Taking Trig to Task

Scott Hendrickson

Teaching Professor
Brigham Young University

NCTM April 2017

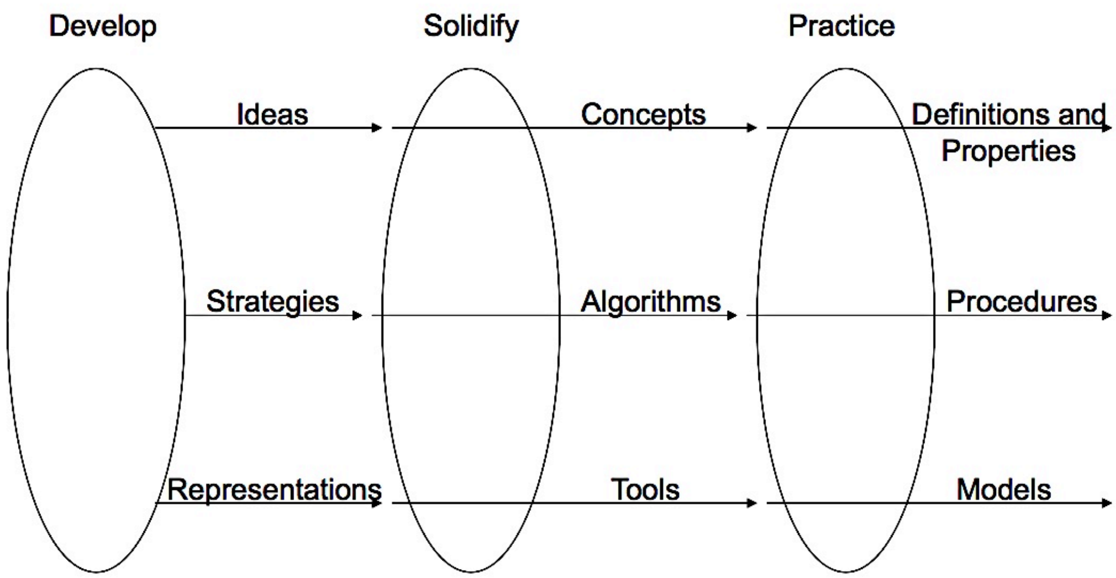
The Comprehensive Mathematics Instruction (CMI) Framework

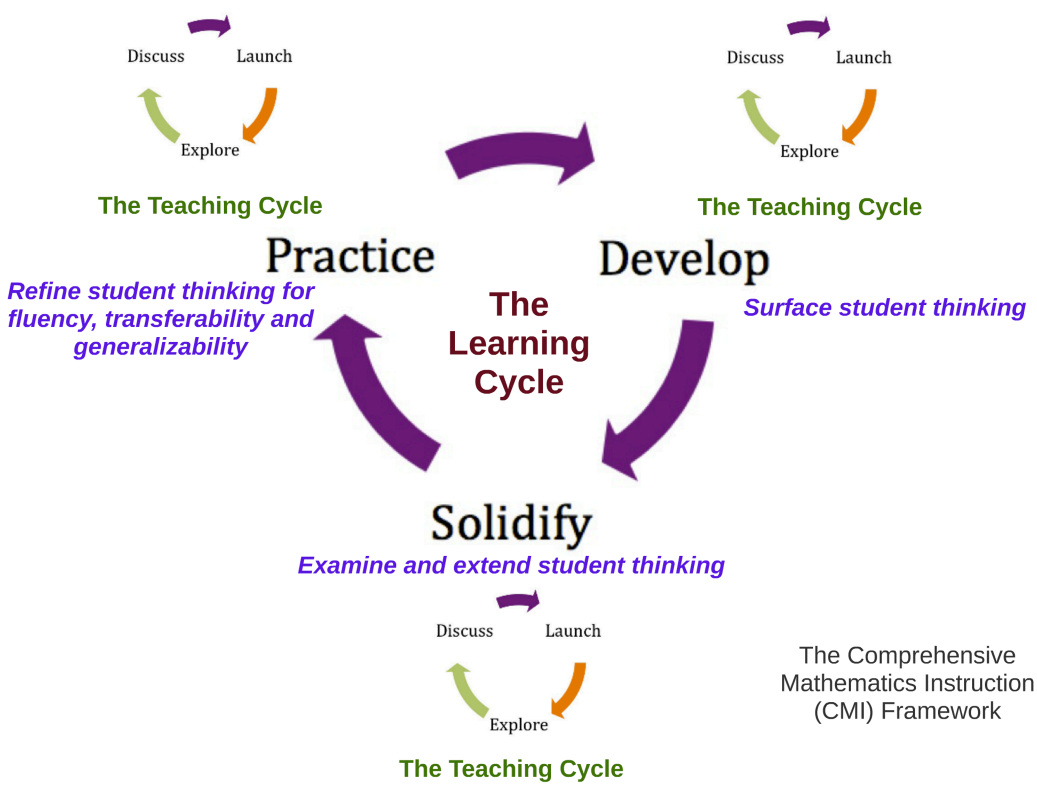
developed by the

Brigham Young University Public School Partnership

Mathematics Vision Project
mathematicsvisionproject.org

Continuum of Mathematical Understanding





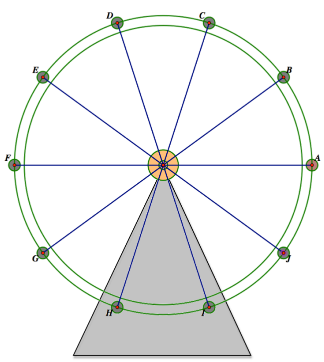
The Comprehensive Mathematics Instruction (CMI) Framework

6.1 George W. Ferris' Day Off

A Develop Understanding Task

Perhaps you have enjoyed riding on a Ferris wheel at an amusement park. The Ferris wheel was invented by George Washington Ferris for the 1893 Chicago World's Fair.

Carlos, Clarita and their friends are celebrating the end of the school year at a local amusement park. Carlos has always been afraid of heights, and now his friends have talked him into taking a ride on the amusement park Ferris wheel. As Carlos waits nervously in line he has been able to gather some information about the wheel. By asking the ride operator, he found out that this wheel has a radius of 25 feet, and its center is 30 feet above the ground. With this information, Carlos is trying to figure out how high he will be at different positions on the wheel.



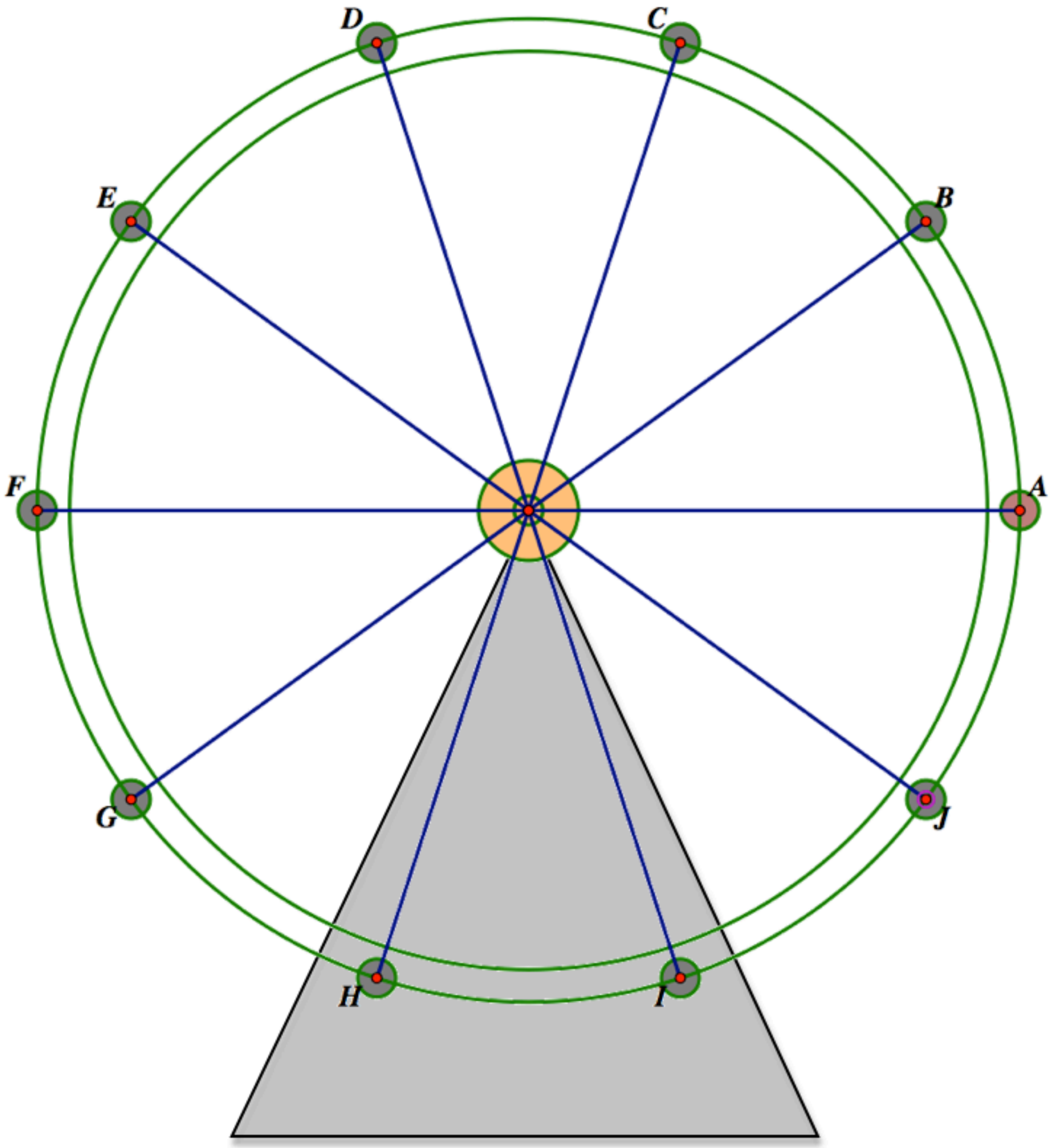
1. How high will Carlos be when he is at the top of the wheel? (To make things easier, think of his location as simply a point on the circumference of the wheel's circular path.)

2. How high will he be when he is at the bottom of the wheel?

3. How high will he be when he is at the positions farthest to the left or the right on the wheel?

Because the wheel has ten spokes, Carlos wonders if he can determine the height of the positions at the ends of each of the spokes as shown in the diagram. Carlos has just finished studying right triangle trigonometry, and wonders if that knowledge can help him.

4. Find the height of each of the points labeled A-J on the Ferris wheel diagram on the following page. Represent your work on the diagram so it is apparent to others how you have calculated the height at each point.

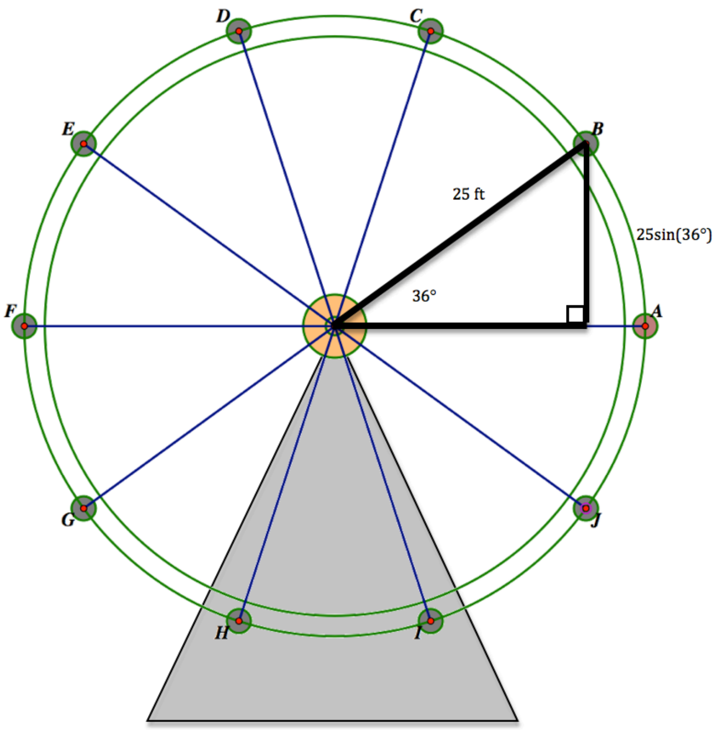


D
C
E
B
F
A
G
J
H
I

6.2 "Sine" Language

A Solidify Understanding Task

In the previous task, *George W. Ferris' Day Off*, you probably found Carlos' height at different positions on the Ferris wheel using right triangles, as illustrated in the following diagram.



Recall the following facts from the previous task:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground

Carlos has also been carefully timing the rotation of the wheel and has observed the following additional fact.

- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

1. How high will Carlos be 2 seconds after passing position A on the diagram?

2. Calculate the height of a rider at each of the following times *t*, where *t* represents the number of seconds since the rider passed position A on the diagram. Keep track of any regularities you notice in the ways you calculate the height. As you calculate each height, plot the position on the diagram.

Elapsed time since passing position A	Calculations	Height of the rider
1 sec		
2 sec		
2.5 sec		
3 sec		
4 sec		
8 sec		
14 sec		
18 sec		
23 sec		
28 sec		
36 sec		
37 sec		
40 sec		

5. Examine your calculations for finding the height of the rider during the first 5 seconds after passing position A (the first few values in the above table). During this time, the angle of rotation of the rider is somewhere between 0° and 90°. Write a general formula for finding the height of the rider during this time interval.

6. How might you find the height of the rider in other "quadrants" of the Ferris wheel, when the angle of rotation is greater than 90°?

6.3 More "Sine" Language
A Solidify Understanding Task

Clarita is helping Carlos calculate his height at different locations around a Ferris wheel. They have noticed that when they use their formula $h(t) = 30 + 25\sin(\theta)$ their calculator gives them correct answers for the height even when the angle of rotation is greater than 90°. They don't understand why since right triangle trigonometry only defines the sine for acute angles.

Carlos and Clarita are making notes of what they have observed about this new way of defining the sine that seems to be programmed into the calculator.

Carlos: "For some angles the calculator gives me positive values for the sine of the angle, and for some angles it gives me negative values."

1. Without using your calculator, list at least five angles of rotation for which the value of the sine produced by the calculator should be positive.

2. Without using your calculator, list at least five angles of rotation for which the value of the sine produced by the calculator should be negative.

Clarita: "Yeah, and sometimes we can't even draw a triangle at certain positions on the Ferris wheel, but the calculator still gives us values for the sine at those angles of rotation."

3. List possible angles of rotation that Clarita is talking about—positions for which you can't draw a reference triangle. Then, without using your calculator, give the value of the sine that the calculator should provide at those positions.

Carlos: "And, because of the symmetry of the circle, some angles of rotation should have the same values for the sine."

4. Without using your calculator, list at least five pairs of angles that should have the same sine value.

Clarita: "Right! And if we go around the circle more than once, the calculator still gives us values for the sine of the angle of rotation, and multiple angles have the same value of the sine."

5. Without using your calculator, list at least five sets of multiple angles of rotation where the calculator should produce the same value of the sine.

Carlos: "So how big can the angle of rotation be and still have a sine value?"
Clarita: "Or how small?"

6. How would you answer Carlos and Clarita's questions?

Carlos: "And while we are asking questions, I'm wondering how big or how small the value of the sine can be as the angles of rotation get larger and larger?"

7. Without using a calculator, what would your answer be to Carlos' question?

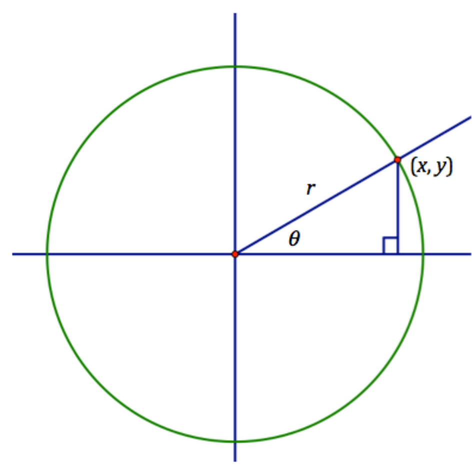
Clarita: "Well, whatever the calculator is doing, at least it's consistent with our right triangle definition of sine as the ratio of the length of the side opposite to the length of the hypotenuse for angles of rotation between 0 and 90°."

Part 2

Carlos and Clarita decide to ask their math teacher how mathematicians have defined sine for angles of rotation, since the ratio definition no longer holds when the angle isn't part of a right triangle. Here is a summary of that discussion.

We begin with a circle of radius *r* whose center is located at the origin on a rectangular coordinate grid. We represent **an angle of rotation in standard position** by placing its vertex at the origin, the *initial ray* oriented along the positive *x*-axis, and its *terminal ray* rotated θ degrees counterclockwise around the origin when θ is positive and clockwise when θ is negative. Let the ordered pair (x, y) represent the point when the terminal ray intersects the circle. (See the diagram below, which Clarita diligently copied into her notebook.)

In this diagram, angle θ is between 0 and 90°; therefore, the terminal ray is in quadrant I. A right triangle has been drawn in quadrant I similar to the right triangles we have drawn in the Ferris wheel tasks.



8. Based on this diagram and the right triangle definition of the sine ratio, find an expression for $\sin\theta$ in terms of the variables *x*, *y* and *r.*

$\sin\theta =$

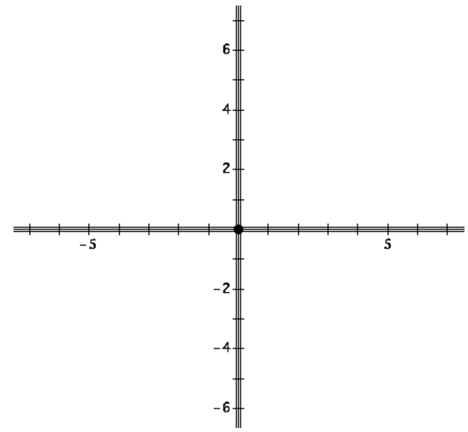
We will use this definition for <u>any</u> angle of rotation. Let's try it out for a specific point on a particular circle.

9. Consider the point (-3, 4), which is on the circle $x^2 + y^2 = 25$.

   a. What is the radius of this circle?

   b. Draw the circle and the angle of rotation, showing the initial and terminal ray.

   c. For the angle of rotation you just drew, what would the value of the sine be if we use the definition we wrote for sine in question 8?



   d. What is the measure of the angle of rotation? How did you determine the size of the angle of rotation?

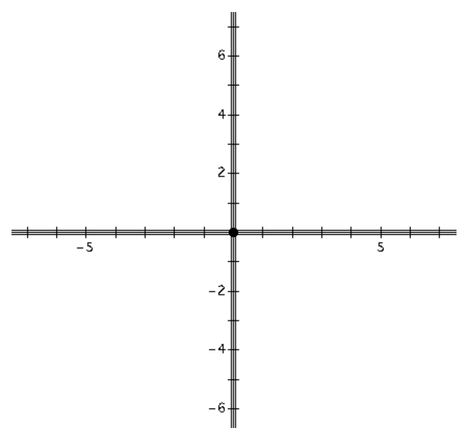
   e. Is the calculated value based on this definition the same as the value given by the calculator for this angle of rotation?

10. Consider the point (-1, -3), which is on the circle $x^2 + y^2 = 10$.

   a. What is the radius of this circle?

   b. Draw the circle and the angle of rotation, showing the initial and terminal ray.

   c. For the angle of rotation you just drew, what would the value of the sine be if we use the definition we wrote for sine in question 8?



   d. What is the measure of the angle of rotation? How did you determine the size of the angle of rotation?

   e. Is the calculated value based on this definition the same as the value given by the calculator for this angle of rotation?

6.4 More Ferris Wheels

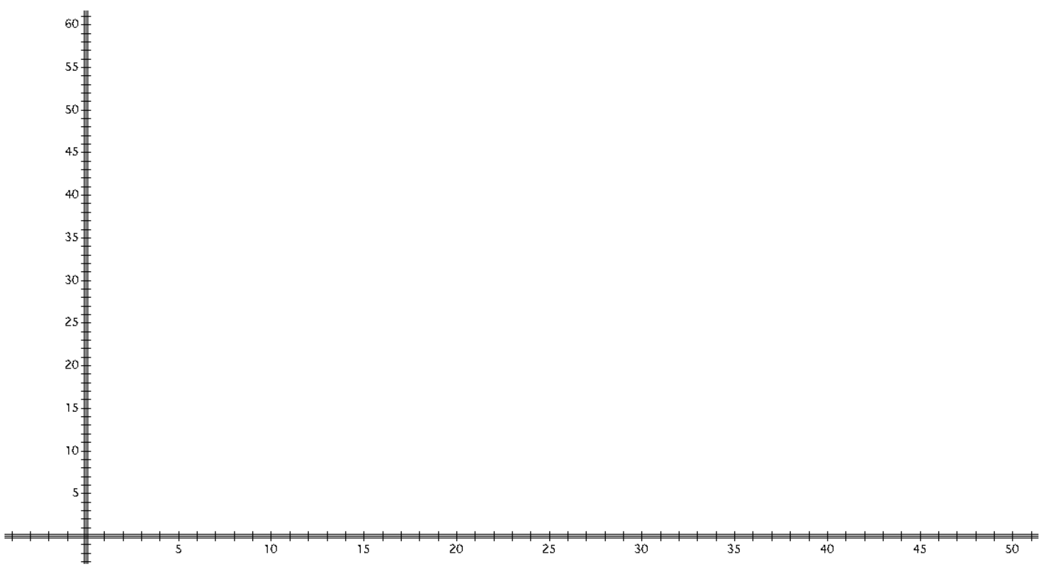
A Solidify Understanding Task

In a previous task, *"Sine" Language*, you calculated the height of a rider on a Ferris wheel at different times t, where t represented the elapsed time after the rider passed the position farthest to the right on the Ferris wheel.

Recall the following facts for the Ferris wheel in the previous tasks:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

1. Based on the data you calculated, as well as any additional insights you might have about riding on Ferris wheels, sketch a graph of the height of a rider on this Ferris wheel as a function of the time elapsed since the rider passed the position farthest to the right on the Ferris wheel. (We can consider this position as the rider's starting position at time t = 0.)

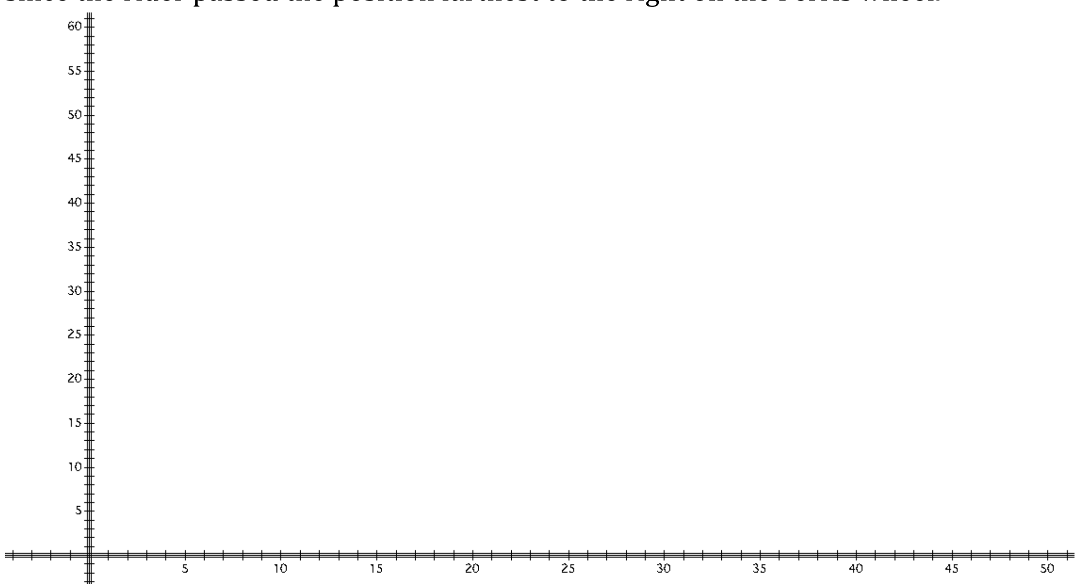


2. Write the equation of the graph you sketched in question 1.

3. Of course, Ferris wheels do not all have this same radius, center height, or time of rotation. Describe a different Ferris wheel by changing some of the facts listed above. For example, you can change the radius of the wheel, or the height of the center, or the amount of time it takes to complete one rotation. You can even change the direction of rotation from counterclockwise to clockwise. If you want, you can change more that one fact. Just make sure your description seems reasonable for the motion of a Ferris wheel.

   Description of my Ferris wheel:

4. Sketch a graph of the height of a rider on your Ferris wheel as a function of the time elapsed since the rider passed the position farthest to the right on the Ferris wheel.



5. Write the equation of the graph you sketched in question 4.

6. We began this task by considering the graph of the height of a rider on a Ferris wheel with a radius of 25 feet and center 30 feet off the ground, which makes one revolution counterclockwise every 20 seconds. How would your graph change if:

- the radius of the wheel was larger or smaller?
- the height of the center of the wheel was greater or smaller?
- the wheel rotates faster or slower?

7. How does the equation of the rider's height change if:

- the radius of the wheel is larger or smaller?
- the height of the center of the wheel is greater or smaller?
- the wheel rotates faster or slower?

8. Write the equation of the height of a rider on each of the following Ferris wheels *t* seconds after the rider passes the farthest right position.

   a. The radius of the wheel is 30 feet, the center of the wheel is 45 feet above the ground, and the angular speed of the wheel is 15 degrees per second counterclockwise.

   b. The radius of the wheel is 50 feet, the center of the wheel is at ground level (you spend half of your time below ground), and the wheel makes one revolution *clockwise* every 15 seconds.

6.5 Moving Shadows
A Practice Understanding Task

In spite of his nervousness, Carlos enjoys his first ride on the amusement park Ferris wheel. He does, however, spend much of his time with his eyes fixed on the ground below him. After a while, he becomes fascinated with the fact that since the sun is directly overhead, his shadow moves back and forth across the ground beneath him as he rides around on the Ferris wheel.

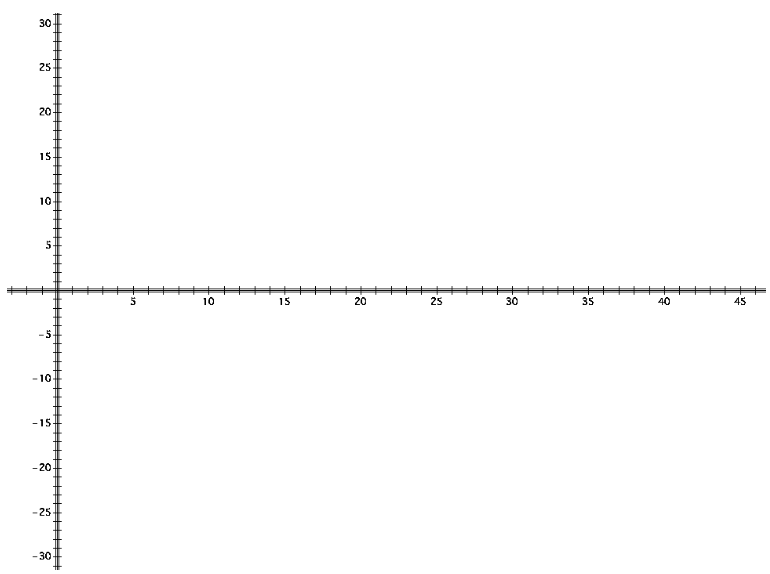
Recall the following facts for the Ferris wheel Carlos is riding:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

To describe the location of Carlos' shadow as it moves back and forth on the ground beneath him, we could measure the shadow's horizontal distance (in feet) to the right or left of the point directly beneath the center of the Ferris wheel, with locations to the right of the center having positive value and locations to the left of the center having negative values. For instance, in this system Carlos' shadow's location will have a value of 25 when he is at the position farthest to the right on the Ferris wheel, and a value of -25 when he is at a position farthest to the left.

1. What would Carlos' position be on the Ferris wheel when his shadow is located at 0 in this new measurement system?

2. Sketch a graph of the horizontal location of Carlos' shadow as a function of time *t*, where *t* represents the elapsed time after Carlos passes position A, the farthest right position on the Ferris wheel.



3. Calculate the location of Carlos' shadow at the times *t* given in the following table, where *t* represents the number of seconds since Carlos passed the position farthest to the right on the Ferris wheel. Keep track of any regularities you notice in the ways you calculate the location of the shadow. As you calculate each location, plot Carlos' position on the diagram.

Elapsed time since passing farthest right position	**Calculations**	**Location of the shadow**
1 sec		
2 sec		
2.5 sec		
3 sec		
4 sec		
5 sec		
8 sec		
9 sec		
10 sec		
14 sec		
15 sec		
18 sec		
20 sec		
23 sec		
28 sec		
36 sec		
37 sec		
40 sec		

5. Write a general formula for finding the location of the shadow at any instant in time.