Using the Concrete-Representational-Abstract Technique to Teach Algebra to Students Who Are Struggling

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What Works Clearinghouse:
Best Evidence Encyclopedias

- Improving Mathematical Problem Solving Recommendations
  - Prepare problems and use them in whole-class instruction—MINIMAL EVIDENCE
  - **Assist students in monitoring and reflecting on the problem-solving process—STRONG EVIDENCE**
  - **Teach students how to use visual representations—STRONG EVIDENCE**
  - Expose students to multiple problem-solving strategies—MODERATE EVIDENCE
  - Help students recognize and articulate mathematical concepts and notation—MODERATE EVIDENCE
  - http://ies.ed.gov/ncee/wwc/PracticeGuide.aspx?sid=16

Early Predictors of High School Math

- Mastery of fractions and early division is a predictor of students' later success with algebra and other higher-level mathematics.
- Need to focus on whole number division and fractions and teaching them better.
- Math scores on national standardized tests among U.S. high school students have not improved in three decades and are significantly behind those in countries such as China, Japan, Finland, the Netherlands and Canada at a time when math proficiency is a requirement for many jobs.
- Noted students who "start ahead in math generally stay ahead" and that those who "start behind generally stay behind" and it looked to find the reason.
- http://www.post-gazette.com/stories/news/education/formula-written-for-math-success-640962/?p=0

Concrete-Representational-Abstract Approach

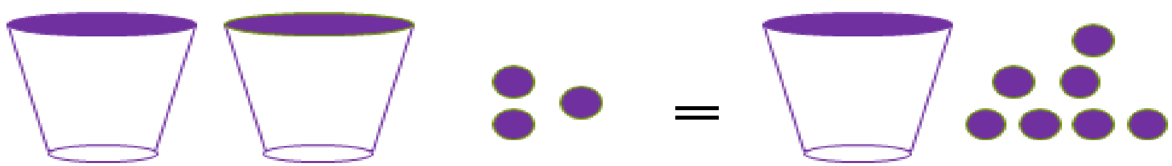
- *Concrete* (Doing Stage). In the concrete stage, the teacher begins instruction by modeling each mathematical concept with concrete materials (e.g., red and yellow chips, cubes, base-ten blocks, pattern blocks, fraction bars, and geometric figures).
- *Representational* (Seeing Stage). In this stage, the teacher transforms the concrete model into a representational (semi-concrete) level, which may involve drawing pictures; using circles, dots, and tallies; or using stamps to imprint pictures for counting.
- *Abstract* (Symbolic Stage). At this stage, the teacher models the mathematics concept at a symbolic level, using only numbers, notation, and mathematical symbols to represent the number of circles or groups of circles. The teacher uses operation symbols (+, –, x, /) to indicate addition, subtraction, multiplication, or division.

Concrete Level

- The CONCRETE LEVEL involves the manipulation of objects. At this level the learner concentrates on both the manipulated objects and the symbolic processes.
  - See the following example (Metz, 2011):

Example: Solve $2x + 3 = x + 7$

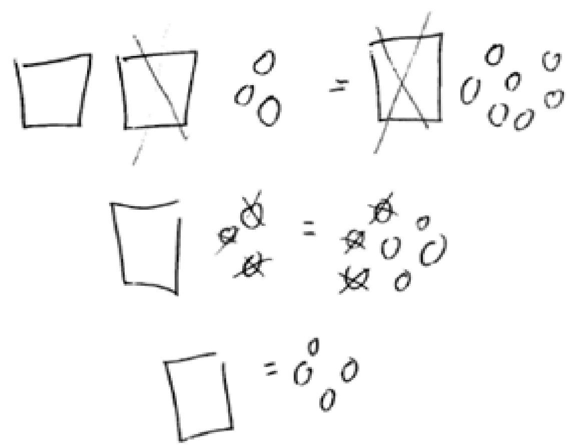
Begin by using concrete objects:



Representational Level

- The REPRESENTATIONAL LEVEL involves working with illustrations of items in performing math tasks.
  - Items include dots, lines, pictures of objects, or nonsense items.
  - The emphasis is on developing associations between visual models and the numerical equations.
  - See the following example (Metz, 2011):

Represent the concrete objects:



Abstract Level

- The ABSTRACT LEVEL involves the use of numerals, which is using only numbers to solve math problems.
  - See the following example (Metz, 2011):

Abstract notation:

$$
\begin{array}{rcl}
2x + 3 & = & x + 7 \\
\underline{-x } & = & \underline{-x } \\
x + 3 & = & +7 \\
\underline{ - 3} & = & \underline{-3} \\
x & = & 4
\end{array}
$$

Programs Using CRA Approach

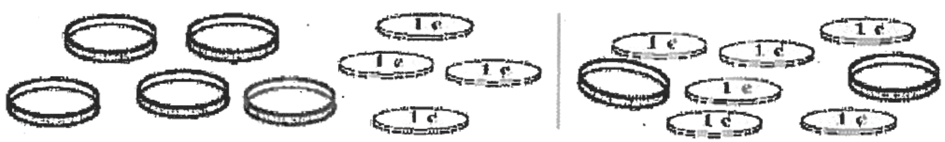
- *Applied Math Made Easy*
- *Solving Equations: An Algebra Intervention*
- *Hands On Equations*
- *Singapore Math*
- *Math-U-See*
- *Algebra Tiles*

Applied Math Made Easy

Solving Equations

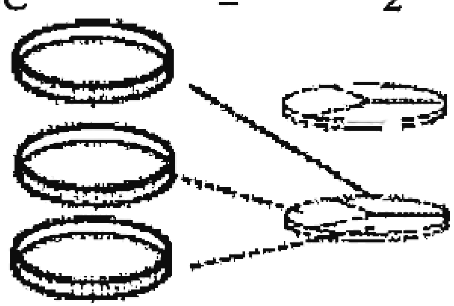
milk caps and pennies

5C + 4 = 2C + 6



3C + 4 - *4* = 6 - *4*

3C = 2

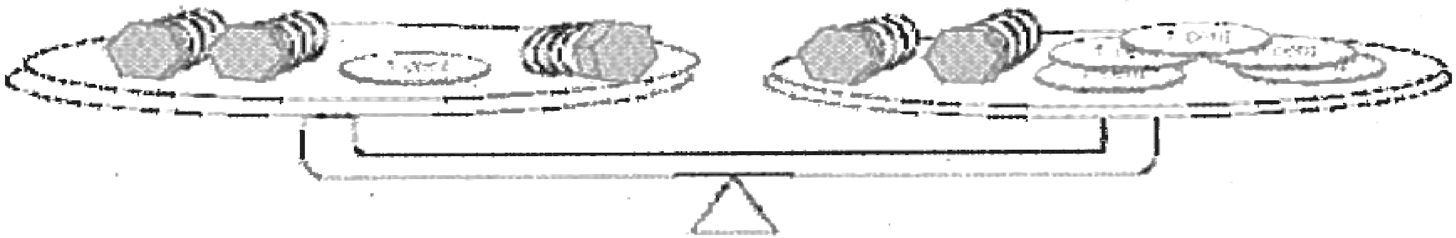


$$\frac{3C}{3} = \frac{2}{3}$$

$$1C = {}^{2}/_{3}$$

Applied Math Made Easy
Solving Equations

On the scale below, make a few mental transfers and you will be able to see **the weight of one bolt.**



a) How many pennies does each bolt weigh?_________

b) What did you do first? I took away two ____________________ from each side.

c) Then I took away ____________________ from ____________________.

d) After that it was easy to see that each bolt was the same weight as _______ pennies.

As you go to each station in this lab, you will **tell me what you did** to find the weight. Here's how to describe your steps using **both words AND algebra** for the one above.

Write the equation: 3 Bolts + 1¢ = 2 Bolts + 5¢ or just **3B + 1 = 2B + 5**

Now I took away 2 bolts from each container. 3B + 1 **- 2B** = 2B + 5 **- 2B**

1B + 1 = 5

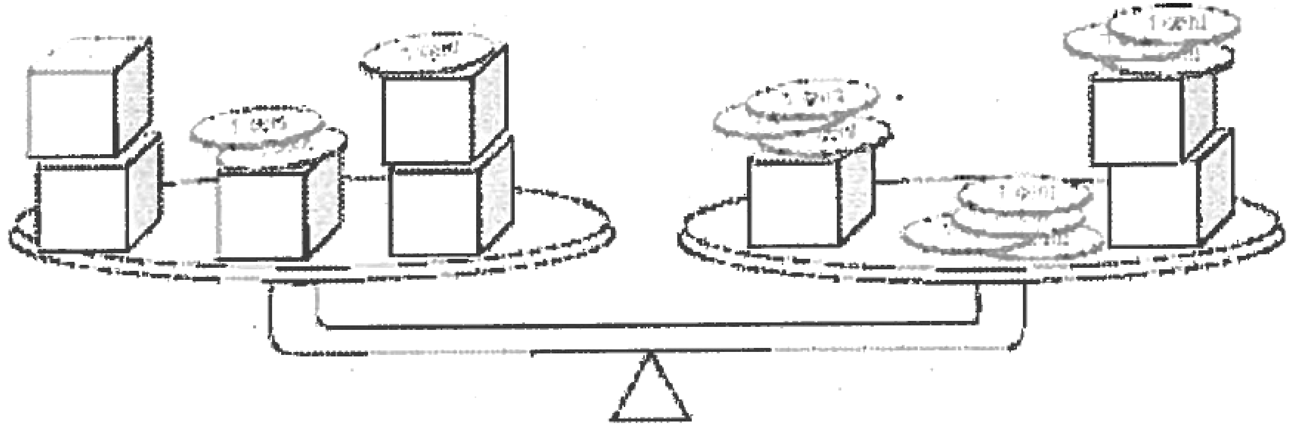
Then just take away a penny from both sides. 1B + 1 **- 1** = 5 **- 1**

1B = 4 ¢ (It's the same as #4d.)

Applied Math Made Easy

Solving Equations

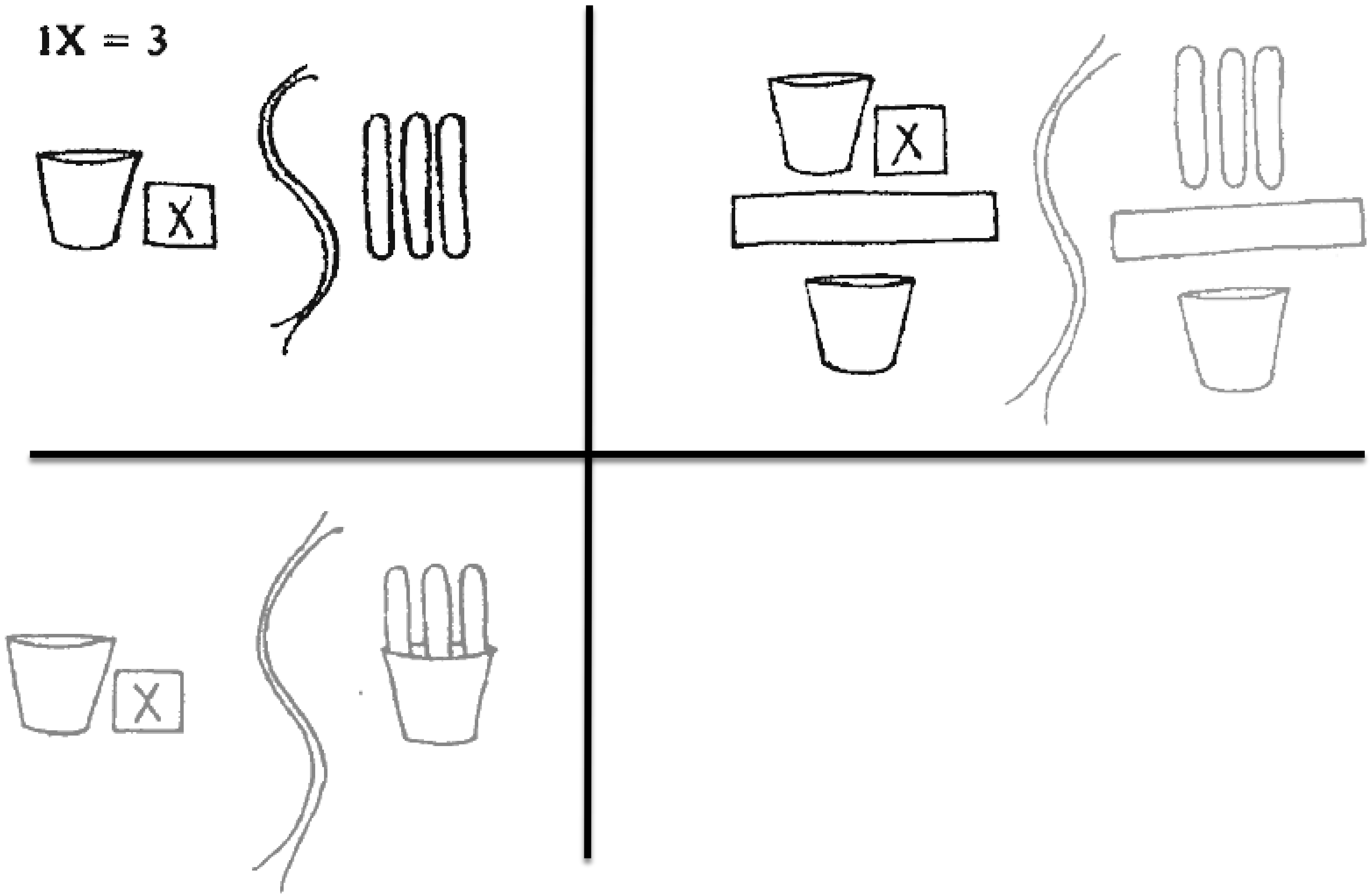
Here are some pennies and cubes on a scale. How many pennies does each cube weigh?



The steps I am thinking:	**Showing my work with algebra:**
Here's the equation:	_______ = _______
I'll take _____ cubes from both sides.	5C + 3 - _____ = 3C + 10 - _____
Here's what I get.	_____ + _____ = _____
Now I need to take away ___ from ______ ______	2C + 3 - _____ = 10 - ___
I get.	_____ = 7
If 2 cubes weigh 7¢, just divide both sides into 2 piles.	$\frac{\_\_\_\_}{2} = \frac{7}{*\_\_\_\_}$
So 1 cube "weighs" ______	_____ = _____ ¢

Solving Equations: An Algebra Intervention

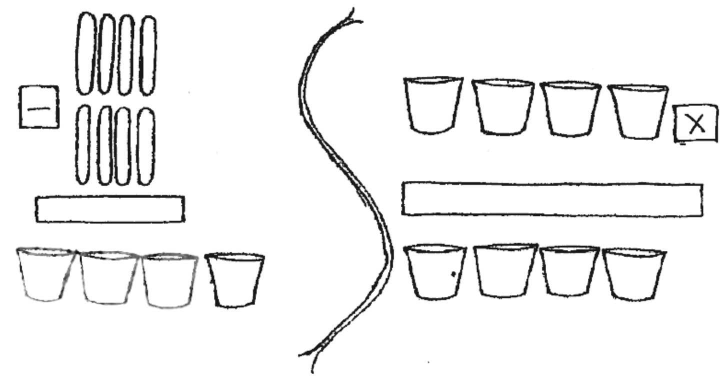
Solving One-Step Equations

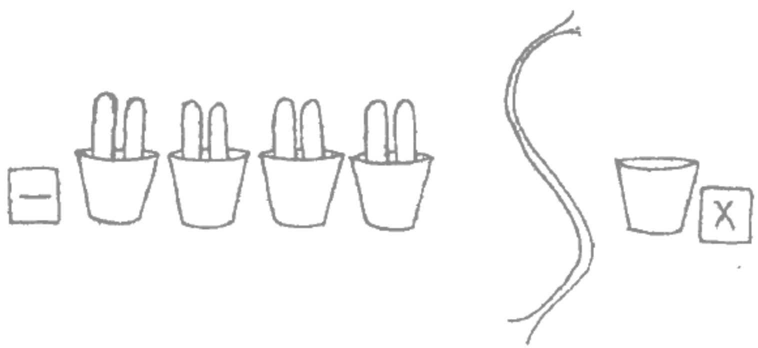


Solving Equations: An Algebra Intervention

Solving One-Step Equations

$-8 = 4X$

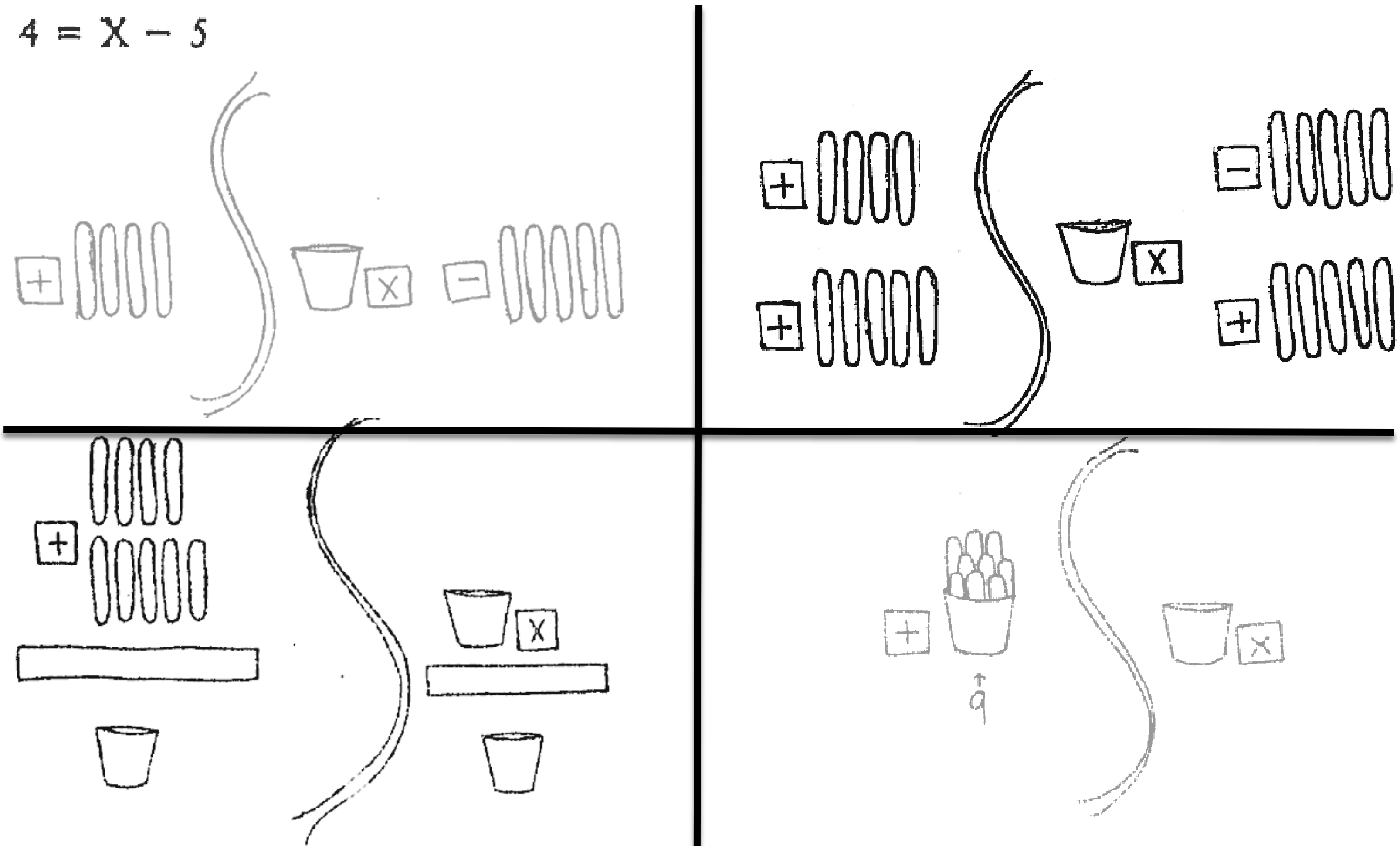




Solving Equations: An Algebra Intervention

Solving Two-Step Equations

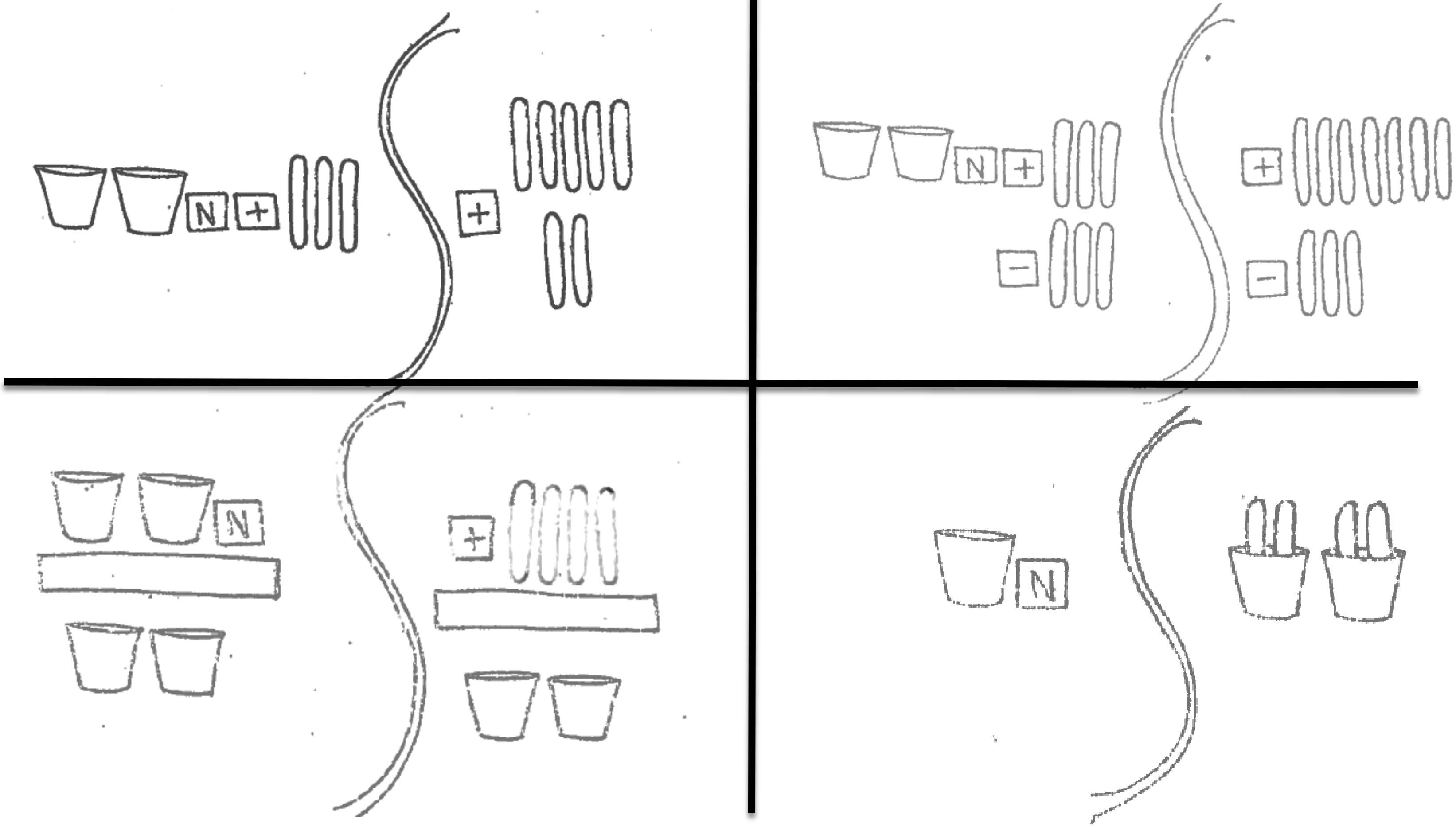
$4 = X - 5$



Solving Equations: An Algebra Intervention

Solving Two-Step Equations

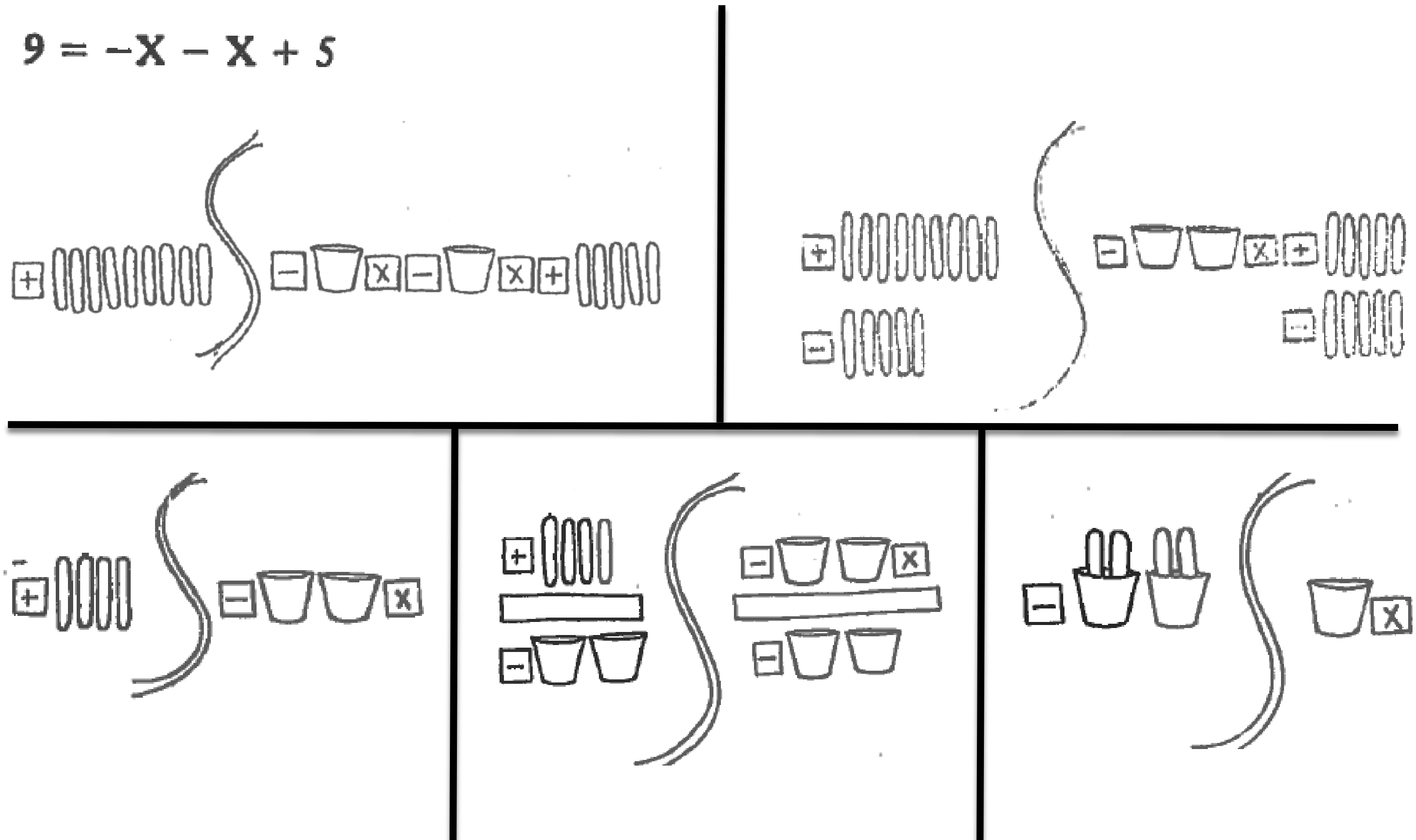
$2N + 3 = 7$



Solving Equations: An Algebra Intervention

Solving Two-Step Equations (Like Variables on the Same Side)

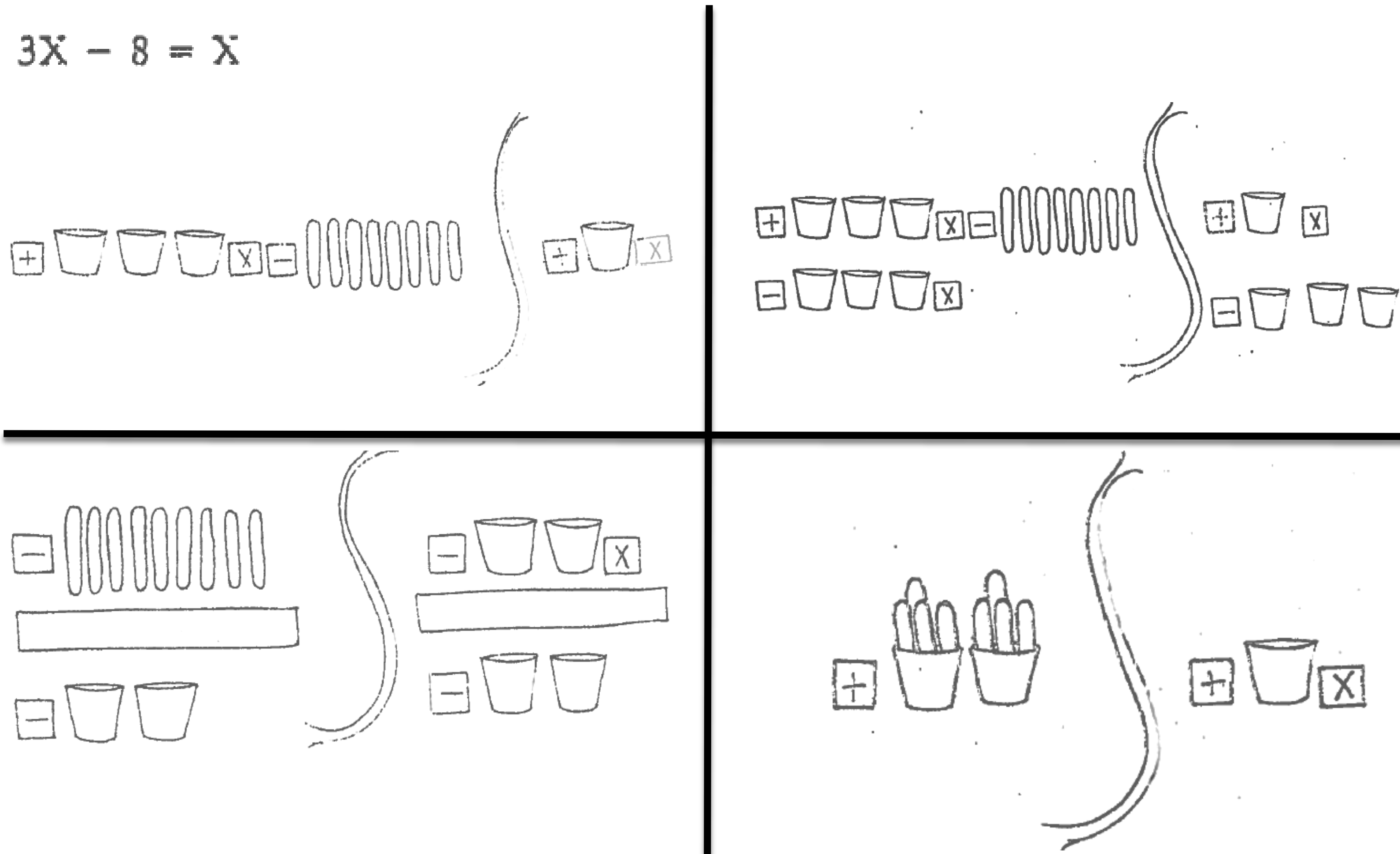
$9 = -X - X + 5$



Solving Equations: An Algebra Intervention

Solving Two-Step Equations—Like Variables on Opposite Sides

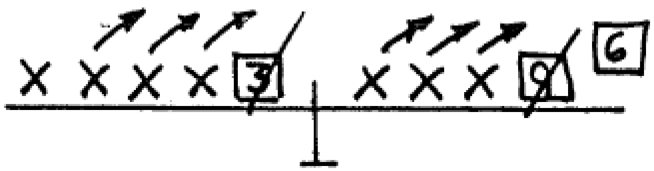
$3X - 8 = X$



Hands-On Equations

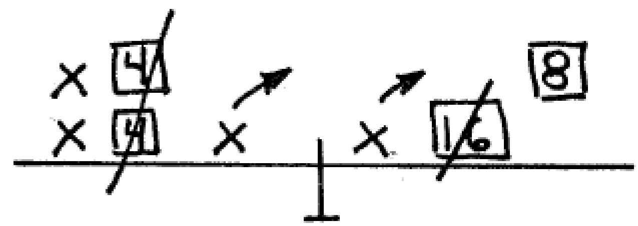
Solving Algebra Equations (Positive Variables and Integers)

Ex. $4x + 3 = 3x + 9$



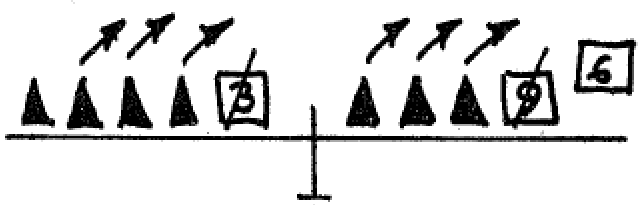
So, $x = 6$. Check: $27 \stackrel{\checkmark}{=} 27$.

Ex. $2(x + 4) + x = x + 16$



So, $x = 4$. Check: $20 \stackrel{\checkmark}{=} 20$.

Ex. $4x + 3 = 3x + 9$

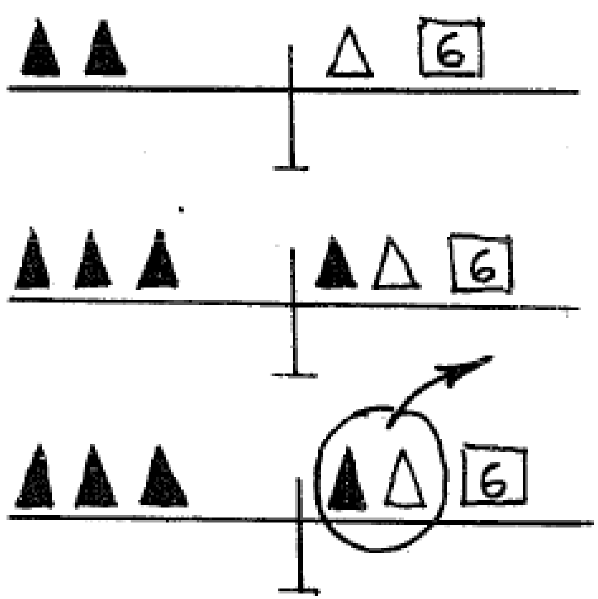


So, $x = 6$. Check: $27 \stackrel{\checkmark}{=} 27$.

Hands-On Equations

Solving Algebra Equations (Positive & Negative Variables and Positive Integers)

Ex. $2x = \ast + 6$



(add x on each side)

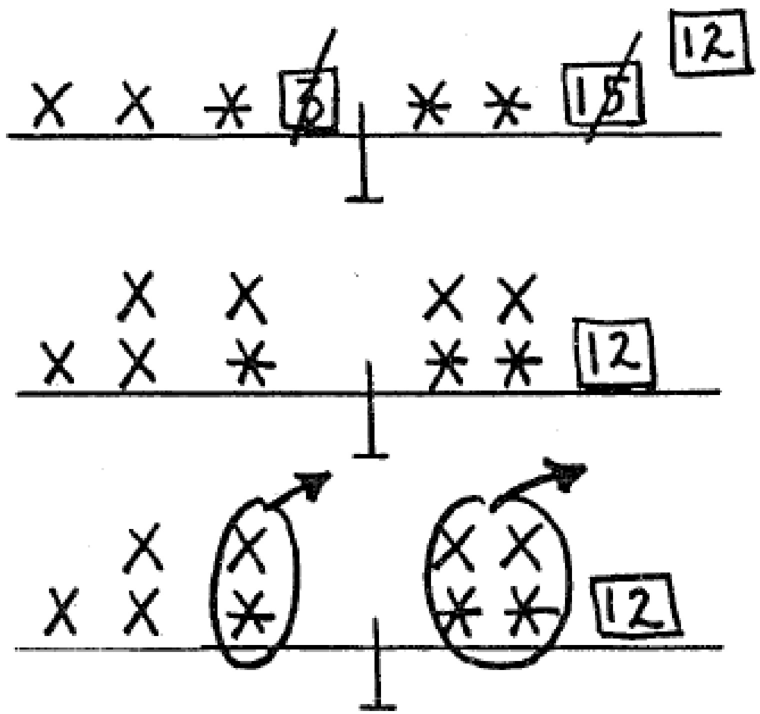
(remove opposites)

So, $x = 2$, $\ast = -2$. Check: $4 \stackrel{\checkmark}{=} 4$.

Hands-On Equations

Solving Algebra Equations (Positive & Negative Variables and Positive Integers)

Ex. $2x + \ast + 3 = 2\ast + 15$



(add 2 x's on each side)

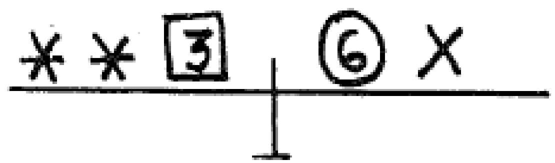
(Remove opposites)

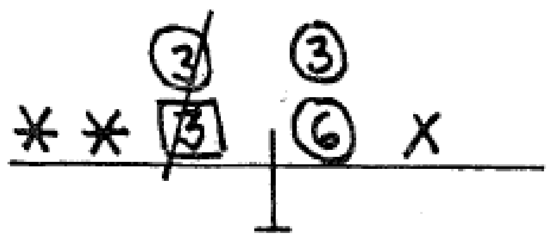
So, $x = 4$, $\ast = -4$. **Check:** $7 \overset{\checkmark}{=} 7$.

Hands-On Equations

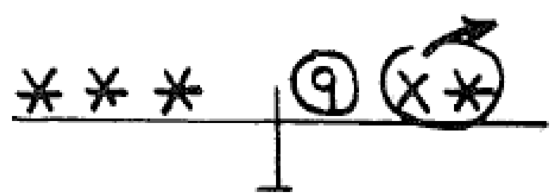
Solving Algebra Equations (Positive & Negative Variables and Positive & Negative Integers)

Ex. $2\ast + 3 = -6 + x$





(add a -3 to each side and remove opposites)



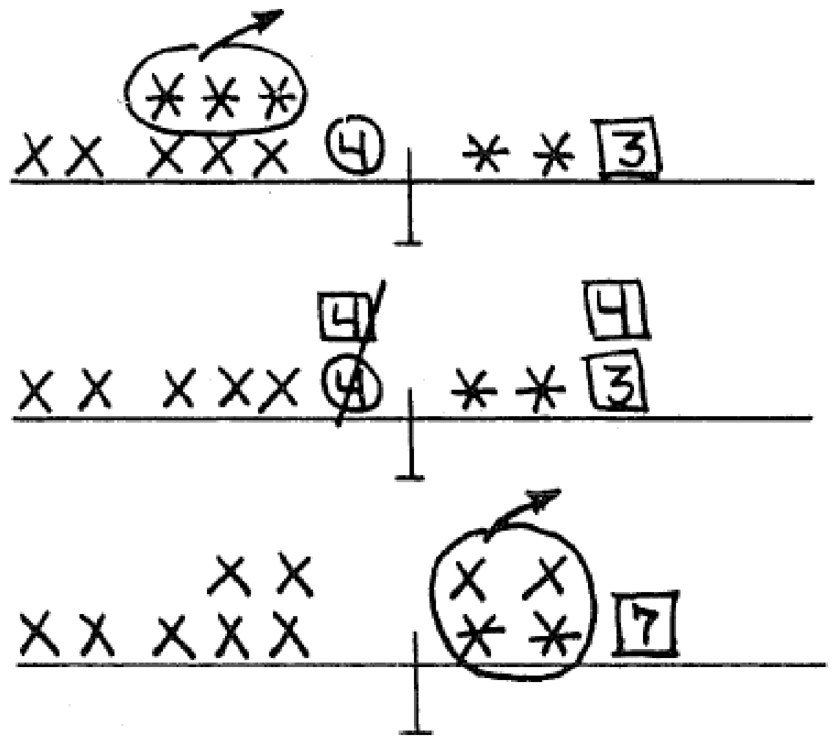
(add a $\ast$ to each side and remove opposites)

So, $\ast = -3$, $x = 3$. Check: $-3 \overset{\checkmark}{=} -3$.

Hands-On Equations

Solving Algebra Equations (Positive & Negative Variables and Positive & Negative Integers

Ex. $2x - 3(-x) + (-4) = 2(-x) + 3$



(remove 3 $(-x)$'s, or 3 ✱'s, as part of the setup)

(add a $+4$ to each side and remove opposites)

(add 2 x's to each side and remove opposites)

So, $x = 1$, $-x = -1$. Check: $1 \stackrel{\checkmark}{=} 1$.

Singapore Math

The Model Method and Algebra

Creating and Solving Algebraic Equations

Example 1

There are 50 children in a dance group. If there are 10 more boys than girls, how many girls are there?

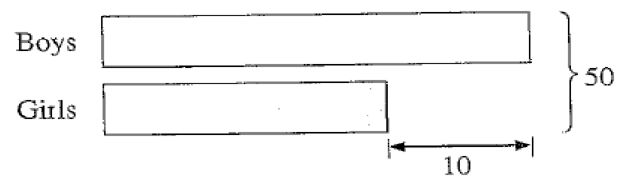
The algebraic method involves using a letter symbol such as x to represent an unknown quantity. For example, let x be the number of girls. As there are 10 more boys than girls, the number of boys is $x + 10$. The total number of boys and girls is $x + (x + 10)$, which is equal to 50. Thus students obtain the following equation to solve the problem:

$$x + (x + 10) = 50$$

The solution of the equation is $x = 20$.

There are 20 girls.

Students can draw the comparison model to represent the problem situation, and use it to solve the problem using the unitary method or the algebraic method.

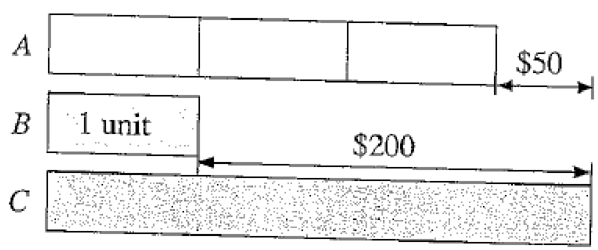


This model shows that the total number of boys and girls is 50, and that the difference between the number of boys and the number of girls is 10.

Singapore Math
The Model Method and Algebra
Creating and Solving Algebraic Equations

Example 2

A has 3 times as much money as B.
B has \$200 less than C.
C has \$50 more than A.
Find the total amount of money that A, B and C have.



2 units = \$200 – \$50 = \$150
1 unit = \$150 ÷ 2 = \$75

A's money = 3 units = 3 × \$75 = \$225
B's money = \$75
C's money = \$225 + \$50 = \$275

Total amount of money = \$225 + \$75 + \$275 = \$575

From the model, students obtain the equation:

$$3x + 50 = x + 200$$

The solution of the equation is $x = 75$.

$$3x + x + (3x + 50) = 7x + 50 = 575$$

The total amount of money is \$575.

Singapore Math

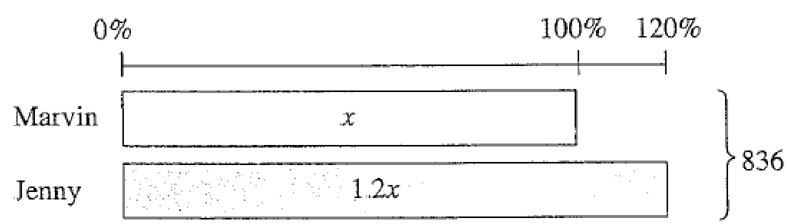
The Model Method and Algebra

Creating and Solving Algebraic Equations

Example 3

Jenny and Marvin have 836 stamps altogether. Jenny has 20% more stamps than Marvin. How many more stamps does Jenny have than Marvin?

The following model shows that the number of Jenny's stamps is 20% more than Marvin's.



Let x be the number of Marvin's stamps. Then the number of Jenny's stamps is 120% of x. This is $1.2x$. From the model, students obtain the equation:

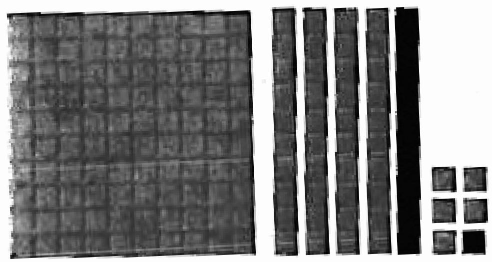
$$x + 1.2x = 836$$

The solution of the equation is $x = 380$.

$$1.2x - x = 0.2x = 76$$

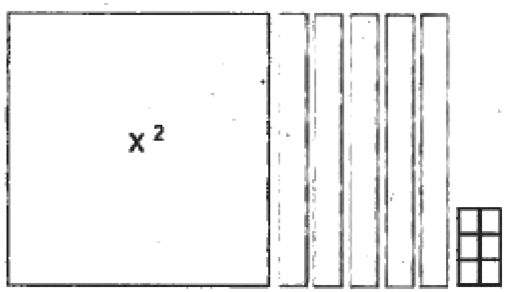
Jenny has 76 more stamps than Marvin.

Math-U-See
Base 10 & Base X



$$100 + 50 + 6$$
$$1 \times 100 + 5 \times 10 + 6 \times 1$$
$$1 \times 10^2 + 5 \times 10^1 + 6 \times 10^0$$

In the decimal system, every value is based on 10. The decimal system is referred to as base 10.



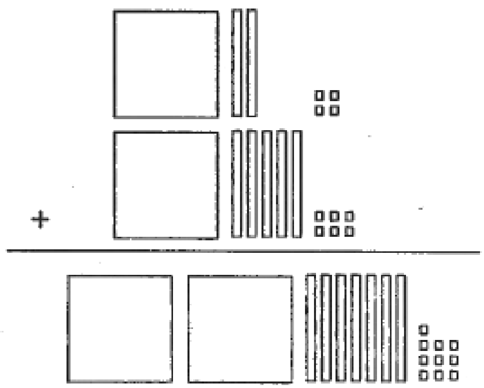
$$X^2 + 5X + 6$$
$$X \cdot X + 5 \cdot X + 6 \cdot 1$$
$$1 \cdot X^2 + 5 \cdot X^1 + 6 \cdot X^0$$

In algebra, every value is based on X. Algebra is arithmetic in base X.

Math-U-See
Addition of Polynomials

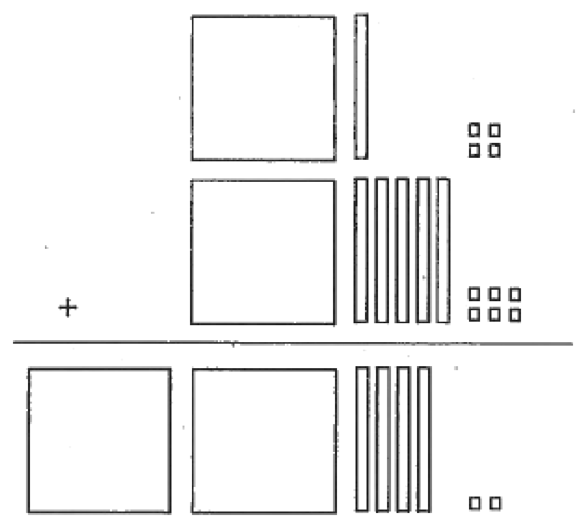
Example 1

$$\begin{array}{r} X^2+2X+\ \ 4 \\ +\,X^2+5X+\ \ 6 \\ \hline 2X^2+7X+10 \end{array}$$



Example 2

$$\begin{array}{r} X^2-\ \ X-4 \\ +\,X^2+5X+6 \\ \hline 2X^2+4X+2 \end{array}$$



Math-U-See
Multiplication of Polynomials

Example 3

$$\begin{array}{r} 13 \\ \times\, 12 \\ \hline \end{array} \;\begin{array}{c} \rightarrow \\ \uparrow \\ \end{array}\; = \;\begin{array}{r} 10+3 \\ \times\, 10+2 \\ \hline 20+6 \\ 100+30 \\ \hline 100+50+6 \end{array}$$

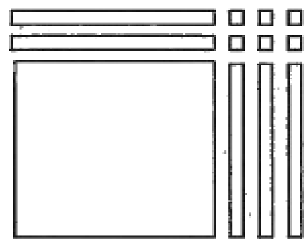
If this were in base X instead of base 10, it would look like this:

$$\begin{array}{r} X+3 \\ \times\, X+2 \\ \hline 2X+6 \\ X^2+3X \\ \hline X^2+5X+6 \end{array}$$

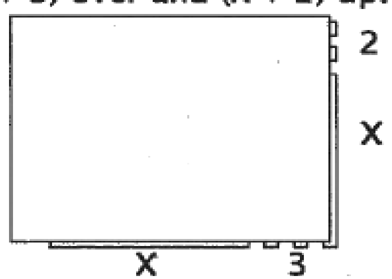
Math-U-See

Multiplication of Polynomials

The area, or product, of this rectangle is $X^2 + 5X + 6$. Do you see it?

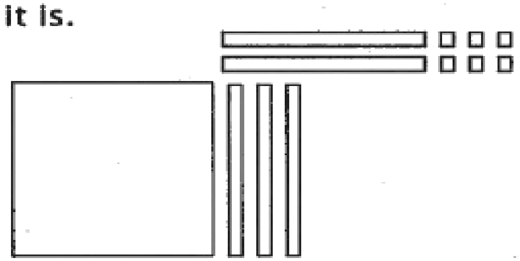


In this rectangle, we cover up most of it to reveal the factors, which are $(X + 3)$ over and $(X + 2)$ up.



The written equivalent of the picture looks just like double-digit multiplication, which it is.

$$\begin{array}{r l} X+3 & \rightarrow \\ \times\, X+2 & \uparrow \\ \hline 2X+6 & \\ X^2+3X & \\ \hline X^2+5X+6 & \end{array}$$

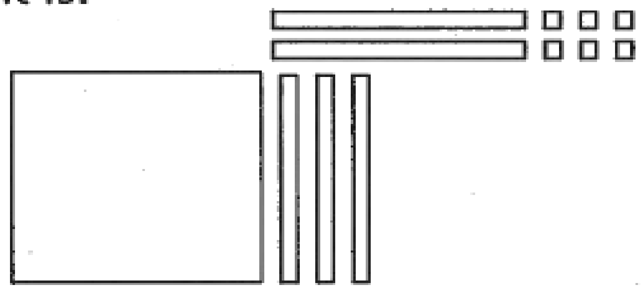


Math-U-See

Multiplication of Polynomials

The written equivalent of the picture looks just like double-digit multiplication, which it is.

$$
\begin{array}{r l}
X+3 & \rightarrow \\
\times\, X+2 & \uparrow \\
\hline
2X+6 & \\
X^2+3X & \\
\hline
X^2+5X+6 &
\end{array}
$$



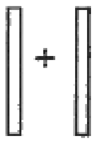
Example 4

$$
\begin{array}{r l c r}
23 & \rightarrow & & 20+3 \\
\times\, 13 & \uparrow & = & \times\, 20+3 \\
\cline{1-1}\cline{4-4}
 & & & 60+9 \\
 & & & 200+30 \\
\cline{4-4}
 & & & 200+90+9
\end{array}
$$

If this were in base X instead of base 10, it would look like this:

$$
\begin{array}{r}
2X+3 \\
\times\, X+3 \\
\hline
6X+9 \\
2X^2+3X \\
\hline
2X^2+9X+9
\end{array}
$$

Math-U-See
Multiplication of Polynomials

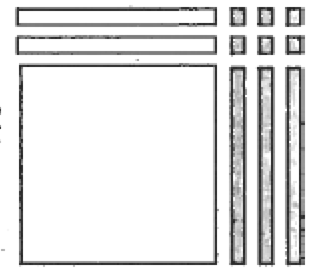


$$X + (-X) = 0$$

X −X

Example 5

$$\begin{array}{r} X-3 \rightarrow \\ \underline{\times X+2} \uparrow \\ 2X-6 \\ \underline{X^2-3X \quad\ } \\ X^2-\ \ X-6 \end{array}$$



X + 2

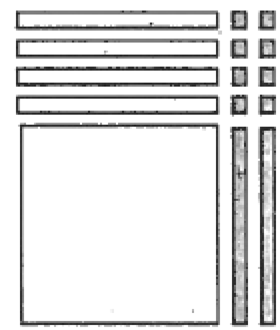
X - 3

Here it is in base 10:

$$\begin{array}{rr} 10-3 & =7 \\ \underline{\times 10+2} & =12 \\ 20-6 & 84 \\ \underline{100-30 \quad\ } & \searrow \\ 100-10-6 & =84 \end{array}$$

Example 6

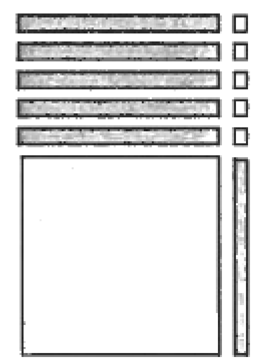
(X − 2)(X + 4)



$$\begin{array}{r} X-2 \rightarrow \\ \underline{\times X+4} \uparrow \\ 4X-8 \\ \underline{X^2-2X \quad\ } \\ X^2-2X-8 \end{array}$$

Example 7

(X − 1)(X − 5)

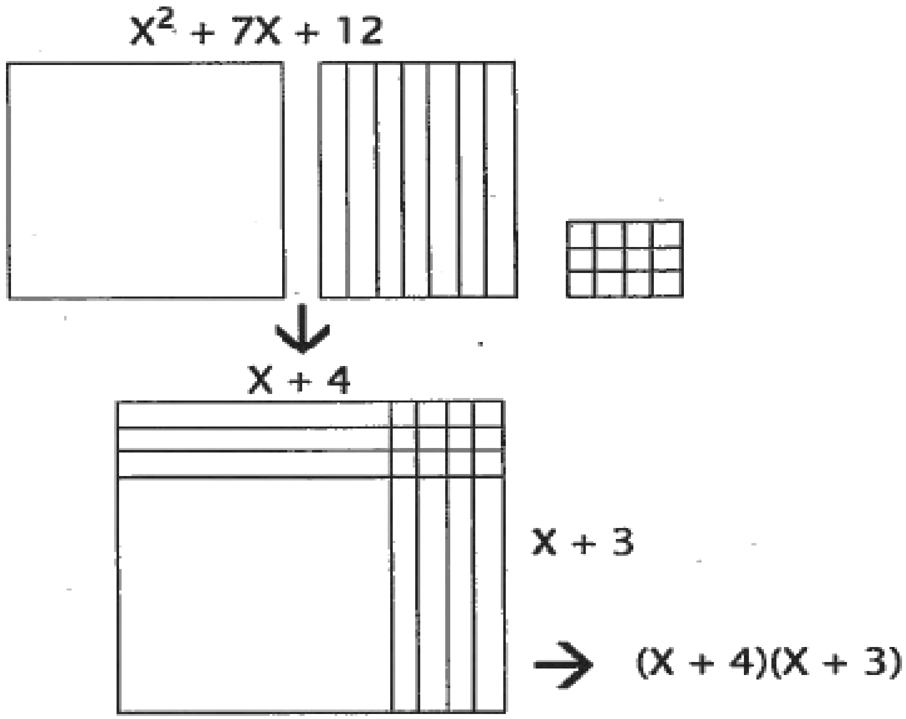


$$\begin{array}{r} X-1 \rightarrow \\ \underline{\times X-5} \uparrow \\ -5X+5 \\ \underline{X^2-\ \ X \quad\ } \\ X^2-6X+5 \end{array}$$

Math-U-See
Factoring of Polynomials

Example 1

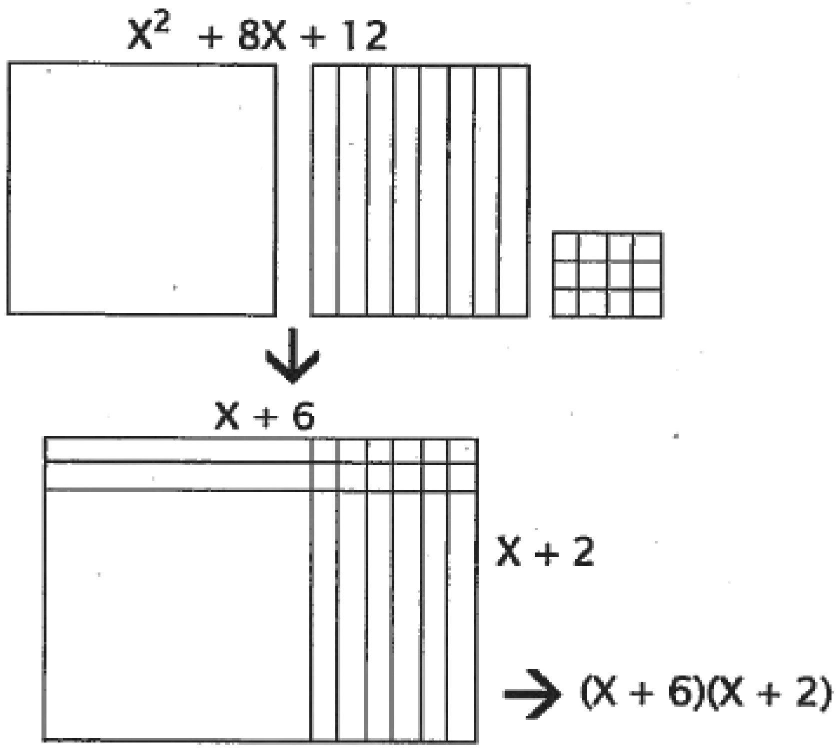
First, build $X^2 + 7X + 12$. This is the product, which is given.
Next, build a rectangle using all the blocks. Then find the factors by reading the lengths of the over dimension and the up dimension.



Math-U-See
Factoring of Polynomials

Example 2

Now find the factors of $X^2 + 8X + 12$. Represent the trinomial with the manipulatives, build a rectangle, and then read the factors.



Math-U-See

Factoring of Polynomials

Example 3

A.

$$\begin{array}{r} 2X+3 \\ \times\ X+2 \\ \hline 4X+6 \\ 2X^2+3X \\ \hline 2X^2+7X+6 \end{array}$$

B. $(X+2)(2X+3) = (X)(2X+3)+(2)(2X+3) = (2X^2+3X)+(4X+6)$

$= 2X^2+7X+6$

Example 4

A.

$$\begin{array}{r} X+\ \ 3 \\ \times\ X+\ \ 4 \\ \hline 4X+12 \\ X^2+3X \\ \hline X^2+7X+12 \end{array}$$

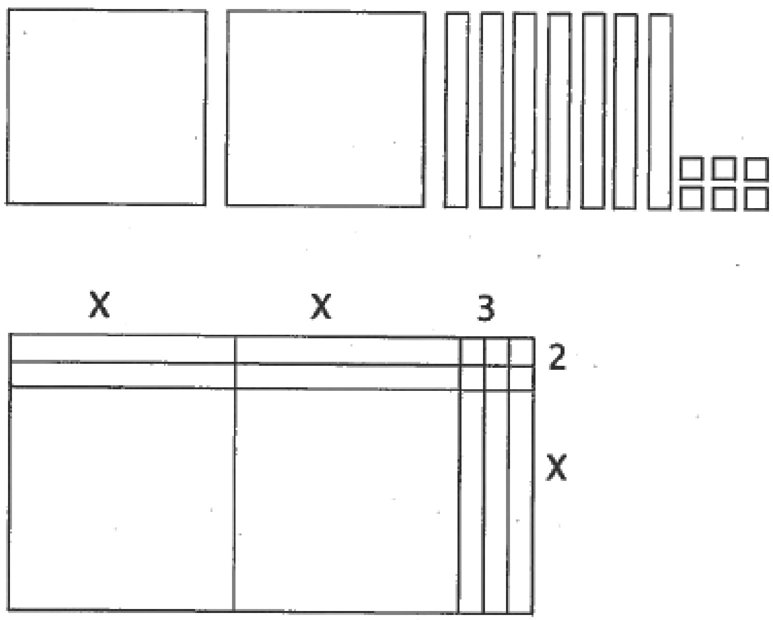
B. $(X+4)(X+3) = (X)(X+3)+(4)(X+3) = (X^2+3X)+(4X+12)$

$= X^2+7X+12$

Math-U-See
Factoring Trinomials

Example 1

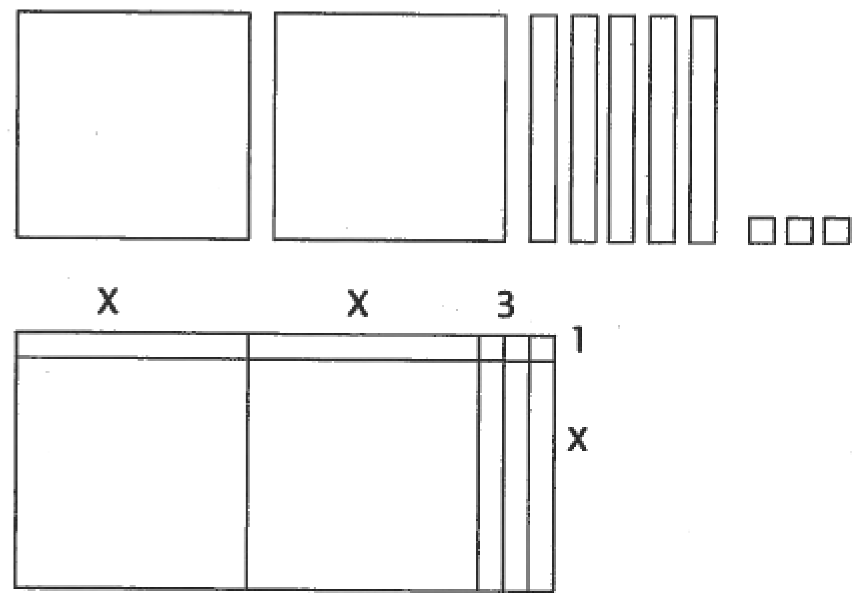
Find the factors of $2X^2 + 7X + 6$.



The factors are $(2X + 3)(X + 2)$.

Example 2

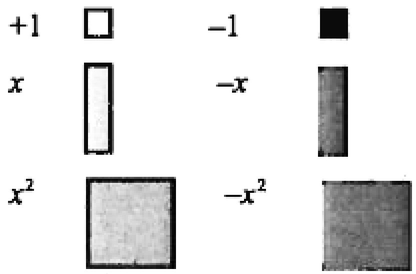
Find the factors of $2X^2 + 5X + 3$.

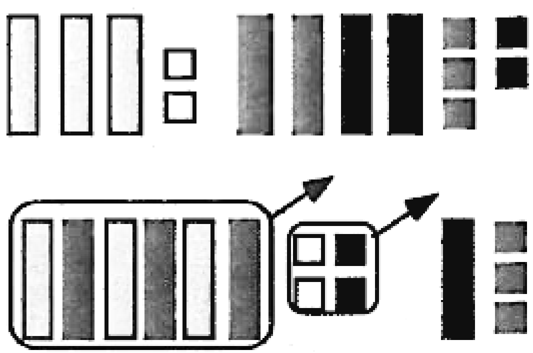


The factors are $(2X + 3)(X + 1)$.

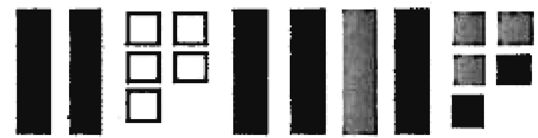
Algebra Tiles
Simplifying Algebraic Expression

Materials:

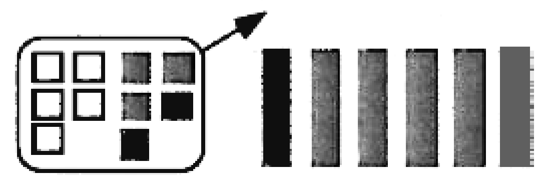




Answer:
$3x + 2 - 4x - 5 = -x - 3$



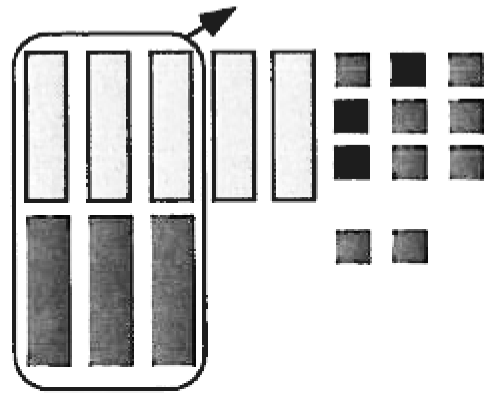
$-2x \quad + 5 \quad - 4x \quad - 5$



Answer: $-6x$

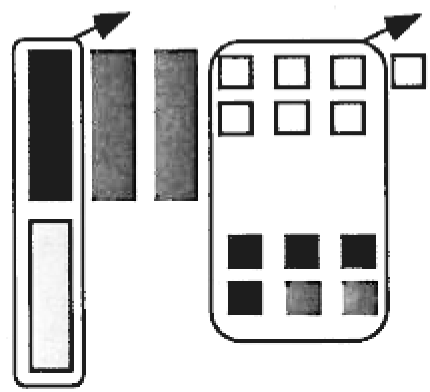
Algebra Tiles
Simplifying Algebraic Expression

$5x - 9 - 2 - 3x$



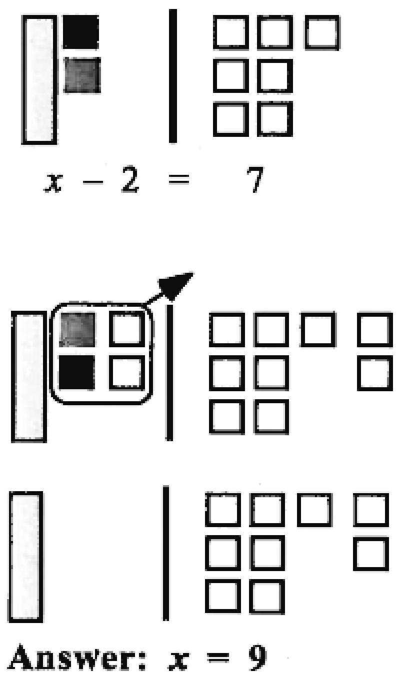
$5x - 9 - 2 - 3x = 2x - 11$

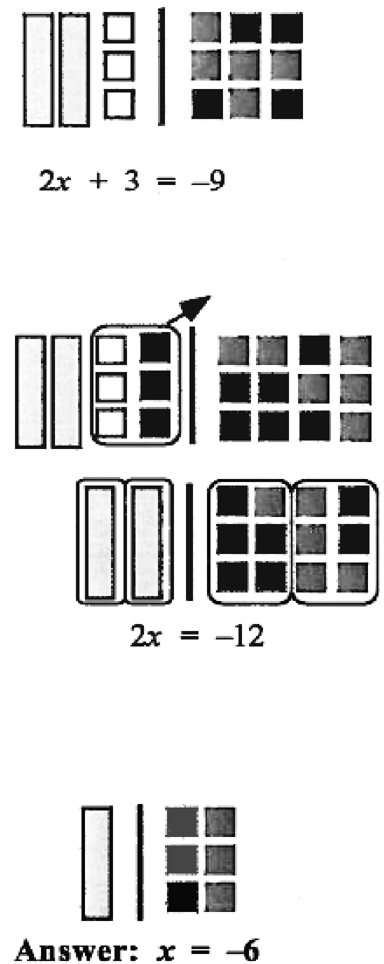
$-3x + 7 + x - 6$



$-3x + 7 + x - 6 = -2x + 1$

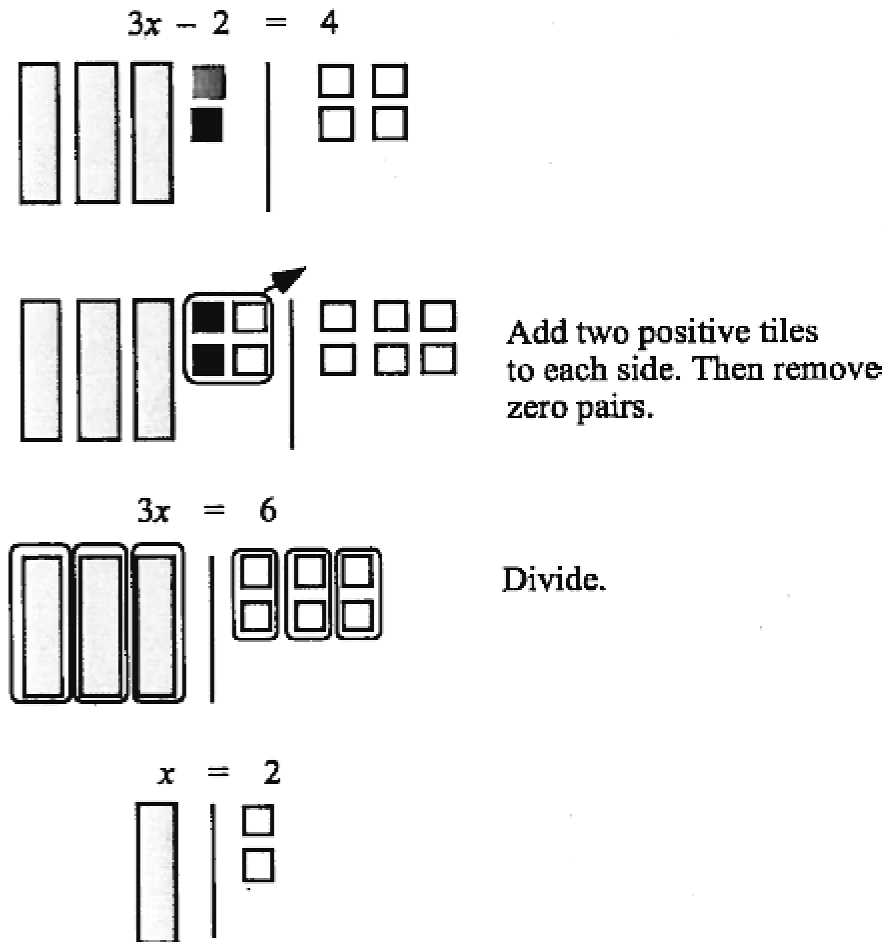
Algebra Tiles
Solving Linear Equations



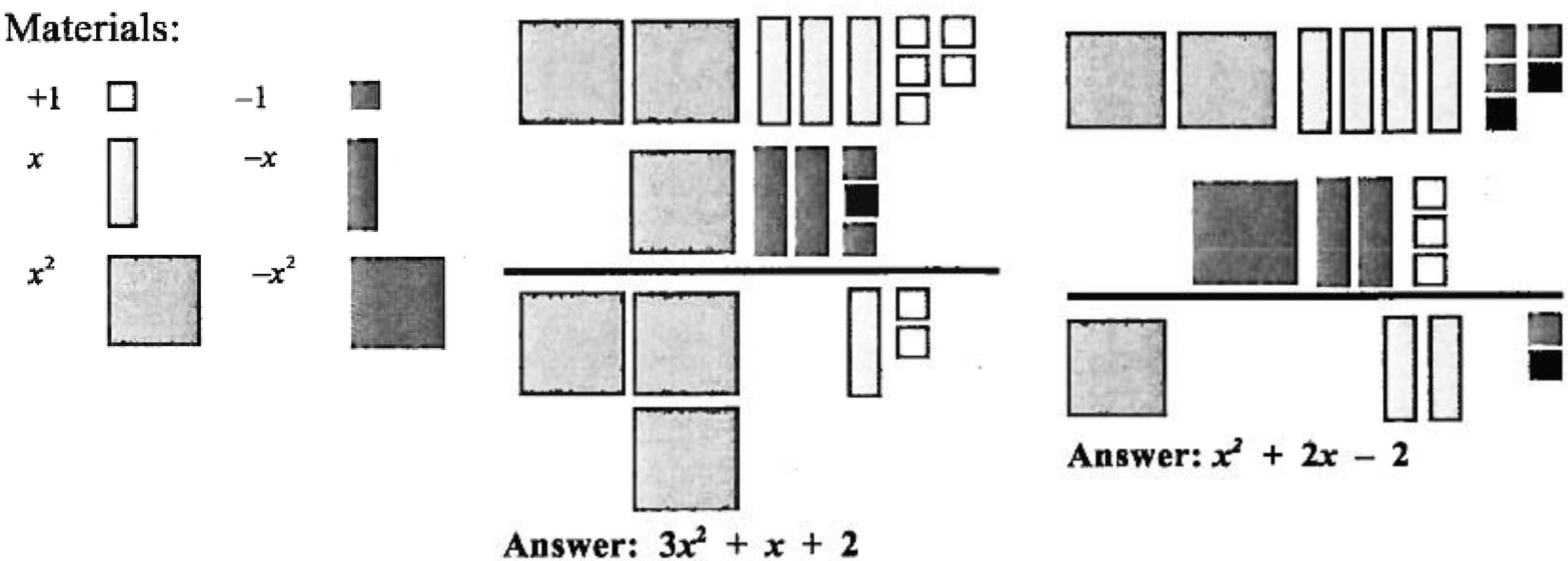


Algebra Tiles
Solving Linear Equations

$3x - 2 = 4$



Algebra Tiles
Adding & Subtracting Polynomials

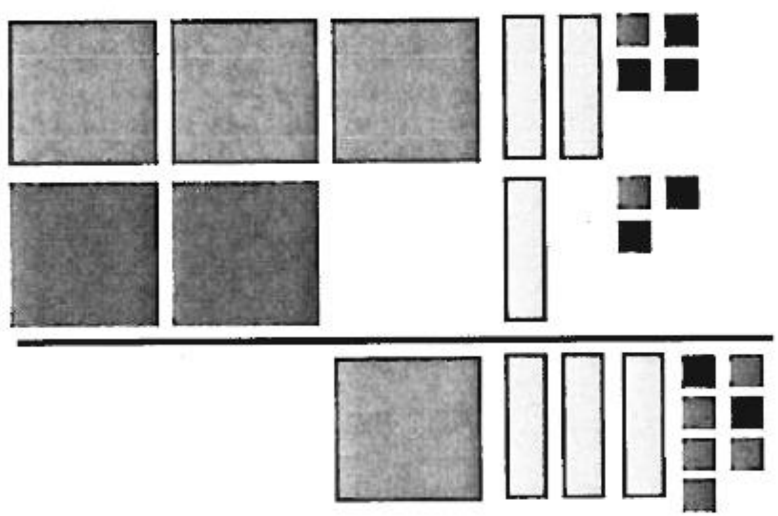


Algebra Tiles
Adding Polynomials

Expression

$$(3x^2 + 2x - 4) + (-2x^2 + x - 3)$$

Model/Answer



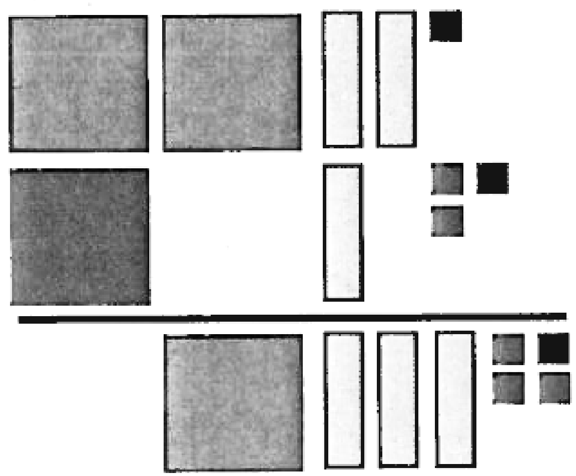
$$(3x^2 + 2x - 4) + (-2x^2 + x - 3) = x^2 + 3x - 7$$

Algebra Tiles
Subtracting Polynomials

Expression

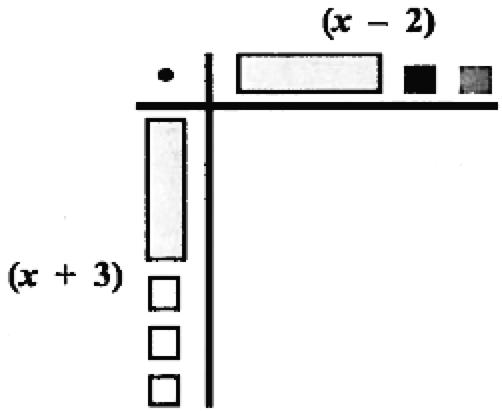
$$(2x^2 + 2x - 1) - (x^2 - x + 3)$$

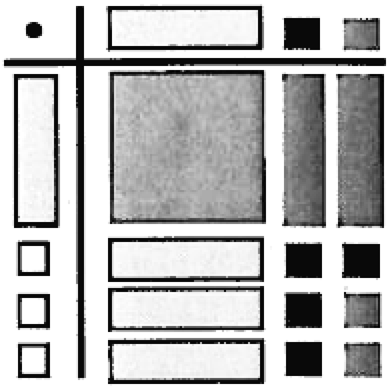
Model/Answer



$$(2x^2 + 2x - 1) - (x^2 - x + 3) = x^2 + 3x - 4$$

Algebra Tiles
Multiplying Polynomials

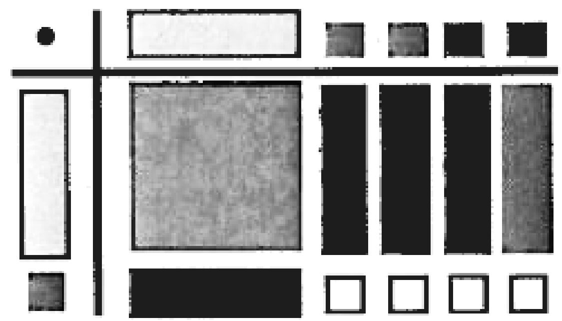




$(x + 3)(x - 2)$

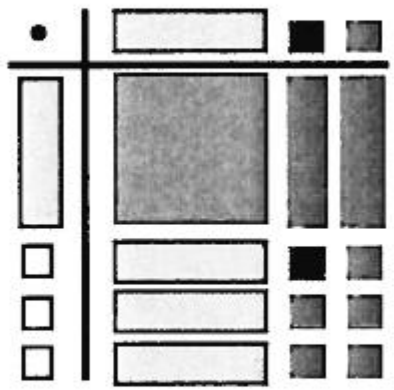
Answer: $x^2 + 3x - 2x - 6 =$
$x^2 + x - 6$

Algebra Tiles
Multiplying Polynomials

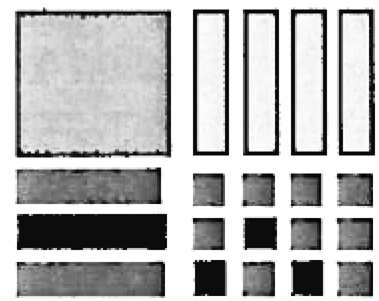
Multiply	Model/Answer
$(x - 1)(x - 4)$	 $x^2 - 5x + 4$

Algebra Tiles

Factoring Trinomials

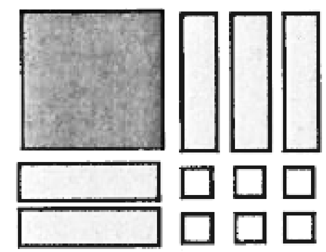
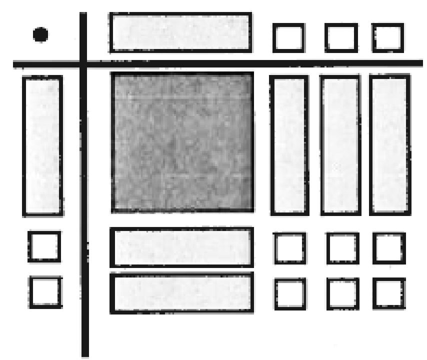


$(x + 3)(x - 2) = x^2 + x - 6$



Answer: $x^2 + 4x - 3x - 12$

Algebra Tiles
Factoring Trinomials

Factor	Model/Answer
$x^2 + 5x + 6$	  Answer: $x^2 + 5x + 6 = (x + 2)(x + 3)$

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