The Hilbert Matrix and its Determinant

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20040928

Abstract

The Hilbert matrix is $A_n \doteq (\frac{1}{i+j+1})$ for $0 \leq i, j \leq n-1$. We show how to evaluate $\det(A_n)$.

The Hilbert Matrix defined above is the matrix $\langle x^i, x^j \rangle$ for $0 \leq i, j \leq n-1$ where the inner product is given by

$$\langle f, g \rangle \doteq \int_0^1 f(t)g(t)dt.$$

Some notation. Of course $n! = \prod_i^n i$. Define $n!! = \prod_1^n i!$.

Theorem 1 *Suppose*

$$A_n = (\frac{1}{i+j+1})_{0 \leq i,j \leq n-1}.$$

Then

$$\det(A) = \frac{(n-1)!!^4}{(2n-1)!!}.$$

Proof: The proof is by induction on n. When $n = 1$ we have

$$A = \begin{pmatrix} \frac{1}{1} \end{pmatrix}$$

and then $\det(A) = \frac{1}{1}$ which is the desired form for $n = 1$.

Suppose the result is true for $n-1$ and consider $\det(A_n)$.

$$\mathbf{A_n} = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{(n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix} \tag{1}$$

Subtract the last (n^{th}) row from each row above it. The entry in the j^{th} column of the i^{th} row is

$$\frac{1}{i+j-1} - \frac{1}{n+j-1} = \frac{n-i}{(i+j-1)(n+j-1)}.$$

The new matrix is then

$$B = \begin{pmatrix} \frac{n-1}{(1)(n)} & \frac{n-1}{(2)(n+1)} & \frac{n-1}{(3)(n+2)} & \cdots & \frac{n-1}{(n)(2n-1)} \\ \frac{n-2}{(2)(n)} & \frac{n-2}{(3)(n+1)} & \frac{n-2}{(4)(n+2)} & \cdots & \frac{n-2}{(n+1)(2n-1)} \\ \frac{n-3}{(3)(n)} & \frac{n-3}{(4)(n+1)} & \frac{n-3}{(5)(n+2)} & \cdots & \frac{n-3}{(n+3)(2n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n-(n-1)}{(n-1)(n)} & \frac{n-(n-1)}{(n)(n+1)} & \frac{n-(n-1)}{(n+1)(n+2)} & \cdots & \frac{n-(n-1)}{(2n-2)(2n-1)} \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

We know that $\det(A_n) = \det(B)$. To compute $\det(B)$ we see that we can factor the term $n-i$ from the t^{th} row for $1 \leq i < n$ and $\frac{1}{n+j-1}$ from the j^{th} column for $1 \leq j \leq n$. Hence we get that

$$\det(A_n) = \frac{(n-1)!^2}{(2n-1)!} \det \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Now subtract the last column from the preceeding columns. The resulting matrix is

$$B = \begin{pmatrix} \frac{n-1}{(1)(n)} & \frac{n-2}{(2)(n)} & \frac{n-3}{(3)(n)} & \cdots & \frac{1}{n} \\ \frac{n-1}{(2)(n+1)} & \frac{n-2}{(3)(n+1)} & \frac{n-3}{(4)(n+1)} & \cdots & \frac{1}{n+1} \\ \frac{n-1}{(3)(n+2)} & \frac{n-2}{(4)(n+2)} & \frac{n-3}{(5)(n+2)} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n-1}{(n-1)(2n-2)} & \frac{n-2}{(n)(2n-2)} & \frac{n-3}{(n+1)(2n-2)} & \cdots & \frac{1}{2n-2} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

As before we can now factor $n-j$ from the j^{th} column and $\frac{1}{n+i+1}$ from the i^{th} row for $1 \leq i, j \leq n-1$ getting

$$\det(A_n) = \frac{(n-1)!^4}{(2n-1)!(2n-2)!} \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-1} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & 1 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-3} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Hence

$$\det(A_n) = \frac{(n-1)!^4}{(2n-1)!(2n-2)!} \det(A_{n-1}).$$

The formula in the statement follows.

So, for example

$$
\begin{aligned}
\det A_1 &= \frac{1}{1} \\
\det A_2 &= \frac{1}{12} \\
\det A_3 &= \frac{1}{2160} \\
\det A_4 &= \frac{1}{6048000} \\
\det A_5 &= \frac{1}{266716800000} \\
\det A_6 &= \frac{1}{186313420339200000} \\
\det A_7 &= \frac{1}{2067909047925770649600000} \\
\det A_8 &= \frac{1}{365356847125734485878112256000000} \\
\det A_9 &= \frac{1}{1028781784378569697887052962909388800000000} \\
\det A_{10} &= \frac{1}{46206893947914691316295628839036278726983680000000000} \\
\det A_{11} &= \frac{1}{3312250489706341375536214362704072710608012767246942208000000000000}
\end{aligned}
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