What is an Infinite Series
…and Why Should I Care?

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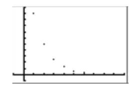
Shameless Plug:
Michigan Council of Teachers of Mathematics Institutes & Conference; July 25 – 27, 2017
https://www.mictm.org/index.php/events-news/mctm-conference-a-institute

Disclaimers

- These activities take between 3 47-minute days to about a week, depending on how much the kids get into it…
- I start Series in January, first I do delta/epsilon and then right into series
- I started thinking about all this while I was teaching History of Math at Towson University; struggling to explain what was so damn important about these ugly things…

How we teach the topic:

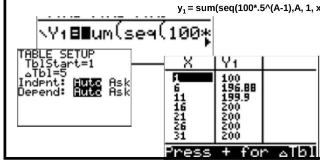
- A sequence is a discrete function
- We represent it graphically, numerically, verbally, & symbolically with a twist
  - recursively; explicitly



A series is a sequence with "+'s" instead of commas…

If the sequence of partial sums converges, so does the original sequence. The sequence of partial sums is a sequence of sums of finite series…

y_1 = sum(seq(100*.5^(A-1),A, 1, x))



Time for a little History Lesson

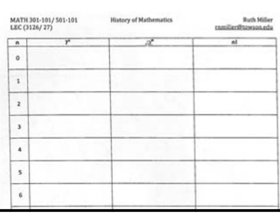
"Seeing that there is nothing (right well beloved students in the Mathematics) that is so troublesome to Mathematicall practise, nor that doth more molest and hinder Calculators, than the Multiplications, Divisions, square and cubical Extractions of great numbers, which besides the tedious expence of time, are for the most part subject to many slippery errors."

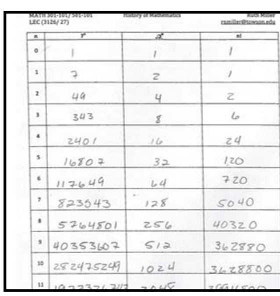
-John Napier's preface to A Description of the Admiral Table of Logarithms

(Note that he's ok with additions & subtractions.)

https://scientificgems.wordpress.com/2013/11/20/plimpton-322-mathematics-3800-years-ago/

SO, let's make some tables… NO CALCULATOR!!!





MATH 301-101/ 501-101
LEC (3126/ 27)
History of Mathematics
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n	7^n	2^n	n!
0	1	1	1
1	7	2	1
2	49	4	2
3	343	8	6
4	2401	16	24
5	16807	32	120
6	117649	64	720
7	823543	128	5040
8	5764801	256	40320
9	40353607	512	362880
10	282475249	1024	3628800
11	[illegible]	[illegible]	[illegible]

Does Napier's expression "tedious expence of time" have any meaning yet?

Now let's use the table…

- Jacob Bernoulli (1655 – 1705) defines what we have come to call the number "e."
- The number ends up being called "e" because Euler (1707 – 1783) called it "e" and nobody was arguing with Euler.
- So all these dates are clearly before the advent of the calculator.
- So ask yourself, how was Euler figuring out e^2?

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$$e^x = 1 + (x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}(x)^n$$

from calculator: $e^2 \approx$ 7.389056099 [A]

n	$e^x = \sum_{n=0}^{n} \frac{1}{n!}(x)^n$, $x = 2$	e^2 (approx)	Error
0	$\frac{1}{0!}(2)^0 = 1(1)$	1	6.389056
1	$\frac{1}{0!}(2)^0 + \frac{1}{1!}(2)^1 = 1 + 2$	3	4.389056
2	$\frac{1}{0!}(2)^0 + \frac{1}{1!}(2)^1 + \frac{1}{2!}(2)^2 = 1 + 2 + \frac{1}{2}4$	5	2.389056
3	$\sum_{n=0}^{2} \frac{1}{n!}x^n = 5 \Rightarrow 5 + \frac{1}{3!}(2)^3 = 5 + 1\frac{1}{3}$	$6\frac{1}{3}$	1.0557227
4	$6\frac{1}{3} + \frac{1}{4!}(2)^4 = 6\frac{1}{3} + \frac{1}{24}(16) =$	7	.389056
5	$7 + \frac{1}{5!}(2)^5 = 7 + \frac{1}{120}(32)$	$7\frac{4}{15}$	.1223894
6	$7\frac{4}{15} + \frac{1}{6!}(2)^6 = 7\frac{4}{15} + \frac{64}{720}$	$7\frac{16}{45}$	.0335005
7	$7\frac{16}{45} + \frac{1}{7!}(2)^7 = 7\frac{16}{45} + \frac{128}{5040}$	$7\frac{8}{21}$	.00810371
8	$7\frac{8}{21} + \frac{1}{8!}(2)^8 = 7\frac{8}{21} + \frac{256}{40320}$	$7\frac{122}{315}$	.001754511
9	$7\frac{122}{315} + \frac{1}{9!}(2)^9 = 7\frac{122}{314} + \frac{512}{362880}$	$7\frac{1102}{2835}$	.00034576
10	$7\frac{1102}{2835} + \frac{1}{10!}2^{10} = 7\frac{1102}{2835} + \frac{1024}{3628800}$	$7\frac{1838}{4725}$	.000061389

We want to be accurate to three decimal places for the AP Test, so how many terms does it take to get that much accuracy?

So now let's use our newly discovered n = 9 series to approximate e^x for other values of x. We already know for e^2 because we just did that to 9 terms…

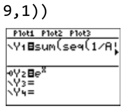
$y_3 = e^x = 1 + (x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}(x)^n$

$y_4 = e^x$

x	$e^x = \sum_{n=0}^{9} \frac{1}{n!}(x)^n$ n=9	e^x	Error
1	$e^1 = 1 + 1 + \frac{1}{2!}(1)^2 + \frac{1}{3!}(1)^3 + \ldots + \frac{1}{9!}(1)^9$	2.718281526	.0000003028 85
2	See other side	$7\frac{1102}{2835}$	.00034576
3	$e^3 = 1 + 3 + \frac{1}{2!}(3)^2 + \frac{1}{3!}(3)^3 + \ldots + \frac{1}{9!}(3)^9$	20.063393	.022144

Here's some hints…

y_1=sum(seq((1/A!)x^A,A,0,9,1))
y_2=e^x
(make y_2 the bouncy ball)



put seq(x,x,1,14,1)->L_1
put $y_1(L_1)$->L_2
put $y_2(L_1)$->L_3
put L_3-L_2->L_4

Wait!! Shouldn't the accuracy at 9 terms be great everywhere?

		e^x	Error
4		54.154144	.444054
5		143.68945	4.7237
6		369.5714	33.857
7		910.7493	185.88377
8	sum(seq((1/x!)*$\boxed{8}$^x), x,0,9,1))	2136.2268	844.7311

Welcome to the radius of convergence…..

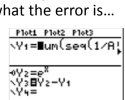
Of course, the radius of convergence for the *infinite* series is infinity:

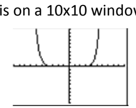
$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0.$$

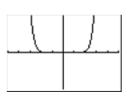
But if I limit my number of terms to 9, how close do I have to be to 0 to be as accurate as I want to be?

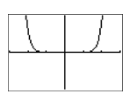
Our question: Out at 9 terms, when is the partial sum about the same as the calculator's graph of e^x?

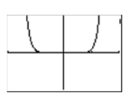
Make $y_3 = y_2 - y_1$; this will allow you to see what the error is… This is on a 10x10 window.

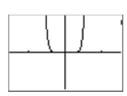




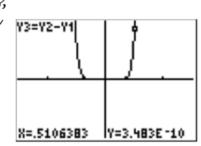

5 x .01


3 x .0004


2 x .000004


1.5 x 4.0 x 10-10

So if we want to be within 9 decimal places, using a 9 term series, I need to be within about .5 of zero.



I have found that the week I spend on this activity

- Helps students to believe that the series they memorize actually do approximate the transcendental functions they represent
- Put the radius of convergence in context
- Help them to remember that while the infinite series may converge, the finite series need proximity & length
- Help them to get a handle on what the Error means, and to better understand the need for error approximations later on…

More on Error~

- If you do this with an alternating series, you can graph the error term as a function and see that the next term is bigger than the error on your $y_1 - y_2$ graph.
- I haven't played much with LaGrange yet, but I imagine there is a way to make that meaningful, too.

WAIT! Update!

- This last Saturday I was visiting with a friend and fellow AP Calc ~~nerd~~ teacher* and look what we figured out…
- Remember that the Lagrange Error Bound is the next term (just like the Alternating Series Bound) except that instead of the actual next term, it's the next term with an "arbitrarily bigger numerator," M.

*Shout out to Greg Timm, from Baltimore…

The next term of the Taylor Series would be

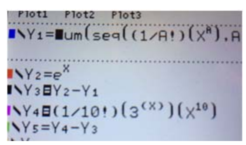
$$A_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-c)^{n+1}$$

And LaGrange informs us that the error is bounded by

$$R_n(x) < \frac{M}{(n+1)!}(x-c)^{n+1}$$

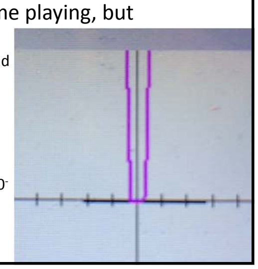
for some M, arbitrarily bigger than $f^{n+1}(c)$

So for our e^x, at 9 terms,
M could be 3^x,
and our c is zero;
y_3 is the actual error,
and y_4 should be a bound on that error.



It took some playing, but

- The black is the error, and the pink is the error bound
- the window is [-1, 1] x [-1x10^{-20}, 1x10^{-20}]



So what is an Infinite Series?

A series is a technique to get an approximate value of a transcendental number by using arithmetic.

- I can be as accurate as I want to be by being close to the center of the series,
- and/ or by adding terms.

…And why do I care?

Likewise, integration is a numerical action, but I don't know how to integrate transcendental functions. If I turn the pesky transcendental function into a polynomial, I can use the power rule to integrate it (or even to take it's derivative).

- I can be as accurate as I want to be by being close to the center of the series
- and/or by adding terms

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