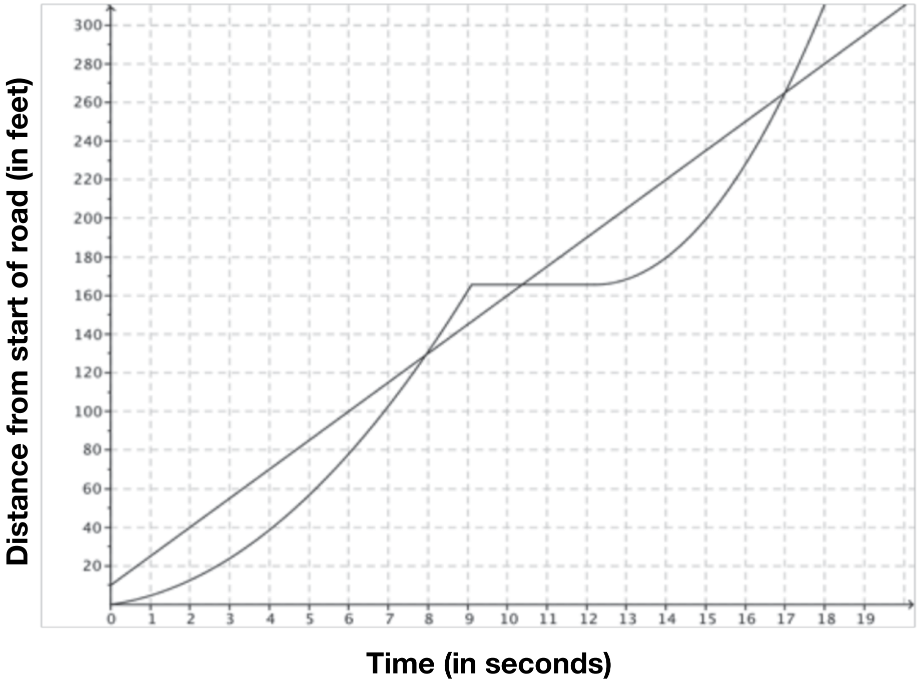
The Bike and Truck Task

A bicycle traveling at a steady rate and a truck are moving along a road in the same direction. The graph below shows their positions as a function of time. Let B(*t*) represent the bicycle's distance and K(*t*) represent the truck's distance.



1. Label the graphs appropriately with B(*t*) and K(*t*). Explain how you made your decision.
2. Describe the movement of the truck. Explain how you used the values of B(*t*) and K(*t*) to make decisions about your description.
3. Which vehicle was first to reach 300 feet from the start of the road? How can you use the domain and/or range to determine which vehicle was the first to reach 300 feet? Explain your reasoning in words.
4. Jack claims that the average rate of change for both the bicycle and the truck was the same in the first 17 seconds of travel. Explain why you agree or disagree with Jack.

Adapted from *Algebra 1 Creating and Interpreting Functions: A Set of Related Lessons* (2015a). Pittsburgh, PA: Institute for Learning, University of Pittsburgh. Lesson guides and student workbooks are available at ifl.pitt.edu.

Exploring Exponential Relationships: The Case of Vanessa Culver

Ms. Culver wanted her students to understand that exponential functions grow by equal factors over equal intervals and that, in the general equation $y = b^x$, the exponent (x) tells you how many times to use the base (b) as a factor. She also wanted students to see the different ways the function could be represented and connected. She selected the Pay It Forward task because it provided a context that would help students in making sense of the situation, it could be modeled in several ways (i.e., diagram, table, graph, and equation), and it would challenge students to think and reason.

The Pay It Forward Task

In the movie *Pay It Forward*, a student, Trevor, comes up with an idea that he thinks could change the world. He decides to do a good deed for three people, and then each of the three people would do a good deed for three more people and so on. He believes that before long there would be good things happening to billions of people. At stage 1 of the process, Trevor completes three good deeds. How does the number of good deeds grow from stage to stage? How many good deeds would be completed at stage 5? Describe a function that would model the Pay It Forward process at *any* stage.

Ms. Culver began the lesson by telling students to find a function that models the Pay It Forward process by any means necessary and that they could use any of the tools that were available in the classroom (e.g., graph paper, chart paper, colored pencils, markers, rulers, graphing calculators). As students began working in their groups, Ms. Culver walked around the room stopping at different groups to listen in on their conversations and to ask questions as needed (e.g., How did you get that? How do the number of good deeds increase at each stage? How do you know?). When students struggled to figure out what to do, she encouraged them to try to visually represent what was happening at the first few stages and then to look for a pattern to see if there was a way to predict the way in which the number of deeds would increase in subsequent stages.

As she made her way around the room, Ms. Culver also made note of the strategies students were using (see fig. 1.3) so she could decide which groups she wanted to have present their work. She decided to have the strategies presented in the following sequence. Each presenting group would be expected to explain what they did and why and to answer questions posed by their peers. Group 4 would present their work first since their diagram accurately modeled the situation and would be accessible to all students. Group 3 would go next because their table summarized numerically what the diagram showed visually and made explicit the stage number, the number of deeds, and the fact that each stage involved multiplying by another 3. Groups 1 and 2 would then present their equations one after the other. At this point Ms. Culver decided that she would give students 5 minutes to consider the two equations and decide which one they thought best modeled the situation and why.

Below is an excerpt from the discussion that took place after students in the class discussed the two equations that had been presented in their small groups.

Ms. C.: So who thinks that the equation $y = 3x$ best models the situation? Who thinks that the equation $y = 3^x$ best models the situation? [*Students raise their hands in response to each question.*]

Ms. C.: Can someone explain why $y = 3x$ is the best choice? Missy, can you explain how you were thinking about this?

Missy: Well, group 1 said that at every stage there are three times as many deeds as the one that came before it. That is what my group (4) found too when we drew the diagram. So the "$3x$" says that it is three times more.

Ms. C.: Does everyone agree with what Missy is saying? [*Lots of heads are shaking back and forth indicating disagreement.*] Darrell, why do you disagree with Missy?

Darrell: I agree that each stage has three times more good deeds than the previous stage, I just don't think that $y = 3x$ says that. If x is the stage number like we said, then the equation says that the number of deeds is three times the stage number—not three times the number of deeds in the previous stage. So the number of deeds is only 3 more, not 3 times more.

Ms. C.: Other comments?

Kara: I agree with Darrell. $y = 3x$ works for stage 1, but it doesn't work for the other stages. If we look at the diagram it shows that stage 2 has 9 good deeds. But if you use the equation, you get 6 not 9. So it can't be right.

Chris: $y = 3x$ is linear. If this function were linear, then the first stage would be 3, the next stage would be 6, then the next stage would be 9. This function can't be linear—it gets really big fast. There isn't a constant rate of change.

Ms. C.: So let's take another look at group 3's poster. Does the middle column help explain what is going on? Devon?

Devon: Yeah. They show that each stage has 3 times more deeds than the previous one. For each stage, there is one more 3 that gets multiplied. That makes the new one three times more than the previous one.

Angela: So that is why I think $y = 3^x$ best models the situation. Stage 1 had 3 good deeds, stage 2 people had three each doing three deeds so that is 3^2, stage 3 had 9 people (3^2) each doing 3 good deeds, so that is 3^3. The x tells how many 3's are being multiplied. So as the stage number increases by 1, the number of deeds gets three times larger.

Ms. C.: If we keep multiplying by another three like Angela described, it is going to get big really fast like Chris said. Chris also said it couldn't be linear, so take a minute and think about what the graph would look like.

At this point Ms. Culver asked group 5 to share their graph and proceeded to engage the class in a discussion of what the domain of the function should be, given the context of the problem. The lesson concluded with Ms. Culver telling the students that the function they had created was called *exponential* and explaining that exponential functions are written in the form of $y = b^x$. She told students that in the 5 minutes that remained in class, they needed to individually explain in writing how the equation related to the diagram, the table, the graph, and the problem context. She thought that this would give her some insight regarding what students understood about exponential functions and the relationship between the different ways the function could be represented.

Boston, Melissa, Fredrick Dillon, Margaret Smith, and Stephen Miller. Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 9-12. Reston, VA: National Council of Teachers of Mathematics, 2017.

Exploring Exponential Relationships: The Case of Steven Taylor

This case was written by Margaret Smith at the University of Pittsburgh, based on patterns of instruction documented in the research literature and observed in dozens of classrooms.

Mr. Taylor wanted to engage his students in a lesson that would give them a chance to reason abstractly and quantitatively. He began the lesson by showing students a small portion of the movie *Pay It Forward* in which the protagonist (Trevor) explains how he could change the world by doing good deeds. Some students in the class had seen the movie, but that didn't seem to matter—all students were intrigued to be watching a movie in math class. They wondered how this was going to relate to algebra.

Mr. Taylor stopped the video before giving away any methods or hinting at a solution. He gave students a copy of the task and had one student read it aloud. He then instructed students to begin work on the problem with their partners.

As students worked on the task, Mr. Taylor walked around the room, stopping at different groups to listen in on their conversations. Students were confused about what was happening at each of the stages and unsure of what it meant to describe a function. He decided to bring the class back together.

Mr. Taylor began by pointing out that the task asked them to describe a function that would model the Pay It Forward process at any stage. He then engaged them in the following exchange:

Mr. T: So who remembers what a function is? Chris? [*Chris did not raise his hand.*]

Chris: I don't know.

Mr. T: What did we write in our notebooks a few weeks ago? Can someone find it? [*A minute passes while students look through their notebooks.*]

Colin: A function relates input and output. For every input, there is only one output.

Mr. T: So what do we need to do here? What is the input?

Samantha: The stage number?

Mr. T: Yes. What is the output?

Derek: The number of deeds?

Mr. T: Good. So we need to describe the relationship between the stage number and the number of deeds. We could do that using an equation. So let's start by looking at what happens at each stage.

Boston, Melissa, Fredrick Dillon, Margaret Smith, and Stephen Miller. Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 9-12. Reston, VA: National Council of Teachers of Mathematics, 2017.

Group 1 (equation—incorrect)

$y = 3x$

At every stage there are three times as many good deeds as there were in the previous stage.

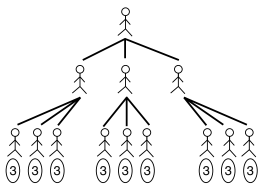
Group 2 (table-like groups 6 & 7 and equation)

$y = 3^x$

Group 3 (diagram-like group 4 and table)

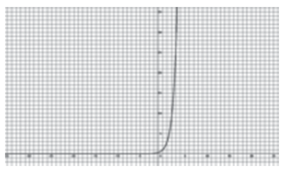
x (stages)		y (deeds)
1	3	3
2	3×3	9
3	$3 \times 3 \times 3$	27
4	$3 \times 3 \times 3 \times 3$	81
5	$3 \times 3 \times 3 \times 3 \times 3$	243

Group 4 (diagram)



So the next stage will be 3 times the number there in the current stage so 27×3. It is too many to draw. You keep multiplying by 3.

Group 5 (table-like groups 6 & 7 and graph)



Groups 6 and 7 (table)

x (stages)	y (deeds)
1	3
2	9
3	27
4	81
5	243

Fig 1.3. Vanessa Culver's students' work

Boston, Melissa, Fredrick Dillon, Margaret Smith, and Stephen Miller. Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 9-12. Reston, VA: National Council of Teachers of Mathematics, 2017.

Mr. Taylor then drew a stick figure to represent Trevor and each of the three people for whom Trevor did good deeds (see fig. 3.5). He then asked students how many good deeds were completed at stage 1, and everyone shouted “3!” He then asked students how many deeds each of the three people would do at stage 2. Again they shouted “3!” He told students to continue the diagram for stages 2 and 3 and to record their information in a table so they could keep track of the number of deeds in each stage. With a clear idea of what to do, students continued to work in their groups to complete the diagram and table.

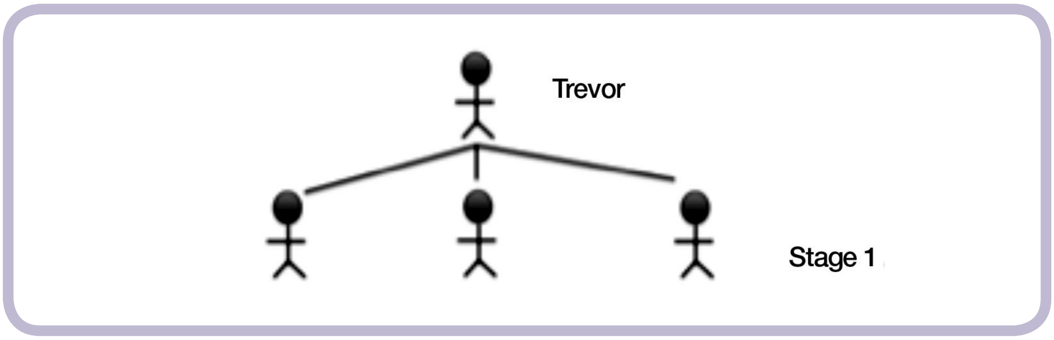
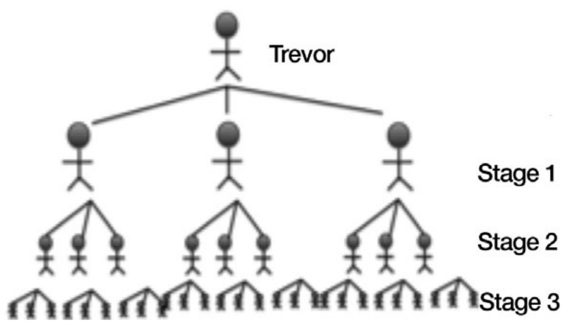


Fig. 3.5. The diagram Mr. Taylor drew on the board

As Mr. Taylor resumed his visits to the groups, he now noticed that they all had completed the diagram (see fig. 3.6 for example) and had made a table showing stage number and the corresponding number of deeds. He began to ask students if they could now write an equation for the function. Most groups said that it would be $y = 3x$, since each stage was three times the one before it.



Stage #	# of Deeds
1	3
2	9
3	27

Fig. 3.6. An example of the diagram and table generated by students

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Once again Mr. Taylor decided to bring the class together for a discussion. He put a copy of the table shown below (minus the last row and the third column) on the document camera. He began:

Mr. T: How much bigger is the number of deeds in stage 2 than in stage 1?

Ss: 3 times bigger!

Mr. T: How much bigger is the number of deeds in stage 3 than in stage 2?

Ss: 3 times bigger!

Mr. T: How much bigger do you think the deeds in stage 4 will be when compared to stage 4?

Ss: 3 times bigger!

Mr. T: How much is 3 times 27?

Ss: 81! [*Mr. Taylor adds stage 4 and 81 deeds to the table.*]

Mr. Taylor went on to explain that at each stage they needed to look at the number of threes they are multiplying together. He added a third column to the table and elicited from students the number of times you multiply three at each stage.

Stage #	# of Deeds	× by 3
1	3	3
2	9	3 × 3
3	27	3 × 3 × 3
4	81	3 × 3 × 3 × 3

He then asked students how they can represent the number of threes that get multiplied together without writing them all out. Camilla yelled out "use an exponent." Mr. Taylor told her, "That is exactly right," and wrote $y = 3^x$ on the board. At that point the bell rang. Mr. Taylor told his class that they would pick up on this tomorrow.

Analyzing Teaching and Learning 9.2

Investigating Teacher Interventions

(Margaret Smith and Victoria Bill [Smith and Bill 2015a] developed dialogues 1–4 in 2015 for an Institute for Learning Professional Development Session in Syracuse, New York. The title of the session was "Supporting Students' Productive Struggle in Learning Mathematics.")

Read the minidialogues shown that follow, then—

- Discuss the nature of each student's struggle.
- Identify what the teacher does to help students move beyond the impasse that they had reached.
- Determine whether the teacher supported the students' productive struggle.

Minidialogues for The Lifeguard Task

Dialogue 1

A student lists the ordered pairs shown below. (2, 24) (4, 48) (9, 108)

Teacher: Where did these numbers come from?

Student: These are the labels for the points on the graph.

Teacher: So how do they help you answer the question?

Student: Maleeka wants to make $80, and I know that's between four hours and nine hours, but I am not sure how to get closer than that.

Teacher: How much money did Maleeka make in two hours?

Student: $24.

Teacher: So, how much money would she make in one hour? *[Student divides 24 by 2 on her paper.]*

Student: $12?

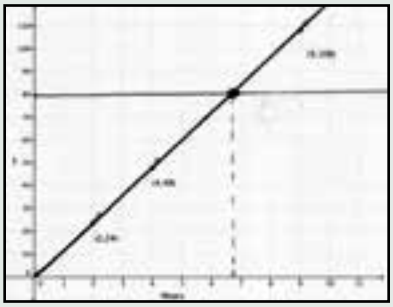
Teacher: That's right. So now you know that she makes $12 in one hour, and we want to find out how many hours she needs to work to make more than $80. Do you remember how we used equations to solve problems like this?

Student: $80 \div 12$?

Teacher: Yes! Solve $12x = \$80$ by dividing 80 by 12 see what you get.

Dialogue 2

A student's graph is shown below.



Teacher: Tell me what you have here.

Student: I drew a line and connected the three points that were given. Then I drew a horizontal line through $80 because that is the least amount of money she wants to make.

Teacher: So, what did this tell you?

Student: The point where the two lines intersect is the number of hours where she makes $80. So it is ($x$, 80). The value of x is somewhere between 6 and 7 hours, but closer to 7.

Teacher: So, how can you figure it out more precisely?

Student: I am not sure.

Teacher: If you look at the three points that are given, how are hours worked related to the amount earned?

Student: The hours worked times 12 gives you the amount she earned.

Teacher: So, can this help you figure out how to find the x-coordinate of your point of intersection?

Student: I can set up an equation $x(12) = 80$ and solve for x.

Teacher: That sounds like a good plan. I will check back in with you later.

Dialogue 3

A student makes the table shown below.

Hours	Earnings
2	24
4	48
9	108

Teacher: Tell me how you got the numbers in your table?

Student: I read the points in the graph.

Teacher: Does that help you answer the question?

Student: Not really. She had to work more than four hours but less than eight. I am not sure where to go from here.

Teacher: I am wondering if you could expand your table to include other values that were not on the graph. How could you get started?

Student: I could put in hours 1 through 9 and then see if I could figure out how much she would make for that number.

Teacher: Okay. So how will you figure out how much she earns for any number of hours?

Student: Well, if I look at the three values in the table, it seems that each of the hours was multiplied by 12 to get the earnings. So that means she must make $12 an hour. So, I will just multiply all the hours in my new table by 12 and see if I can find what number of hours get her more than $80.

Teacher: Sounds good!

Dialogue 4

A student cannot get started.

Teacher: What have you figured out so far?

Student: Nothing. I am not sure what to do.

Teacher: The first thing you need to do is to figure out how much money she makes every hour. Do you remember what operation you need to use to do that?

Student: Divide?

Teacher: What are you going to divide?

Student: 2 ÷ 24?

Teacher: No, 24 ÷ 2.

Student: So, it's 12.

Teacher: So this is the amount she makes per hour. Now you need to divide 80 by 12.

Dialogue 5

A student cannot get started.

Teacher: What have you figured out so far?

Student: Nothing. I am not sure what to do.

Teacher: Why don't you start by reading the problem again and then looking really closely at the graph. I will be back in a few minutes.