Making Connections from a Simplex Lock to the Binomial Theorem

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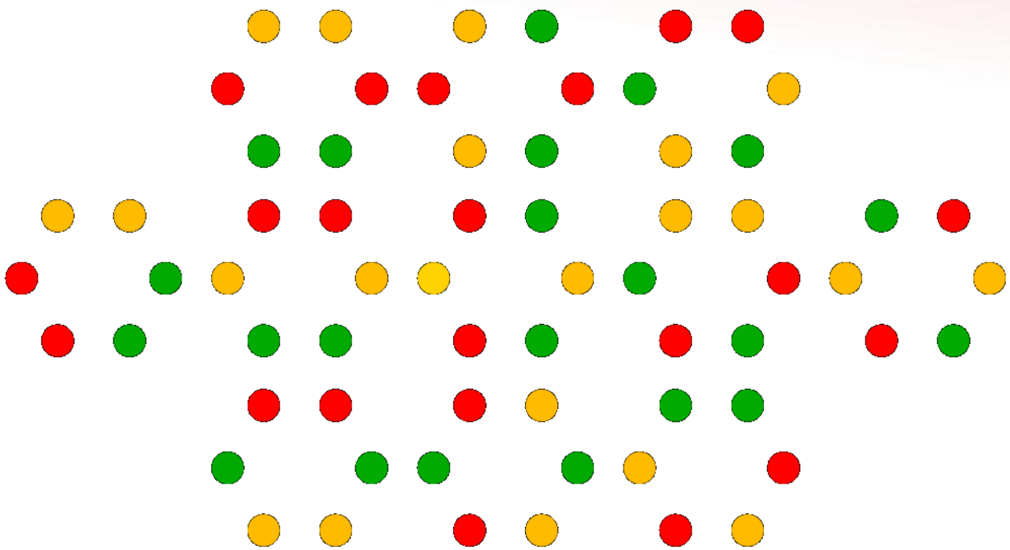
Combinatorics

Combinatoric is the branch of mathematics concerning the study of finite or countable discrete structures.



Combinations

In mathematics, a combination is a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter. For example, when we want to find the number of combinations of size 2 without repeated letters that can be made from the three letters in the word CAT, order doesn't matter; AT is the same as TA. We can write out the three combinations of size two that can be taken from this set of size three:

CA CT AT

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The Binomial Theorem

$$\left(x+a\right)^n = {}_nC_0x^n + {}_nC_1ax^{n-1} + {}_nC_2a^2x^{n-2} + \cdots + {}_nC_na^n$$

The *x*'s start out to the *n*th power and decrease by 1 in power each term. The *a*'s start out to the 0 power and increase by 1 in power each term. The binomial coefficients are found by computing the combination symbol. Also the sum of the powers on *a* and *x* is *n*.

Find the 5th term of $(x + a)^{\boxed{6}}$

$$\binom{6}{4}$$

1 less than term number

5th term will have a^4

(power on *a* is 1 less than term number)

So we'll have x^2

(sum of two powers is 6)

The Binomial Theorem & Pascal Triangle

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

$$
\begin{array}{c}
{}_{0}C_{0} \\
{}_{1}C_{0} \quad {}_{1}C_{1} \\
{}_{2}C_{0} \quad {}_{2}C_{1} \quad {}_{2}C_{2} \\
{}_{3}C_{0} \quad {}_{3}C_{1} \quad {}_{3}C_{2} \quad {}_{3}C_{3} \\
{}_{4}C_{0} \quad {}_{4}C_{1} \quad {}_{4}C_{2} \quad {}_{4}C_{3} \quad {}_{4}C_{4} \\
{}_{5}C_{0} \quad {}_{5}C_{1} \quad {}_{5}C_{2} \quad {}_{5}C_{3} \quad {}_{5}C_{4} \quad {}_{5}C_{5}
\end{array}
$$

$$
\begin{array}{c}
\binom{1}{1} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}
\end{array}
$$

$$1 = 2^0$$

$$1 + 1 = 2 = 2^1$$

$$1 + 2 + 1 = 4 = 2^2$$

$$1 + 3 + 3 + 1 = 8 = 2^3$$

$$1 + 4 + 6 + 4 + 1 = 16 = 2^4$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$$

$$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6$$

$$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128 = 2^7$$

$$1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256 = 2^8$$

$$1 + 9 + 36 + 84 + 126 + 126 + 84 + 36 + 9 + 1 = 512 = 2^9$$

$$1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1 = 1{,}024 = 2^{10}$$

$$1 + 11 + 55 + 165 + 330 + 462 + 462 + 330 + 165 + 55 + 11 + 1 = 2048 = 2^{11}$$

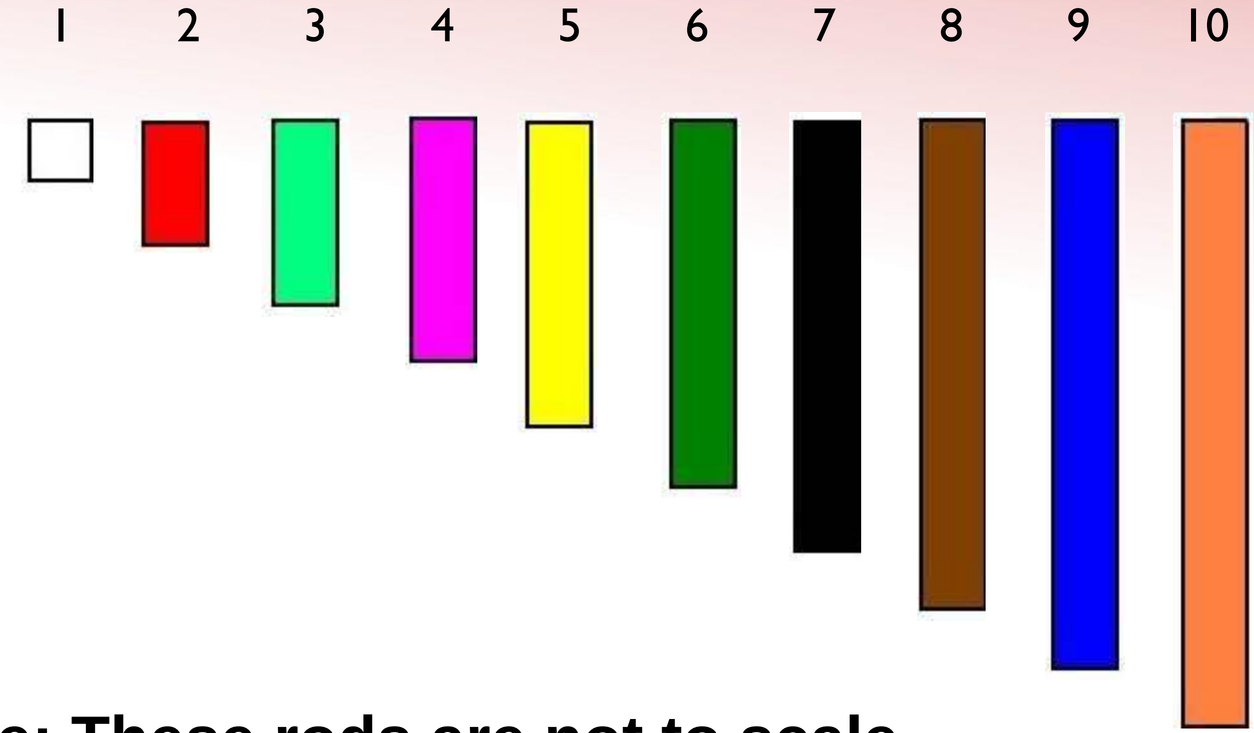
$$1 + 12 + 66 + 220 + 495 + 792 + 924 + 792 + 495 + 220 + 66 + 12 + 1 = 4{,}096 = 2^{12}$$

$$1 + 13 + 78 + 186 + 715 + 1287 + 1716 + 1716 + 1287 + 715 + 186 + 78 + 13 + 1 = 8{,}192 = 2^{13}$$

$$1 + 14 + 91 + 364 + 1001 + 2002 + 3003 + 3432 + 3003 + 2002 + 1001 + 364 + 91 + 14 + 1 = 16{,}384 = 2^{14}$$

$$1 + 15 + 105 + 455 + 1365 + 3003 + 5005 + 6435 + 6435 + 5005 + 3003 + 1365 + 455 + 105 + 15 + 1 = 32{,}768 = 2^{15}$$

Getting Acquainted with Cuisenaire Rods for Another Problem



***Note: These rods are not to scale.**

Choo Choo Train

Let us assume a "train of length 5" whose combined length is 5.

Here are some examples:

<table>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td colspan="2">2</td><td>1</td><td colspan="2">2</td></tr>
<tr><td colspan="2">2</td><td colspan="3">3</td></tr>
<tr><td colspan="3">3</td><td colspan="2">2</td></tr>
<tr><td colspan="3">3</td><td>1</td><td>1</td></tr>
</table>

Choo Choo Train

Notice that the 3-2 train and the 2-3 train contain the same type of train. But they are linked differently because of the order.

Choo Choo Train

How many trains of length 4 are there?

How many trains of length 5 are there?

Find a formula for the number of trains of length n. Come up with a convincing reason that your rule is correct.

Create an algorithm that will generate all the trains of length n.

Choo Choo Train

How many trains of length n are there that use only cars of lengths 1 and 2?

Find a general rule and explain why your rule works.

How many trains of length 11 are there which use only cars of length 1, 2, 3 and 4?

Choo Choo Train

A train of length 1
1
A train of length 2
2 11
A train of length 3
3 12 21 111
A train of length 4
4 13 31 112 121 211 22 1111
A train of length 5
5 14 41 113 131 311 32 23
2111 1211 1121 1112 122 212
221 11111

Choo Choo Train

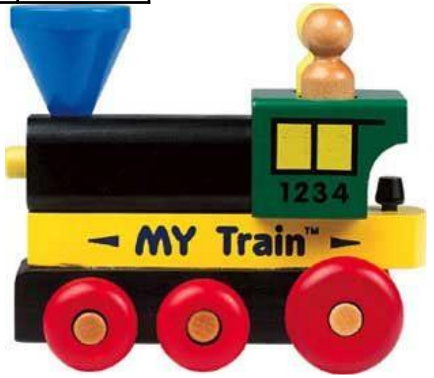
A train of length 6

6 51 15 114 141 411 1113 1131
1311 3111 123 132 312 321 231
213
33 222 1122 1221 1212 2121 2112
2211
21111 12111 11211 11121 11112 42
24 111111

Choo Choo Train

This model is using all possible cars to generate the given length of train:

# of length	1	2	3	4	5	6	……...	n	
# of trains	1	2	4	8	16	32	……...	2^{n-1}	



Choo Choo Train

Another way to look this problem

T_0=1 Engine+0 car
T_1=1 Enhine+1 car
T_2=2 Engine+1+1 cars or Engine+2 car
T_3=4 (see the diagram above)
T_4=8 (see the diagram above)
T_5=16 (see the diagram above)
T_6=32 (see the diagram above)

So $T_2=T_1+T_0$
$T_3= T_2+T_1+T_0$
$T_4= T_3+T_2+T_1+T_0$
$T_5=T_4+ T_3+T_2+T_1+T_0$
$T_n=T_{n-1}+T_{n-2}+T_{n-3}+\ldots.T_4+ T_3+T_2+T_1+T_0$

(n is the length of train and T_n is total # of ways to link the train with the cars)

Choo Choo Train

How many trains of length ***n*** are there if we only use cars of length 1 and length 2?

# of length	0	1	2	3	4	5	6
# of trains	1	1	2	3	5	8	13

$T_2=T_0+T_1$
$T_3=T_2+T_1$
$T_4=T_3+T_2$

$T_5=T_4+T_3$
$T_6=T_5+T_4$
$T_7=T_6+T_5$

There is a recursive pattern $\mathbf{T_n=T_{n-1}+T_{n-2}}$

Choo Choo Train

How many trains of length 11 are there if we only use cars of length 1, length 2 and length 3?

# of length	0	1	2	3	4	5	6	……..
# of trains	1	1	2	4	7	13	24	…….

$T_3=T_2+T_1+T_0=4$
$T_4=T_3+T_2+T_1 =7$
$T_5=T_4+T_3+T_2=13$
$T_6=T_5+T_4+T_3 =24$
$T_7=T_6+T_5+T_4 =44$
$T_8=T_7+T_6+T_5 = 81$
$T_9=T_8+T_7+T_6=149$
$T_{10}=T_9+T_8+T_7=274$
$T_{11}=T_{10}+T_9+T_8= 504$

Choo Choo Train

How many trains of length ***n*** are there if we only use cars of length 1, length 2 and length 3?

# of length	0	1	2	3	4	5	6
# of trains	1	1	2	4	7	13	24

There is another recursive pattern

$\mathbf{T_n=T_{n-1}+T_{n-2}+T_{n-3}}$

Choo Choo Train

How many trains of length ***n*** are there if we only use cars of length 3 and length 5?

How many trains of length ***n*** are there if we only use cars of length 1, length 3 and length 5?

How many trains of length ***n*** are there if we only use cars of length 2, length 4 and length 6?

What did you learn from the Choo Choo Train about a recursive pattern?

A Push Button Lock

A Push Button Lock

Several companies makes a combination lock that is used in many public buildings. Here is a 5-button device which is purely mechanical. We can set the combination using the following rules.

1. A combination is a sequence of 0 or more pushes, each push involving at least one button.
2. Each button may be used at most once (once you press it, it stays in).
3. Each push may include any of the buttons that have not been pushed yet, up to and including all remaining buttons.
4. The combination does not need to include all buttons.
5. When two or more buttons are pushed at the same time, order does not matter.

A Push Button Lock

One more thing:
There is one possible no-push combination; this means the door is already unlocked -- not a good idea but it counts.

Here are some possible combinations

{{1,2},{4}}   {{1,2,4},{3,5}}   {1,2,3}

{{2},{4}}   { }   {{1}, {2}, {3,4}}

A Push Button Lock

At least one company advertises that there are thousands of combinations, and the question is: "Is the company telling the truth?"

Our Problem:

How many combinations are there on a 5-button lock?

A Push Button Lock

Buttons

Classify the combinations by how many buttons they use.

There are five combinations that use only one button.

Combinations that use only two buttons come in two types: press one button then press another, or press two buttons together.

Pushes

Break down the possible combinations into how many "pushes" they have. One push means pushing down one or more buttons simultaneously.

So, the combination 2|3 4 5|1 has three pushes. The combination 1|2|3 also has three pushes.

If a combination uses five pushes, it must use all five of the buttons in some order. 1|2|3|4|5, 2|1|3|4|5, etc.

A Push Button Lock

Fewer Buttons

A problem-solving strategy that can be employed is to work on an easier and related problem. That is, work on locks with fewer buttons.

It's easy to count the combinations for a 1-button lock; there are two combinations: push the button, or don't push it (the door is unlocked). Then count the combinations for a 2-button lock.

Another problem-solving strategy is to look for patterns in how the number of combinations grows.

A Push Button Lock

Make a list of all the combinations on a four-button lock and come up with a systematic way to extend these to the combinations on a five button lock.

A Push Button Lock

Everyone should understand there are two different methods of computing the number of combinations in this problem (buttons and pushes).

Understanding how to solve the choo choo train problem is very critical to unlocking this problem.

A Push Button Lock

0 Button……..……………………………...1

{ }

1 Button………………………………….2

{ }, {1}

2 Buttons………………………….....…..6

{ }, {1}, {2}, {1,2}

{{1}, {2}}, {{2}, {1}}

3 Buttons……………………………...26

{ }, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}

{{1}, {2}}, {{2}, {1}}, {{1},{3}}, {{3},{1}}, {{2},{3}},
{{3},{2}}, {{1,2}, {3}}, {{1,3}, {2}}, {{2,3}, {1}}, {{3}, {1,2}},
{{2},{1,3}}, {{1},{2,3}}

{{1}, {2}, {3}}, {{1}, {3}, {2}} {{2}, {1}, {3}} {{2}, {3}, {1}},
{{3}, {1}, {2}}, {{3}, {2}, {1}}

3-Button Lock: Organize (by # buttons)

Buttons used (k)	Some Examples	Notations	$\sum(k)$	Totals
0	unlocked	$\binom{3}{0}=1$	1	26
1	{1}, {2}, {3}	$\binom{3}{1}=3$	3	
2	{{1},{2}}, {{1},{3}}, …	$\binom{3}{1}\binom{2}{1}=6$	9	
	{1,2}, {2,3}, …	$\binom{3}{2}=3$		
3	{{1}, {2}, {3}} , …	$\binom{3}{1}\binom{2}{1}\binom{1}{1}=6$	13	
	{{1,2},{3}}, {{1,3}, {2}}, {{2,3},{1}}, {{1}, {2,3}}, {{2},{1,3}}, {{3},{1,2}}	$2\binom{3}{1}\binom{2}{2}=2(3)=6$		
	{1,2,3}	$\binom{3}{3}=1$		

3-Button Lock: Organize (by # pushes)

Pushes used (k)	Some Examples	Notations	$\sum(k)$	Totals
0	unlocked	$\binom{3}{0}=1$	1	26
1	{1}, {2}, {3}	$\binom{3}{1}=3$	7	
	{1,2}, {1,3}, {2,3}	$\binom{3}{2}=3$		
	{1,2,3}	$\binom{3}{3}=1$		
2	{{1},{2}}, {{3}, {2}}, …	$\binom{3}{1}\binom{2}{1}=6$	12	
	{{1,2},{3}}, {{2},{1,3}}, …	$2\binom{3}{1}\binom{2}{2}=2(3)=6$		
3	{1}, {2}, {3}…..	$\binom{3}{1}\binom{2}{1}\binom{1}{1}=6$	6	

A Push Button Lock

A combination for a five button lock may start with one push of 3 buttons, leaving 2 buttons not pushed.

How many ways can you choose that 3-button push?

$$C_3^5 = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\times 4}{2\times 1} = 10$$

How many ways can you complete the combination after 3 buttons push? 6

{ }, {1}, {2}, {1,2}

{{1}, {2}}, {{2}, {1}}

How many combinations start with a 3-button push?

$$10\times 6 = 60$$

A Push Button Lock

Find the number of combinations for less than five buttons.

Investigate the patterns that emerge.

Make conjectures & generalizations.

Shapes for *n-button combinations*
Combinations using ***n*** *buttons*
Combinations using ***k*** *pushes*

Two-Button Lock

Let $\boldsymbol{A_n}$ be the set of all combinations for a lock with $\boldsymbol{n}$ buttons.
Also, let $\boldsymbol{S_n}$ be the number of combinations for a lock with $\boldsymbol{n}$ buttons.

n=0 A_0={{ }} so S_0= 1
n=1 A_1={{ }, {1}} so S_1= 2
n=2 A_2={{ }, {1}, {2}, {1,2}, {{1}, {2}}, {{2}, {1}}} so S_2= 6

Why S_2= 6? S_2= 1+4+1=6

1 because of "one" push with 0 button (no push counts).
4 comes from pushing just one button (2 of them) and pushing one button per push for 2 pushes.
1 because of one push with 2 buttons.

For n=2, there are $C_1^2 = \binom{2}{1}$ =2 ways to select the first push with one button.

Three-Button Lock

After the first push, there are 2 ways to decide the second push (another push or no push ; A_1={{ }, {1}}. Therefore, the decision to be made for the second push is reduced to a one-button lock problem. $4 = \binom{2}{1} \bullet S_1 = 2 \times 2$

$S_2 = 1 + \binom{2}{1} \cdot S_1 + 1 = 6$

If it's a three-button lock problem, the first push can be 0, 1, 2 or 3 buttons.

There is one combination to open the lock if the first push uses 0 button.

There are 3 ways to choose if the first push uses only 1 button. ……… $\binom{3}{1} = 3$
After the first push, there are 2 buttons left to decide the second push. The decision to be made for the second push is really a two-button lock problem. ……...
$\binom{3}{1} \cdot S_2 = 3x6 = 18$

There are 3 ways to choose if the first push uses exactly 2 buttons. … $\binom{3}{2} = 3$
After the first push, there is 1 button left to decide the second push. The decision to be made for the second push is really a one-button lock problem. ………
$\binom{3}{2} \cdot S_1 = 3x2 = 6$

Lastly, there is one combination to open the lock if the first push uses all 3 buttons.

Three-Button Lock

$S_3 = 1 + \binom{3}{1} \bullet S_2 + \binom{3}{2} \bullet S_1 + 1 = 1+18+6+1 = 26$

Four-Button Lock

$S_4 = 1 + \binom{4}{1} \bullet S_3 + \binom{4}{2} \bullet S_2 + \binom{4}{3} \bullet S_1 + 1$

$= 1 + (4x26) + (6x6) + (4x2) + 1 = 1+104+36+8+1 = 150$

What is the total number of combinations for a lock with ***5*** buttons ?

What is the total number of combinations for a lock with ***n*** buttons ?

Problem: Unlocking Locks

Since we have studied simplex locks, what is the total number of combinations for a lock with ***5*** buttons?

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What is the total number of combinations for a lock with ***n*** buttons (Recurrence Formula)?

$$S_n = \sum_{k=1}^{n-1} \binom{n}{k} \bullet S_{n-k} + 2$$

Using the Pascal's Triangle

For "n choose k," find row n entry k.
The first row is numbered 0.
The first entry in a row is numbered 0.

Example: $\binom{4}{2} = 6$

1																
1	1															
1	2	1														
1	3	3	1													
1	4	6	4	1												
1	5	10	10	5	1											
1	6	15	20	15	6	1										
1	7	21	35	35	21	7	1									
1	8	28	56	70	56	28	8	1								
1	9	36	84	126	126	84	36	9	1							
1	10	45	120	210	252	210	120	45	10	1						
1	11	55	165	330	462	462	330	165	55	11	1					
1	12	66	220	495	792	924	792	495	220	66	12	1				
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1			
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1		
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1	
1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16	1

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THANK YOU

for coming!