# Hands-on Geometry for Deeper Understanding For All Learners in Grades 3-5

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CHARTERED 1693

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# The van Hiele Levels

**Level 0: Pre-recognition.** Geometric figures are not recognized. For example, students cannot differentiate between three-sided and four-sided polygons.

**Level 1: Visualization.** Geometric figures are recognized as entities, without any awareness of parts of figures or relationships between components of the figure. A student should recognize & name figures and distinguish a given figure from others that look somewhat the same.

**Level 2: Analysis.** Properties are perceived, but are isolated and unrelated. A student should recognize and name properties of geometric figures.

**Level 3: Abstraction.** Definitions are meaningful, with relationships being perceived between properties and between figures. Logical implications and class inclusions are understood, but the role and significance of deduction is not understood.

**Level 4: Deduction.** The student can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. A student should be able to supply reasons for steps in a proof.

**Level 5: Rigor.** The standards of rigor and abstraction represented by modern geometries characterize level 4. Symbols without referents can be manipulated according to the laws of formal logic. A student should understand the role and necessity of indirect proof and proof by contrapositive.

## Additional Points

1. The learner can not achieve one level without passing through the previous levels.

2. Progress from one level to another is more dependent on educational experience than on age or maturation.

3. Certain types of experiences can facilitate or impede progress within a level or to a higher level.

# GEOMETRY: QUADRILATERALS

You will explore quadrilaterals and their properties through the use of various manipulatives such as sorting pieces and geo-strips. The sequence of activities is designed to facilitate an increase in a learner's van Hiele level of thinking about quadrilaterals from Level 1 to Level 3. First, you will learn how to determine the van Hiele levels of your own students by analyzing how they sort a set of quadrilateral pieces. Then you play the game "What's My Rule?" which develops the ability to classify quadrilaterals by various attributes and to focus on more than one attribute at a time. You will also construct parallelograms, rectangles, rhombi, and squares using geo-strips and make observations while the figures are flexed (Level 2). Finally, you will identify relationships between parallelograms, rectangles, rhombi, squares, trapezoids, kites, and darts through a lab that culminates in the creation of a quadrilateral family tree (Level 3). While these activities are presented with quadrilaterals, most of them are easily adapted to triangles and other polygons.

## Activity 1: Quadrilateral Sort

1) Divide yourselves into groups of two or three. Use a cutout set of quadrilaterals you made for homework for your group. Lay out the pieces with the letters up. Don't call them quadrilaterals when you lay them out for your own students. These objects can be grouped together in many different ways. For example, if we sorted the shapes that make up the American flag (the red stripes, the white stripes, the blue field, the white stars), we might sort by color and put the white stripes and the stars together because they are white, the red strips in another group because they are red, and the blue field by itself because it is the only blue object. Another way the flag parts could be grouped would be all the strips and the blue field together because they are all rectangles and all the stars together because they are not rectangles. Sort the shapes into groups that belong together, recording the letters of the pieces you put together and the criteria you used to sort. Sort two or three times, recording each sort.
2) Compare your ways with those of other groups.
3) How do you think your students would sort these figures?

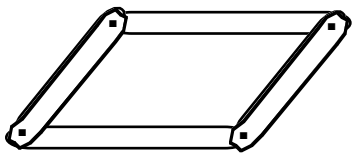
## Activity 2: What's My Rule?

1) Divide yourselves into groups of three or four. Use a cutout set of quadrilaterals you made for homework for your group. Lay out the pieces with the letters up.
2) One participant in each group is the sorter. The sorter writes down a "secret rule" to classify the set of quadrilaterals into two or more piles and uses that rule to slowly sort the pieces as the other players observe.
3) At any time, the players can call "stop" and guess the rule. After the correct rule identification, the player who figured out the rule becomes the sorter. The correct identification is worth 5 points. A correct answer, but not the written one, is worth 1 point. Each incorrect guess results in a 2-point penalty. The winner is the first one to accumulate 10 points.

# Activity 3: Quadrilateral Properties Laboratory

1) Divide yourselves into groups of three or four. Experiment with the geo-strips or ang-legs as Dr. Mason asks questions.

2) Pick two pairs of congruent segments and connect them as shown below. Have them flex the figure to different positions.



What stays the same? What changes? What do you notice about the opposite sides of this quadrilateral?

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A **parallelogram** is a quadrilateral with opposite sides parallel.

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What is the sum of the interior angles of this quadrilateral?

What do you notice about the opposite angles?

What if you turn the strips so that they cross, forming two triangles? Is this figure is a polygon. What are the essential elements of a polygon?

3) Make one of the angles a right angle (You can use the square corner of a piece of paper or a file card to check your accuracy.)

- What happens to the other angles?
- Will this always be true when you make one angle of a parallelogram a right angle? How do you know?
- Is it still a parallelogram?
- Is it still a quadrilateral?
- Is it still a polygon?
- What other name, besides polygon, quadrilateral, and parallelogram, can be given to it now?

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A **rectangle** is a parallelogram with four right angles.

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4) Make a parallelogram that has all four sides equal in length. What is another name for this parallelogram?

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A **rhombus** is a parallelogram with four congruent sides.

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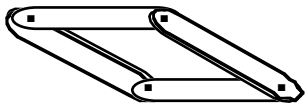
5) Flex the figure to different positions.



- What stays the same? What changes?
- What is the sum of the interior angles of this quadrilateral?
- What do you notice about the opposite angles?
- Is it still a quadrilateral? Is it still a polygon?

6) Make one of the angles of this rhombus a right angle, checking with your square corner. What happens to the other angles? Is it still a parallelogram? What other name, besides polygon, quadrilateral, parallelogram, and rhombus, can be given to this new figure?

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A **square** is a parallelogram with four congruent sides and four right angles.

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Is it a rectangle? How do you know?

7) Discuss the definitions for quadrilateral, parallelogram, rectangle, rhombus, and square given above. Discuss the examples of each given on the quadrilateral definitions sheet, noticing their orientations and how each example fits the definition even though they aren't necessarily the stereotype figure usually seen. What are the implications of the fact that a Level 1 student recognizes shapes by comparing them to a known shape has for teaching?

# Activity 4: Quadrilateral Sorting Laboratory

1) Divide yourselves into groups of three or four. Spread out your Quadrilateral Set with the letters facing up so you can see them.

2) Find all of the quadrilaterals having 4 right angles. List them by letter alphabetically in the corresponding row of the Table.

3) Consider all of the quadrilaterals again. Find all of the quadrilaterals having exactly one pair of parallel sides. List them by letter alphabetically in the corresponding row of the Table.

4) Continue in this manner until the Table is complete.

5) Which category is the largest? What name can be used to describe this category?

6) Which lists which are the same? What name can be used to describe quadrilaterals with these properties?

7) Are there any lists that are proper subsets of another list? If so, which ones?

8) Are there any lists that aren't subsets of one another that have some but not all members in common? If so, which ones?

9) Which lists have no members in common?

10) Label each of the categories in the Table with the most specific name possible using the labels kite, quadrilateral, parallelogram, rectangle, rhombus, square, and trapezoid. For example, #1 - a quadrilateral that has 4 right angles is a rectangle. (Having 4 right angles isn't enough to make it a square; it would need 4 congruent sides as well.)

11) Compare your results to that of the other Lab Groups. Then fill out the family tree by inserting the names kites, rectangles, squares, and trapezoids into the appropriate places on the diagram.

**Quadrilateral Table**

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1. has 4 right angles

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2. has exactly one pair of parallel sides

---

3. has two pair of opposite sides congruent

---

4. has 4 congruent sides

---

5. has two pair of opposite sides parallel

---

6. has no sides congruent

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7. is convex with two pair of adjacent sides congruent, but not all sides congruent

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8. has perpendicular diagonals

---

9. has opposite angles congruent

---

10. is concave

---

11. is convex

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12. its diagonals bisect one another

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13. has four sides

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14. has four congruent angles

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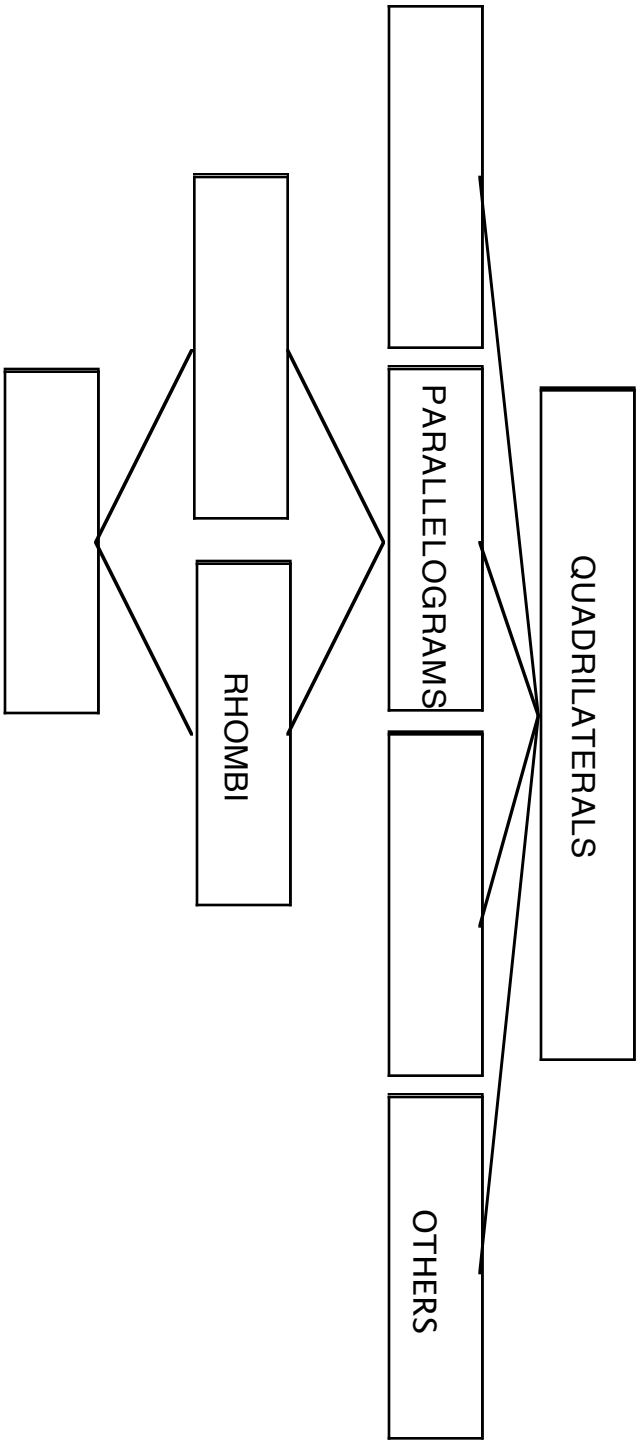
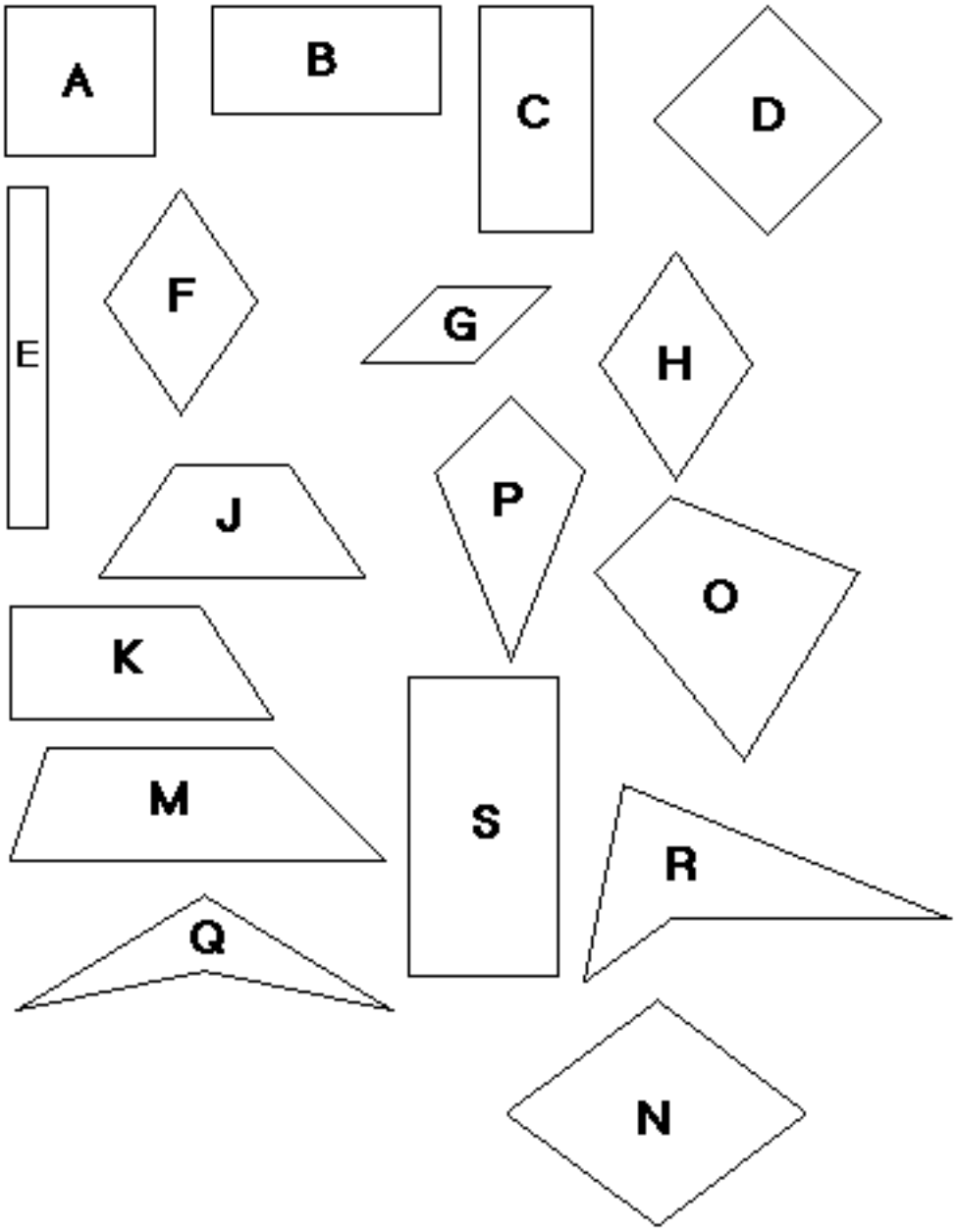
15. has four congruent sides and four congruent angles

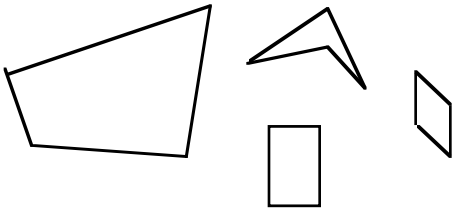

QUADRILATERALS
PARALLELOGRAMS
RHOMBI
OTHERS

# Quadrilateral Sorting Pieces

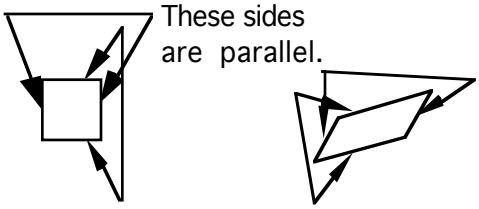


# TYPES OF QUADRILATERALS

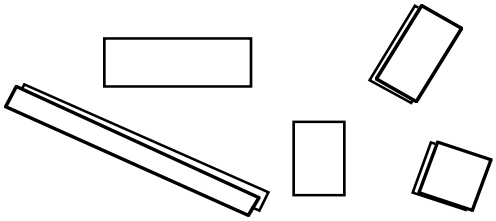
A **quadrilateral** is a **four** sided polygon.



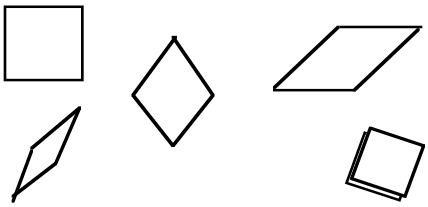
A **parallelogram** is a quadrilateral with **opposite** sides **parallel** .



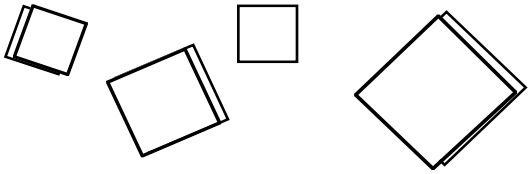

A **rectangle** is a quadrilateral with **all right angles.**



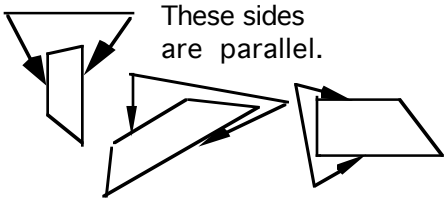
A **rhombus** is a quadrilateral with **all sides congruent.**



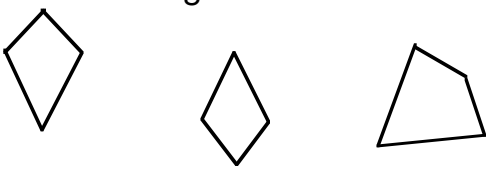
A **square** is a quadrilateral with 4 **right angles** and 4 **congruent sides**.



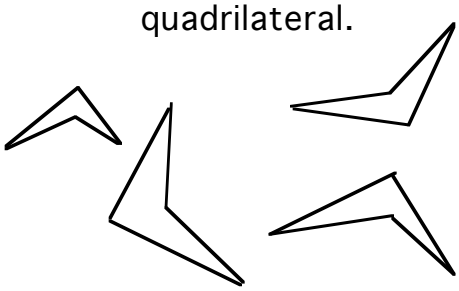
A **trapezoid** is a quadrilateral with **exactly one** pair of parallel sides.




A **kite** is a quadrilateral with 2 pair of **consecutive congruent sides** but not all congruent sides.

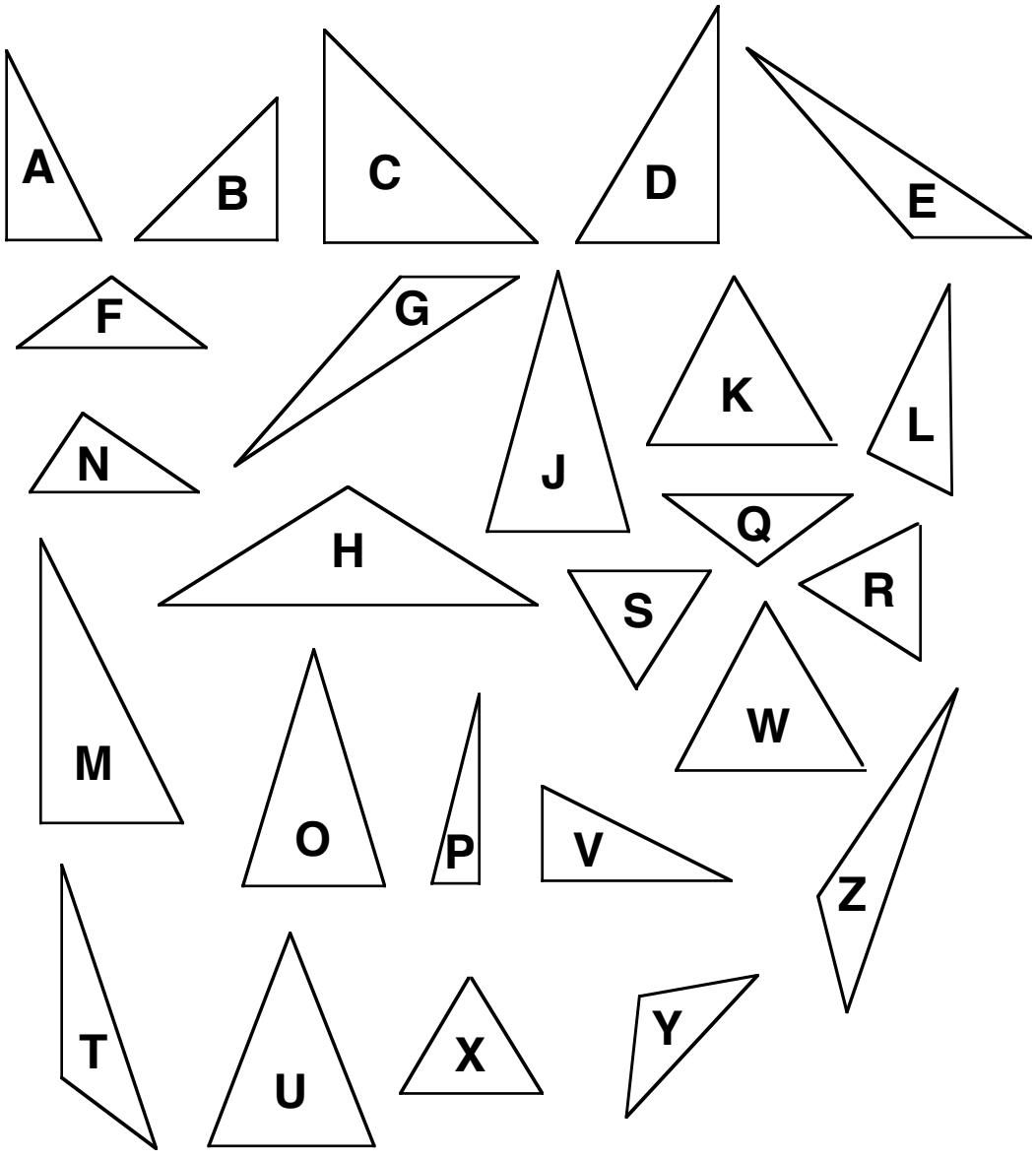


A **dart** is a **concave** quadrilateral.



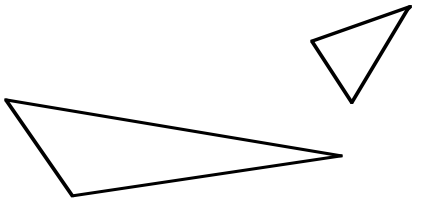
Check the definitions for Trapezoid and Kite used in your state.

# Triangle Sorting Pieces

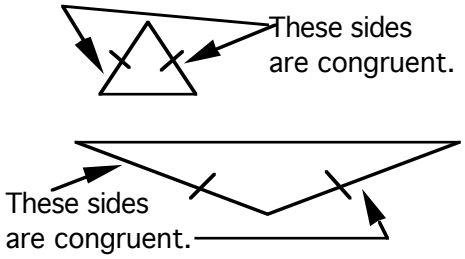


# TYPES OF TRIANGLES

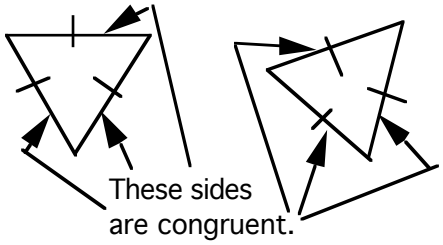
A **scalene** triangle has **no** congruent sides.



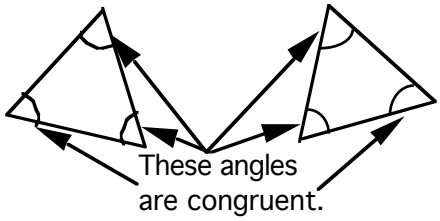
An **isosceles** triangle has **two or more** sides congruent.



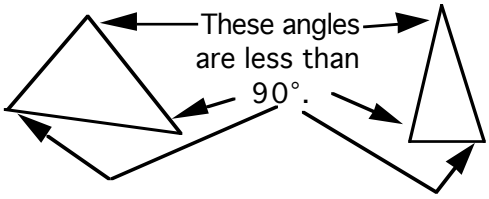

A **equilateral** triangle has all three sides congruent.



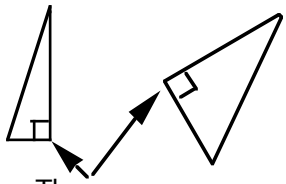

A **equiangular** triangle has all three angles congruent.




A **acute** triangle has all three angles acute.

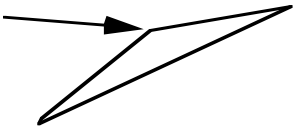



A **right** triangle contains a right angle.



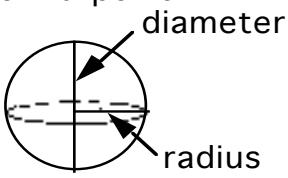

An **obtuse** triangle contains an obtuse angle.

This angle is greater than 90° and less than 180°.

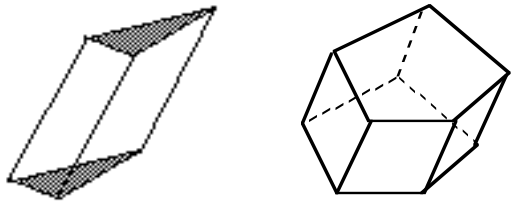


# TYPES OF SOLIDS

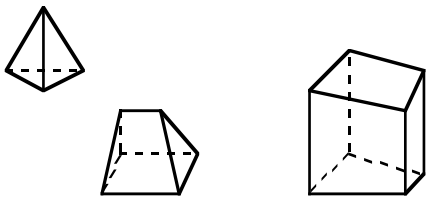
A **sphere** is the set of points in space that are equidistant from a point.



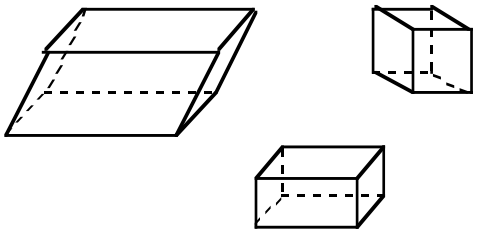

A **prism** is a polyhedron formed by 2 congruent parallel polygons & the connecting parallelograms.



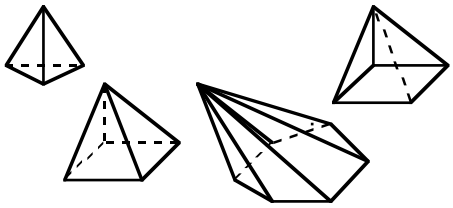
A **polyhedron** is a closed surface made up of polygons.



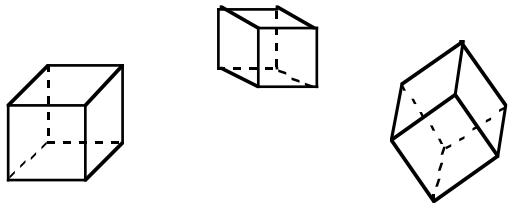
A **rectangular prism** is a prism whose bases are rectangles.



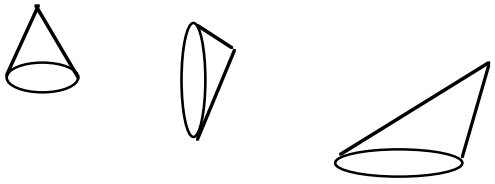
A **pyramid** is a polyhedron with a polygonal base & triangular sides meeting in a point.



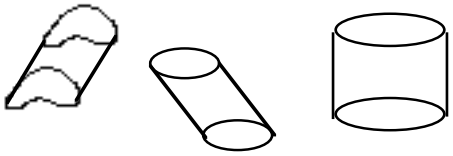
A **cube** is a prism whose bases and sides are **squares**.



A **cone** is a closed surface formed by a curve (the base) and all the line segments joining it to a point.



A **cylinder** is a closed surface formed by two parallel closed curves and the line segments connecting them.



# Activity 5: 13 Knots in a Rope

Make 13 equally spaced knots on a rope.

- Connect knot #1 to knot #13.
- Make a triangle using knots 1, 4, and 8 as the vertices.
- Pull tightly at the vertices. What kind of triangle does this make?
- The Egyptians were called "rope stretchers." How might the Egyptians have used such a rope?

- What is a polygon?
- Make a polygon with your rope.
- What is the definition of a polygon?
- Translate, rotate, reflect the polygon.
- Is the polygon still a polygon? How do you know?
- Are the properties still the same?

Make a quadrilateral with opposite sides two units in length and four units in length.

- Make angle A into a right angle. What happens to the other angles?
- Make a parallelogram with four sides of the same length. What is another name for this shape?
- Make one corner of the rhombus a right angle. What happens to the other angles?
- What is a diagonal? How many are there for each polygon? Can you show them?
- Make a concave quadrilateral. What does "concave" mean?
- Make a quadrilateral whose sides are 1-1-5-5 units.
- Make another whose sides are 2-2-4-4.
- These are examples of a kite. How would you define "kite?"
- What appears to be true about the diagonals of a kite?
- Can you make a trapezoid with your rope? What properties of a trapezoid can you name?

There are many other geometry concepts that can be investigated with a rope such as these. . .

- Triangle types (equilateral, isosceles, scalene, obtuse, acute)
- Make all the hexagons you can
- Pythagorean triples
- The Triangle Inequality Theorem

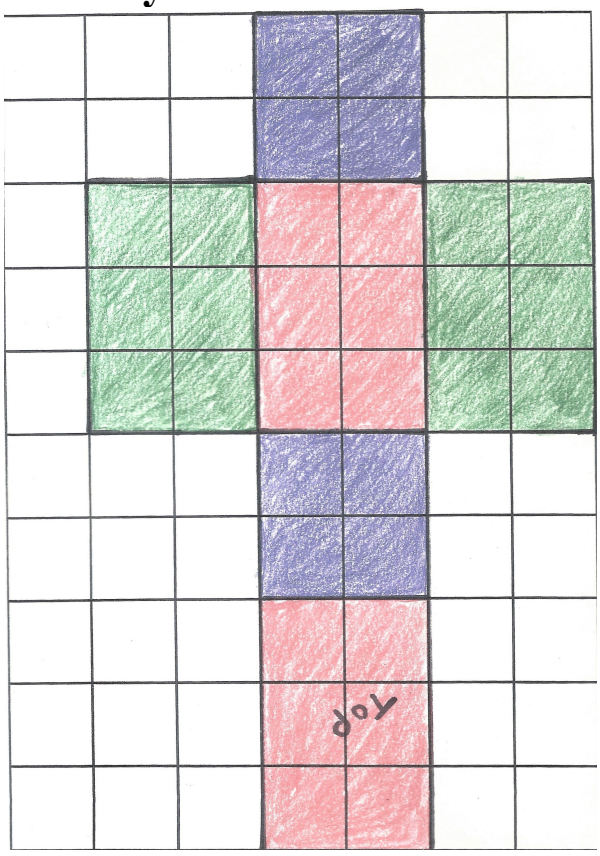
# Activity 6: The perimeter is 24. What's the area?

Working in groups of two or three, cut off a strip of adding machine paper that is at least 24 inches long. Glue it together so that it forms a paper collar that is about 24.1 inches around. Count out 38 one-inch cubes. Using as many of these cubes as needed, construct **rectangular** arrays that fill the paper collar you just made. Record the dimensions of each array and record its area and perimeter. Note any patterns you find.

# The area is 24. What's the perimeter?

Working in groups of two or three, count out 24 one-inch cubes. Form these 24 cubes into a **rectangular** array, using all 24 cubes. Record the dimensions of each array and record its area and perimeter. Note any patterns you find.

# Activity 7: Volume and Surface Area




Construct several copies of this prism. The first one can be filled with centimeter cubes if needed to calculate the volume. Then put two together to double the height and fill in the chart. Stack a third one to triple the height and record.

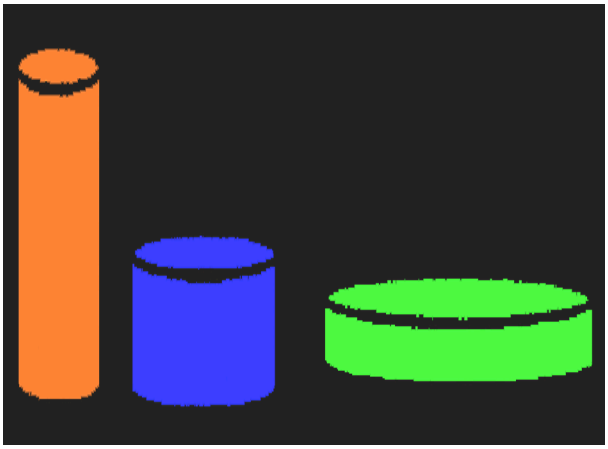
Start with one prism again. Then put two together to double the length and fill in the chart. Add a third one to triple the length and record.

Repeat with the width. What generalizations can you make?

| Scale Factor | Length | Width | Height | Surface Area in 1cm squares |
|---|---|---|---|---|
| Starting prism | | | | |
| Doubling | | | | |
| Tripling | | | | |
| Quadrupling | | | | |
| | | | | |
| Scale Factor | Length | Width | Height | Surface Area in 1cm squares |
| Starting prism | | | | |
| Doubling | | | | |
| Tripling | | | | |
| Quadrupling | | | | |
| | | | | |

## Activity 8: Tubes and Rice

Do 3 tubes with the same surface area have the same volume?



- Take one transparency & tape sides to make a tube (first tube).
- Take another transparency, cut into two equal pieces & tape sides to make a second tube.
- Take a third transparency, cut into three equal pieces & tape sides to make a third tube.(The 3 tubes have equal surface areas.)

Which tube has the greatest volume or do they all hold the same amount?

- Fill the tall tube with plastic markers or rice.
- Put the second tube around the first.
- Pull up the first to see how the plastic markers "fit" the second tube.

What do you think would happen with the third tube?