represents the area of the surface element $d\sigma$, and in addition, its direction is perpendicular to the plane of that surface element. (This description is still ambiguous because there are two directions that are perpendicular to a plane, and it reflects the fact that there are several ways to represent the surface $S$ with an equation of the form $g(x, y, z) = 0$.)

**Example.** (16.5.29) To find the flux of $\mathbf{F}(x, y, z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ across the portion of the cylindrical surface $S$ given by $y = e^x$, whose projection on the $yz$-plane is the rectangular region $R_{yz} = [1, 2] \times [0, 1]$,

we must evaluate

$$I = \iint_S \mathbf{F} \cdot d\mathbf{\sigma} = \int \int_{R_{yz}} \mathbf{F} \cdot \frac{\nabla g}{|\nabla g|} \cdot \mathbf{i} \, dy \, dz.$$ 

over the $yz$-plane. We let $\mathbf{p} = \mathbf{i}$, and turn the different parts of this integral in terms of the variables $y$ and $z$, replacing all appearances of $e^x$ by $y$.

- With $g(x, y, z) = e^x - y$, we have $\nabla g = e^x \mathbf{i} - \mathbf{j} = y\mathbf{i} - \mathbf{j}$, and
- $\nabla g \cdot \mathbf{i} = y$.
- $\mathbf{F} \cdot \nabla g = -4y$.

This yields

$$I = \int_0^1 \int_1^2 \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{i}|} \, dy \, dx = \int_0^1 \int_1^2 -4 \, dy \, dz = -4.$$ 

3
Surface Integrals in Parametrized Coordinates

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Last time, we saw that, over a surface $S$ given by an equation of the form $g(x, y, z) = 0$, we could compute surface integrals

$$I = \int_S f(y, z, z) \, d\sigma,$$

or

$$J = \int_S F(x, y, z) \cdot d\sigma$$

where $f$ and $F$ are scalar and vector fields respectively by letting

$$d\sigma = \left| \nabla f \right| |\nabla f \cdot p| \, dA$$

or

$$d\sigma = \frac{\nabla f}{|\nabla f \cdot p|} \, dA,$$

where $dA$ is an area element in one of the three standard planes, and $p$ is perpendicular to $dA$. In this lesson, we continue our exploration of surface integrals by considering surfaces which are represented by a parametrization.

A surface $S$ is said to be parametrized by a function $r(u,v) = f(u,v)i + g(u,v)j + h(u,v)k$ if the range of $r$ is the surface $S$. The parametrization can also be written in coordinate form:

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$

In general, the parametrization of a surface must be given using exactly two variables ($u$ and $v$), just like the parametrization of a line was given using exactly one variable.

**Example.** (Parametrization of a plane) Suppose that $\Pi$ is a plane that goes through three non-collinear points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$. Then $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$ and $\overrightarrow{AC} = (c_1 - a_1, c_2 - a_2, c_3 - a_3)$ are non-parallel, though not necessarily perpendicular, vectors which determine the tilt of the plane $\Pi$ by the parametrization

$$r(u,v) = u \overrightarrow{AB} + v \overrightarrow{AC} + (a_1, a_2, a_3)$$

or

$$\begin{cases} x = (b_1 - a_1)u + (c_1 - a_1)v + a_1 \\ y = (b_2 - a_2)u + (c_2 - a_2)v + a_2 \\ z = (b_3 - a_3)u + (c_3 - a_3)v + a_3 \end{cases}$$
In this parametrization, \( u \) and \( v \) determine how far from point \( A \) a general point \( x \) is along the vectors \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \). In this sense, \( u \) and \( v \) can be thought of as new coordinates inside the plane \( \Pi \).

**Example.** (Parametrization of a sphere) A sphere of radius 2 centered at \( C(c_1, c_2, c_3) \) can be parametrized, based on spherical coordinates, by

\[
\begin{align*}
  x &= 2 \sin \phi \cos \theta + c_1 \\
  y &= 2 \sin \phi \sin \theta + c_2 \\
  z &= 2 \cos \phi + c_3
\end{align*}
\]

\( \phi \) and \( \theta \) represent spherical angles, not with respect to the origin, but with respect to the center \( C \) of the sphere.

To integrate over a surface parametrized by \( r(u, v) = (f(u, v), g(u, v), h(u, v)) \), let

\[
d\sigma = |r_u \times r_v|, \quad \text{and} \quad d\sigma = r_u \times r_v,
\]

where \( r_u = \left\langle \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right\rangle \) and \( r_v = \left\langle \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right\rangle \). In other words, \( r_u = \frac{\partial r}{\partial u} \). The vector \( d\sigma \) is always normal to the surface, but it may point to either side of the surface depending on whether we choose to let it equal \( r_u \times r_v \) or \( r_v \times r_u \). (Recall also that in the last section, \( d\sigma \) depended on the ambiguous direction of \( \nabla g \).) Example 3 shows how to choose the correct form for \( d\sigma \).

**Example 1.** (Area of a cone frostrum) A cone frostrum is a section of a cone which does not contain the vertex. Let a cone frostrum \( S \) be given by the equation

\[
z = \frac{1}{3} \sqrt{x^2 + y^2}, \quad 1 \leq z \leq \frac{4}{3}.
\]

Clearly (because of the expression \( x^2 + y^2 \)), we should let \( x = r \cos \theta \) and \( y = r \sin \theta \), so that the equation becomes \( z = r/3 \). Therefore the parametrization of \( S \) is

\[
r(r, \theta) = \left\langle r \cos \theta, r \sin \theta, \frac{r}{3} \right\rangle, \quad \text{where } 3 \leq r \leq 4.
\]

The area of \( S \) is given by the surface integral

\[
\int_S d\sigma = \int_S |r_r \times r_\theta| \, dr \, d\theta.
\]

Since

\[
|r_r \times r_\theta| = \left| \left\langle \cos \theta, \sin \theta, \frac{1}{3} \right\rangle \times \left\langle -r \sin \theta, r \cos \theta, 0 \right\rangle \right| = \left| \left\langle -\frac{1}{3} r \cos \theta, -\frac{1}{3} r \sin \theta, r \right\rangle \right| = \frac{r \sqrt{10}}{3},
\]

the area of \( S \) is

\[
\int_3^4 \int_0^{2\pi} \frac{r \sqrt{10}}{3} \, dr \, d\theta = \frac{7\pi \sqrt{10}}{3}.
\]
Notice that the parameters $r$ and $\theta$, although inspired by cylindrical coordinates, do not represent cylindrical coordinates, since $z$ depends on $r$. They are not polar coordinates either. The surface element is $r\sqrt{10/3} \, dr \, d\theta$ rather than simply $r \, dr \, d\theta$. The factor of $\sqrt{10/3}$ accounts for the fact that $S$ is not a flat surface.

**Example 2.** (Surface integral over a cone) To integrate $F(x, y, z) = z - x$ over the cone $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq 1$, we use cylindrical coordinates, with $z = r$. Therefore the parametrization of the cone is

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= r.
\end{align*}
\]

with,

\[
|r_r \times r_\theta| = |(\cos \theta, \sin \theta, 1) \times (-r \sin \theta, r \cos \theta, 0)| = |(-r \cos \theta, -r \sin \theta, r)| = r\sqrt{2}.
\]

This implies that the surface integral computes as

\[
\iint_S F(x, y, z) \, d\sigma = \int_0^{2\pi} \int_0^1 (r - r \cos \theta) r \sqrt{2} \, dr \, d\theta = \frac{2\pi \sqrt{2}}{3}.
\]

**Example 3.** (Outward flux through a cone) The outward flux of the vector field $F = y^2i + xzj - k$ over the cone $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq 1$ is given by

\[
I = \iiint_S F \cdot d\sigma.
\]

Now comes the time to decide whether $d\sigma = r_r \times r_\theta \, dr \, d\theta$ or $d\sigma = r_\theta \times r_r \, dr \, d\theta$. As in the previous example,

\[
r_r \times r_\theta = \langle -r \cos \theta, -r \sin \theta, r \rangle,
\]

which, according to the $z$-component, is an upwards pointing vector. On a cone, this is also an inwards pointing vector. To compute the outward flux, we need its opposite

\[
r_\theta \times r_r = \langle r \cos \theta, r \sin \theta, -r \rangle.
\]

Therefore

\[
F \cdot d\sigma = \left[ (r^2 \sin^2 \theta)i + (r^2 \cos \theta)j - k \right] \cdot \left[ r \cos \theta i + r \sin \theta j - r k \right] \, dr \, d\theta
\]

\[
= r^3 \sin^2 \theta \cos \theta + r^3 \cos \theta \sin \theta + r \, dr \, d\theta,
\]

and the outward flux of $F$ across the cone is

\[
I = \int_0^{2\pi} \int_0^1 (r^3 \sin^2 \theta \cos \theta + r^3 \cos \theta \sin \theta + r) \, dr \, d\theta = \pi.
\]
Example 4. (Center of mass and moment of inertia) We will now use surface integrals to find the center of mass and moment of inertia with respect to the $z$-axis of the cone frustum $S$ given by $x^2 + y^2 - z^2 = 0$ for $1 \leq z \leq 2$, with a constant density $\delta$. Since the cone is symmetric with respect to the $z$-axis, the center of mass is located on the $z$-axis. We also expect its height to be between 1 and 2, and independent of $\delta$, because it is constant. We parametrize the cone as in the previous two examples, with $d\sigma = r \sqrt{2} \, dr \, d\theta$. So the mass of the cone is

$$M = \iint_S dm = \iint_S \delta \, d\sigma = \int_0^{2\pi} \int_1^2 \delta r \sqrt{2} \, dr \, d\theta = \frac{14\pi\delta \sqrt{2}}{3},$$

and its first moment with respect to the $xy$-plane is

$$M_{xy} = \iint_S z \, dm = \iint_S r \delta (r \sqrt{2}) \, dr \, d\theta = \frac{14\pi\delta \sqrt{2}}{3},$$

so that the $z$-component of the center of mass is

$$\bar{z} = \frac{M_{xy}}{M} = \frac{14}{9} \approx 1.55.$$

The moment of inertia with respect to the $z$-axis is found in a similar fashion, namely,

$$I_z = \iint_S (x^2 + y^2) \, dm = \iint_S r^2 \delta (r \sqrt{2}) \, dr \, d\theta = \frac{15\pi\delta \sqrt{2}}{2},$$

and the radius of gyration is

$$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{15}{2}}.$$

Example 5. (Tangent plane at a point) To find the tangent plane at $P(\sqrt{2}, \sqrt{2}, 2)$ to the cone $z = r$, we need a normal vector at that point. In the last lesson, since the cone is parametrized by $x = r \cos \theta, y = r \sin \theta, z = r$, we can, as in Example 3, get a normal vector by computing

$$\mathbf{r}_\theta \times \mathbf{r}_r (r, \theta) = \langle r \cos \theta, r \sin \theta, -r \rangle.$$

Of course, we can also use $\mathbf{r}_r \times \mathbf{r}_\theta$ as the normal vector. Using the fact that the parametrization comes from cylindrical coordinates and the fact that $x = y$ at $P$, we conclude that $\theta = \pi/4$ and that $r = \sqrt{x^2 + y^2} = 2$. Therefore,

$$\mathbf{n} = (\mathbf{r}_\theta \times \mathbf{r}_r) (2, \pi/4) = \left\langle \sqrt{2}, \sqrt{2}, -2 \right\rangle.$$

Now, recall that the equation of a plane is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, so the equation of the plane at $P$ is

$$\left\langle \sqrt{2}, \sqrt{2}, -2 \right\rangle \cdot \left[ (x, y, z) - \left\langle \sqrt{2}, \sqrt{2}, 2 \right\rangle \right] = 0,$$

or

$$\sqrt{2}x + \sqrt{2}y - 2z = 0.$$
Stoke’s Theorem

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Recall that by Green’s Theorem, the circulation of a field $\mathbf{F}$ (computable by a line integral) on a closed curve $\mathcal{C}$ is also given by a double integral over the region $R$ enclosed by $\mathcal{C}$:

$$\oint_{\mathcal{C}} M \, dx + N \, dy = \iint_{R} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$$

We also mentioned that, the integrand of the double integral $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ can be found by computing the $z$-component of the mnemonic:

$$\text{curl} \, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}.$$}

This mnemonic is called the curl of the vector field. Using the language of curl, Green’s Theorem states that the circulation of a vector field is equal to the double integral of the $z$-component of the curl.

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA.$$

Since the curl of a vector field is itself another vector field, the flux of the curl may be computed. In fact, Green’s Theorem directly generalizes to a surface integral, by computing all three components of the curl:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{\sigma},$$

which expands out, in rectangular coordinates, to

$$\oint_{\mathcal{C}} M \, dx + N \, dy + P \, dz = \iint_{S} \left[ \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) i - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) j + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) k \right] \cdot d\mathbf{\sigma}.$$

This generalization, called Stoke’s Theorem, states that the circulation of a vector field $\mathbf{F}$ along the boundary $\mathcal{C} = \partial S$ of a surface $S$ is equal to the flux of the curl of $\mathbf{F}$ across the surface $S$. While Green’s Theorem is concerned with a flat region in the plane, Stoke’s Theorem applies to a general surface in space. But both allow to translate between a one-variable integral and a double-integral.
Example 1. (16.7.3) To compute the circulation of the vector field \( \mathbf{F} = yi + xzj + x^2k \) counterclockwise along the boundary \( \partial T \) of the triangle \( T \) cut out by the plane \( x + y + z = 1 \) by the first octant, we could evaluate the line integral
\[
\oint_{\partial T} y \, dx + xz \, dy + x^2 \, dz = \int_{s_1} y \, dx + xy \, dy + x^2 \, dz + \int_{s_2} y \, dx + xy \, dy + x^2 \, dz + \int_{s_3} y \, dx + xy \, dy + x^2 \, dz
\]
which would require that we parametrize three different curves, one for each side of the triangle. So instead, we apply Stoke’s Theorem, which states that
\[
I = \int_{\partial T} y \, dx + xz \, dy + x^2 \, dz = \iint_{T} \nabla \times \mathbf{F} \cdot d\mathbf{\sigma}.
\]
As usual, the first step in the evaluation of the surface integral is to parametrize it. Let the triangular surface be parametrized by \( \mathbf{r}(x, y) = xi + yj + (1 - x - y)k \) or
\[
\begin{align*}
x &= x \\
y &= y \\
z &= 1 - x - y
\end{align*}
\]
Then
\[
\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x \mathbf{i} - 2x \mathbf{j} + (z - 1) \mathbf{k} = -x \mathbf{i} - 2x \mathbf{j} - (x + y) \mathbf{k},
\]
and
\[
d\mathbf{\sigma} = (\mathbf{r}_x \times \mathbf{r}_y) \, dy \, dx = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \, dy \, dx = (i + j + k) \, dy \, dx,
\]
so that
\[
\nabla \times \mathbf{F} \cdot d\mathbf{\sigma} = (-4x - y) \, dy \, dx,
\]
and therefore, integrating over the projection of \( T \) onto the \( xy \)-plane by setting \( z = 0 \) in the equation of the plane, the domain of integration is bounded by the line \( x + y = 1 \), and the \( x \)- and \( y \)-axes, we conclude that
\[
I = \int_0^1 \int_0^{1-x} (-4 - y) \, dy \, dx = \frac{1}{2}.
\]

Example 2. (16.7.7) The present example illustrates how Stoke’s Theorem can be used in the opposite direction. We will compute the flux of the curl of
\[
\mathbf{F} = \left< y, x^2, (x^2 + y^4)^{3/2} \sin e^{\sqrt{x^2y^2}} \right>
\]
over the portion of the ellipsoid $S$ given by $4x^2 + 9y^2 + 36z^2 = 36$ above the $xy$-plane by rewriting it as a line integral. By letting $z = 0$, we find that the boundary of this surface is the ellipse $C$ given by $4x^2 + 9y^2 = 36$, which can be parametrized as

$$
\begin{cases}
x = 3 \cos t, \\
y = 2 \sin t,
\end{cases} \quad 0 \leq t \leq 2\pi.
$$

This implies that on $C$,

$$
\mathbf{F} \cdot d\mathbf{r} = \left\langle 2 \sin t, 9 \cos^2 t, \left(9 \cos^2 t + 16 \sin^4 t\right)^{3/2} \sin 1 \right\rangle \cdot \left\langle -3 \sin t, 2 \cos t, 0 \right\rangle dt \\
= \left( -6 \sin^2 t + 18 \cos^3 t \right) dt
$$

Therefore, by Stoke’s Theorem, the flux of the curl of $\mathbf{F}$ is

$$
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left( -6 \sin^2 t + 18 \cos^3 t \right) dt = -6\pi.
$$

**Example 3.** (16.7.17) Finally, we take a look at an example in which Stoke’s Theorem is used twice, once in each direction. The goal is to find the flux of the curl of

$$
\mathbf{F} = \langle 3y, 5 - 2x, z^2 - 2 \rangle
$$

over the surface $S$ parametrized by

$$
r(\phi, \theta) = \left\langle \sqrt{3} \sin \phi \cos \theta, \sqrt{3} \sin \phi \sin \theta, \sqrt{3} \cos \phi \right\rangle, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.
$$

By Stoke’s Theorem, the flux of the curl of $\mathbf{F}$ on $S$ is equal to its circulation on the boundary of $S$, which is a circle $C$ of radius $\sqrt{3}$ in the $xy$-plane. In turn, since the boundary $C$ is planar, by Green’s Theorem (the 2-dimensional version of Stoke’s Theorem), the circulation is equal to the flux through the disk $D$ of radius $\sqrt{3}$ in the $xy$-plane:

$$
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \int_0^{\sqrt{3}} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \\
= \int_0^{2\pi} \int_0^{\sqrt{3}} (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \cdot r \ dr \ d\theta = 2\pi \left( -5 \right) \frac{3}{2} = -15\pi.
$$

Note that the eventual double integral is evaluated in polar coordinates. As illustrated by this example, a consequence of Stoke’s Theorem, is the flux of the curl of a vector field through two different surfaces (with the same orientation) that share the same boundary are equal. (If the surfaces have opposite orientations, then the flux of the curl will be equal and opposite.)

As a final remark, please note the difference between the gradient of a scalar field $f$, which is denoted by $\text{grad } f = \nabla f$ and the curl of a vector field $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$. 

3
Divergence Theorem

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We come to the last generalization of the Fundamental Theorem of Calculus, which is concerned with computing the outward flux of a 3-dimensional vector field across a closed surface that is the boundary of a region in space: if \( \mathbf{F} \) is a vector field in \( \mathbb{R}^3 \), then the outward flux across the boundary of a region \( R \) is given by

\[
\text{flux} = \iint_{\partial R} \mathbf{F} \cdot d\sigma = \iiint_R \nabla \cdot \mathbf{F} \, dV
\]

where \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) and

\[
\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M, N, P \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.
\]

Note that the expression for \( \nabla \) is only this simple in rectangular coordinates. The second example in these notes illustrates how to handle an integrand that is best integrated in a different coordinate system.

**Example** (16.8.5). By the Divergence Theorem, the outward flux of the vector field

\[
\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}
\]

across the boundary of the cube \( C \) bounded by the planes \( x = \pm 1, y = \pm 1 \) and \( z = \pm 1 \) is given by

\[
I = \iint_{\partial D} \mathbf{F} \cdot d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV.
\]

Now \( \nabla \cdot \mathbf{F} = -1 - 1 + 0 = -2 \), so by pulling the constant out of the triple integral,

\[
I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} -2 \, dV = -2 \cdot 2^3 = -16.
\]

**Example** (16.8.13). To calculate the outward flux of the vector field

\[
\mathbf{F} = \sqrt{x^2 + y^2 + z^2} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})
\]

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across the boundary of the region $R$ given by $1 \leq x^2 + y^2 + z^2 \leq 2$, we will apply the Divergence Theorem. It is tempting to convert all quantities to spherical coordinates immediately, but we should not do so until we find the divergence. This is because

$$\nabla \neq \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \right).$$

Instead, letting $M, N$ and $P$ be the $x, y,$ and $z$-components of $F$, take we take the partials with respect to the rectangular coordinates and then convert to spherical coordinates, yielding

$$\frac{\partial M}{\partial x} = \frac{x^2}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{x^2 + y^2 + z^2} = \rho \sin^2 \phi \cos^2 \theta + \rho,$$
$$\frac{\partial N}{\partial y} = \frac{y^2}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{x^2 + y^2 + z^2} = \rho \sin^2 \phi \sin^2 \theta + \rho,$$
$$\frac{\partial P}{\partial z} = \frac{z^2}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{x^2 + y^2 + z^2} = \rho \cos^2 \phi + \rho.$$

Therefore, summing these up,

$$\nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \left( \rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta \rho \cos^2 \phi \right) + 3 \rho = 4 \rho,$$

we find that the outward flux is given by

$$\text{flux} = \int \int_{\partial R} F \cdot d\sigma \quad \text{by the definition of flux}$$
$$= \int \int \int_{R} \nabla \cdot F \ dV \quad \text{by the Divergence Theorem}$$
$$= \int_{0}^{2\pi} \int_{0}^{\pi} \int_{1}^{\sqrt{2}} 4 \rho \cdot \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta$$
$$= 2 \pi \cdot 2 \cdot \left( \left( \sqrt{2} \right)^4 - 1 \right) = 12 \pi.$$

For those who are curious, $\nabla$ called the “del” operator can also be expressed in other coordinates systems. For example, in spherical coordinates,

$$\nabla \cdot F = \frac{1}{\rho \sin \phi} \left( \frac{\partial}{\partial \phi} (F_\theta \sin \phi) - \frac{\partial F_\phi}{\partial \theta} \right) e_\rho + \frac{1}{\rho} \left( \frac{1}{\sin \phi} \frac{\partial F_\phi}{\partial \theta} - \frac{\partial}{\partial \rho} (\rho F_\phi) \right) e_\theta + \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho F_\phi) - \frac{\partial F_\rho}{\partial \phi} \right) e_\phi$$

where $e_\rho, e_\phi$ and $e_\theta$ are the unit vectors in the $\rho, \phi$ and $\theta$ directions, and $F_\rho, F_\phi$ and $F_\theta$ are the components of $F$ in those directions.