Physics 1A, Fall 2003
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23 The Vector Cross Product

23.1 Definition

We want to define the vectors $\mathbf{L}$ and $\boldsymbol{\tau}$ more carefully. $\mathbf{L}$ is a vector whose magnitude is $|\mathbf{r}||\mathbf{F}||\sin\theta|$, where θ is the angle between $\mathbf{r}$ and $\mathbf{p}$. Similarly $\boldsymbol{\tau}$ is a vector whose magnitude is $|\mathbf{r}||\mathbf{F}||\sin\theta|$, where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$.[1]

The *direction* of $\mathbf{L}$ is perpendicular to the plane in which $\mathbf{r}$ and $\mathbf{F}$ lie, and in the sense given by the right-hand rule. The *direction* of $\boldsymbol{\tau}$ is perpendicular to the plane in which $\mathbf{r}$ and $\mathbf{F}$ lie, and in the sense given by the right-hand rule.

Notice that it makes sense to speak of the angular momentum of a single particle, or the torque on it, even if the particle is not part of a solid. In fact, logically, that is where we should have started. The total angular momentum of a rotating solid object is the sum of the angular momenta of each little piece that makes it up.

23.2 Cross Product

The **cross product** is a new mathematical operation. Define $\mathbf{A}\times\mathbf{B}$ as the vector perpendicular to the plane in which $\mathbf{A}$ and $\mathbf{B}$ lie, and with magnitude $|\mathbf{A}||\mathbf{B}|\sin\theta$. Geometrically, this is the area of the parallelogram described by $\mathbf{A}$ and $\mathbf{B}$, and the vector perpendicular to that parallelogram.

The rule is satisfied as follows:

$$(\mathbf{A}\times\mathbf{B})_x = A_yB_z - A_zB_y$$
$$(\mathbf{A}\times\mathbf{B})_y = A_zB_x - A_xB_z$$
$$(\mathbf{A}\times\mathbf{B})_z = A_xB_y - A_yB_x$$

Why is this right? First, notice that

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$$

and

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{B}) = 0$$

since all the terms cancel in pairs. So $\mathbf{A}\times\mathbf{B}$ is indeed perpendicular to $\mathbf{A}$ and $\mathbf{B}$.

[1] Read Giancoli Section 11-2.

23.2.1 Some properties of the cross product

What is its magnitude? For simplicity, take $A_z = B_z = 0$. Then $\mathbf{A} \times \mathbf{B}$ has only a z-component. Work it out:[2]

$$\begin{aligned} |\mathbf{A} \times \mathbf{B}|^2 &= (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = \left[(\mathbf{A} \times \mathbf{B})_z \right]^2 = (A_x B_y - A_y B_x)^2 \\ &= \left(A_x^2 + A_y^2\right)\left(B_x^2 + B_y^2\right) - (A_x B_x + A_y B_y)^2 = |\mathbf{A}|^2 |\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2 \\ &= |\mathbf{A}|^2 |\mathbf{B}|^2 \left(1 - \cos^2\theta\right) = |\mathbf{A}|^2 |\mathbf{B}|^2 \sin^2\theta \end{aligned}$$

So, take the square root:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}||\sin\theta|$$

as desired.

All the usual distributive rules hold, e.g.

$$(\alpha\mathbf{A} + \beta\mathbf{B}) \times \mathbf{C} = \alpha\mathbf{A} \times \mathbf{C} + \beta\mathbf{B} \times \mathbf{C}$$

but the vector product is not commutative! $\mathbf{A} \times \mathbf{B}$ has the same magnitude as $\mathbf{B} \times \mathbf{A}$, but it points in the opposite direction:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

In particular, for any vector, $\mathbf{A} \times \mathbf{A} = 0$.

Nor is the rule associative. In general

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

Because of the distributive rules, it is enough to know the cross products of the unit vectors along the axes:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

and so forth. As an illustration of the failure of the associative rule, notice that

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = 0$$

whereas

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{j} \times \mathbf{k} = -\mathbf{i}$$

23.3 Angular Momentum and Torque as Vector Products

The angular momentum of an object whose coordinate is $\mathbf{r}$ and whose momentum is $\mathbf{p} = m\mathbf{v}$ is defined to be[3]

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$$

while the torque on it is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = m\mathbf{r} \times \mathbf{a}$$

Many properties follow simply from these definitions:

[2] You have to provide some of the intermediate steps.

[3] Read Giancoli Sections 11-2,11-3, 11-4.

1. The torque is the rate of change of angular momentum:
$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$
2. On a composite object, like a solid, the total torque is the rate of change of the total angular momentum.
3. The angular momentum about the center of mass satisfies a theorem similar to the one satisfied by the kinetic energy: Let $\mathbf{R}$ and $\mathbf{P}$ be the position and momentum of the center of mass, and let $\mathbf{r}'_i$ and $\mathbf{v}'_i$ be the positions and velocities of the constituents in the center of mass frame, that is, in the coordinate system in which the center of mass is at rest. Then
$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_i m_i \mathbf{r}'_i \times \mathbf{v}'_i$$
4. There is a similar rule for the torque:
$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} + \sum_i \mathbf{r}'_i \times \mathbf{F}_i$$
5. The two terms above satisfy the rule relating torque to angular momentum independently, in the sense that
$$\frac{d}{dt}\mathbf{R} \times \mathbf{P} = \mathbf{R} \times \mathbf{F}$$
6. Finally, an important theorem about how gravity acts on extended objects. The torque due to gravity is
$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F}$$
where $\mathbf{F}$ is the force of gravity on a mass m as if it we located at the center of mass.

   This last rule works because the mass m that appears in $\mathbf{F} = m\mathbf{a}$ is the same as the mass m that appears in $\mathbf{F} = -mg\mathbf{k}$. It's not a general rule, rather it holds when only gravity is the external force. (It won't work for electric forces.) That's why the center of mass is also called the center of gravity!

23.3.1 Proofs of the Above

Here are the details of the demonstrations of the above six propositions:

1. Since for any vector, $\mathbf{v} \times \mathbf{v} = 0$,
$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}.$$
2. The total angular momentum is $\mathbf{L} = \sum_i \mathbf{L}_i$, so
$$\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \frac{d\mathbf{L}_i}{dt} = \frac{d\mathbf{L}}{dt}$$
Therefore if the total torque on a system is zero, the total angular momentum is a constant.

3. Write

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i$$

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i$$

where $\mathbf{R}$ is the center of mass and M is the total mass. $\mathbf{r}'_i$ is the coordinate in the CM frame. Then

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i m_i \left(\mathbf{R} + \mathbf{r}'_i\right) \times \mathbf{v}_i = \mathbf{R} \times \sum m_i \mathbf{v}_i + \sum m_i \mathbf{r}'_i \times \mathbf{v}_i$$

The first term is $\mathbf{r} \times \mathbf{P}$. In the second, write $\mathbf{v}_i = \mathbf{V} + \mathbf{v}'_i$, where $\mathbf{V} = d\mathbf{R}/dt$ is the velocity of the center of mass, so $\mathbf{v}'_i$ is the velocity of a particle in the CM frame. Then

$$\sum_i m_i \mathbf{r}'_i \times \mathbf{v}_i = \sum m_i \mathbf{r}'_i \times \mathbf{V} + \sum_i m_i \mathbf{r}'_i \times \mathbf{v}'_i$$

But

$$\sum_i m_i \mathbf{r}'_i = \sum_i m_i \mathbf{r}_i - \sum_i m_i \mathbf{R} = 0$$

and so

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_i m_i \mathbf{r}'_i \times \mathbf{v}'_i$$

4. Write

$$\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \mathbf{R} \times \sum_i \mathbf{F}_i + \sum_i \mathbf{r}'_i \times \mathbf{F}_i = \mathbf{R} \times \mathbf{F} + \sum_i \mathbf{r}'_i \times \mathbf{F}_i$$

5.

$$\frac{d}{dt}\mathbf{R} \times \mathbf{P} = \mathbf{V} \times M\mathbf{V} + \mathbf{R} \times \frac{d}{dt}\mathbf{P} = \mathbf{R} \times \mathbf{F}$$

because of the earlier theorem about how the CM moves.

6. The torque due to gravity is

$$\begin{aligned}\boldsymbol{\tau} &= \sum_i \boldsymbol{\tau}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = -\sum_i \mathbf{r}_i \times m_i g \mathbf{k} \\ &= -g \sum_i m_i r_i \times \mathbf{k} = -gK\mathbf{R} \times \mathbf{k} = \mathbf{R} \times \mathbf{F}\end{aligned}$$