

The Model Shop

Volume 1: Functions from Geometry

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Introduction

Consider this typical math-book problem:

A 5-meter ladder is leaning against a wall. The base of the ladder is 1 meter from the base of the wall. How high is the top of the ladder?

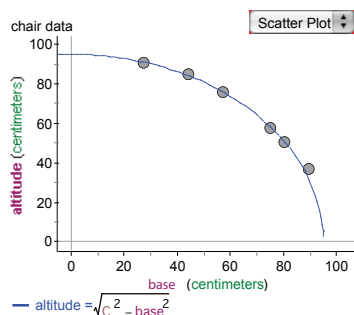
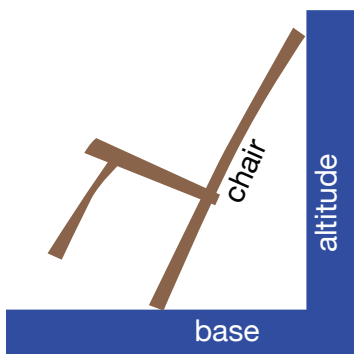
The point of this problem is to get students to use the Pythagorean Theorem. It demands that they combine geometry and algebra: geometry to recognize a right triangle and recall the appropriate formula, and algebra to solve that formula for the relevant side.

The activities in this book help make these connections. And they do it in a special way: through data. Students won't *calculate* the height of the top of the ladder, they'll *measure* it, *with the ladder in different places*. Here's the basic idea:

1. We present a situation with some geometry in it.
2. Students take some measurements, record them in a data table, and plot them on a graph.
3. They figure out a mathematical function that has the same pattern as the data (this is a *mathematical model* of the data).
4. They use the function to answer some sort of question about the situation.

The third step is the hard one. As you will see, there are two ways this can work:

- ❖ You use your understanding of the geometry and of the situation to come up with your model.
- ❖ You find a model using the data, and use that model to help you understand the geometry of the situation.



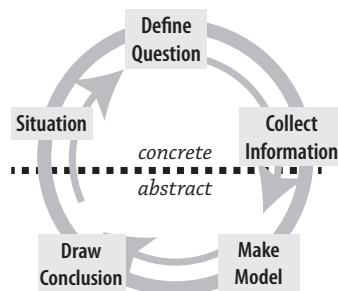
You don't have a 5-meter ladder, so instead of using a ladder, students tilt a chair against a wall (*Tilting Chairs*, page 104). They measure the distance from the wall to the bottom of the chair, and from the floor to the top of the chair—for different tilts. Then they graph those distances and find a function that relates the two quantities. In a way, this is the reverse of the traditional ladder problem. Instead of going from the formula to a specific number, they go from numbers to the formula.

That means that the point is not a particular answer, but rather a *relationship*.

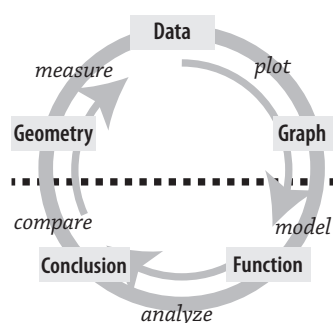
We hope the formulas and the geometry will make more sense because they're about something real.

Real data is messy and confusing. Still, learning to handle real data is important—and it can be fun. This is partly because we will use technology to do a lot of the graphing in this book, and that will take the drudgery out of the process.

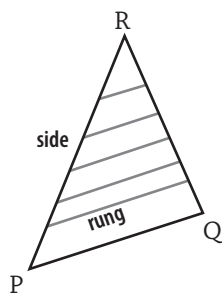
Modeling in This Book



A sample, general modeling cycle.



The modeling cycle, more as it appears in this book.



Modeling has become a big deal in mathematics education. People mean lots of different things by “modeling,” however, so let’s be clear where the activities in this book fit in.

Modeling mavens will often present a diagram of a “modeling cycle” like the one in the illustration. The details vary, but these diagrams capture the essence of modeling. I’ve added a dotted line that separates reality from symbolic math land. That transition—from reality to abstraction—is the vital “modeling step.”

There are lots of ways to make models. In this book, we practice one of the most common ways to use models: we take *measurements*; we treat the measurements as *data*; and we model the data with mathematical *functions*.

This is only one approach, and it’s only part of the cycle. For example, the activities in this book tell the students up front what variables are relevant. That’s very important, but here, students do *not* get to decide what to measure.

That’s OK. When students do a bigger project, they’ll have to do that. But in this book, we want to focus on connecting geometry to data to graphs to functions, for two main reasons:

- ❖ In geometry, there’s not a lot of variability. The model functions are clear, they make sense, and you can make good predictions.
- ❖ Students in intermediate algebra and beyond often have trouble solving problems in geometrical contexts. It can be hard to resurrect spatial understanding from your geometry course and connect it to symbolic expressions.

Then we do two things, quite deliberately. First, we slow the process down and make it more concrete by having students *measure*. And second, instead of simply translating geometry into symbolic expressions, we have designed these activities to go all the way to *functions*—because the advanced courses are about functions.

In a way, it’s the long way around: when you’re more advanced, you look at a figure and say, well, the **side** distance, from R to the rung, is k times the **rung** length. But that may be too mysterious when you’re starting out. So instead we say, let’s measure the figure in a bunch of places. We’ll use those measurements as data. We’ll plot the data on a graph. Then we’ll find a function to fit the data. Huh! They’re proportional! Why would that be? OMG, the triangles are *similar*.

Or you might say, the triangles *look* similar, but are they really? If they are, corresponding sides are supposed to be proportional. What does that mean about the relationship between **side** and **rung**? Do the data fit this theory?

The point here is for students to see that math *works*, even when you approach it from different directions. Students get the same ideas in different guises: multiple perspectives, multiple representations. This makes these activities useful and challenging to students in Algebra 2, Precalculus, and beyond.

Nuts and Bolts **Book Pages.** The activities are numbered. Each activity has a student page followed by one or more teacher pages. Teacher pages include “answers,” or at least a sample graph. Some activities also have templates, or other pages suitable for copying.

Instead of photocopying the student page, you could just project it, and have students work in notebooks if you wish. We have also posted PDFs of the “template” pages at <http://www.eeps.com/modelshop>. Look for resources.

Materials. Students will need rulers and grid paper. Some activities need protractors. Activities about circles may need compasses, or a variety of round objects, e.g., jar lids, bicycle wheels, etc. Other activities need simple objects such as books or cups.

Technology. You need software or a device that lets you enter data and plot points, and then plot functions on the same graph. Fathom and Logger Pro are good examples of standalone software. Wolfram Alpha and Desmos are good examples of web-based tools. Then there are calculators: The TI-Nspire™’s data module is based on Fathom, so most Fathom techniques work, with modifications, on that device. That said, you can enter data into lists, and plot functions, on most graphing calculators. But there are too many models for us to include specific instructions.

In this edition, pictures of example graphs are from Fathom and Desmos.

How These Activities Fit In. There are several ways to assign these activities.

First, I have used them as in-class activities. All students take measurements, enter data, and find models. The instructor circulates during this time, nudging and asking questions. Then, at the end, you lead a discussion to wrap up the experience, helping the students discuss what challenges they faced and what mathematics they learned. The teacher notes have suggestions for discussion topics or closing tasks.

Second, the activities can be mostly homework. I have used them as weekly write-ups, kind of like the “problems of the week” that many teachers use. That way students have more time and have a chance to write mathematically. You can also give students the discussion questions as examples of issues they might address in their write-ups.

This second plan can be especially appealing. We look for ways to do larger pieces of work, like projects, but a more open-ended “math project” can be difficult and frustrating—for everybody. These activities, in contrast, help students develop their mathematical chops, but are contained. Since they’re smaller than projects, you can do more than one in a term, giving students a chance to improve—and giving you a chance to figure out what their work says about their understanding.

Prediction The student pages of this book often ask them to predict. We usually ask, “What will the relationship look like?” In these activities, that means,

Before you take any measurements, think about the situation and sketch the graph you think you will get when you actually plot measurements.

Prediction is hard for students. To make sure it gets done, one strategy is to collect the predictions the day before students begin to take measurements.

These predictions serve several purposes:

- ❖ They help students think about the situation beforehand.
- ❖ They give students a chance to be surprised when their predictions are wrong.
- ❖ They give you, the teacher, a chance to make sure students understand what’s being asked.

Student predictions often start out being terrible, but they improve with practice. Help students decide what makes a good prediction. This can start with something as simple as “the graph has labeled axes” or “the graph shows realistic values.” As students get more experienced, they get better at following graphing conventions, estimating distances, and inferring the shape of relationships. Eventually, they can even start predicting equations for their models.

As a teacher, you can help make prediction meaningful and effective.

- ❖ At the end, always ask students to compare their predictions to reality. If you do reflective writing, this is a good topic.
- ❖ Be sure to ask students what was *right* about their predictions. Only then ask how they could have made better ones.
- ❖ Finally, ask about surprises: where did the graph turn out differently than they thought?

Good Modeling Habits Working with models and data is emerging as an important skill. There are important “habits of mind” to adopt that may never have been part of your math curriculum. Here is a partial list:

- ❖ **Check limiting cases.** It often helps to measure and reason about special cases, especially ones at the edges of possibility. The model you create must work properly at these special cases—and it’s easy to check that out.

So if you’re doing that Pythagorean ladder problem, your formula had better work when the ladder is up against the wall, and when the ladder is flat on the ground.

- ❖ **Look at residuals.** When you create a mathematical model to fit data, look at the residuals. If the model is good, there will be no pattern in the residuals—they will look random—and they will be centered around zero.

Fathom or the N-spire will make a residual plot for you automatically. But even with Desmos or other tools, you can pay attention to whether there is a pattern in the way points are above or below your model function.

- ❖ **Test your function.** If you can, use your function to predict a measurement you have not yet made. It might be the next element in a pattern, or the length of some segment you have not yet drawn.
- ❖ **Enter only the data.** Whenever possible, enter actual measurements, and let technology do all the calculations. For example, if you're working with circles, and you actually measure the diameter, enter the diameter. Do not divide by two and enter the radius. Instead, make a new column for radius in your table and make the computer divide by two.
- ❖ When working symbolically, **leave your equations “in letters”** as long as possible. More about this shortly.

“Parent” Functions and Transformations

You can introduce students to a wide variety of functions through their “parent” functions. These are prototypes for a whole family of functions. For example, quadratic functions are based on the simplest of all quadratics, $f(x) = x^2$. Sinusoids are based on $\sin x$ and $\cos x$. You might begin learning about exponentials by looking at $g(x) = 2^x$, but when you're more sophisticated, you'll probably use $h(x) = e^x$ as the root, parent function.

If students learn the overall properties and characteristic shapes of these functions, they can judge, from the shape of a graph of real data, what functions might fit.

Of course, the parent function itself rarely fits. It might have the right general shape, but the details will be wrong. So you have to transform it: give it translations and dilations in x and/or y .

This is not the place to describe all of them in detail, but just so you remember what I'm talking about, at left is a sine expression with transformations labeled.

A sine function expression with transformations labeled: $A \sin(b(x-h)) + k$. The labels are: **A** (vertical stretch), **b** (horizontal squish), **h** (horizontal slide), and **k** (vertical slide).

The activities in this book let students apply their growing understanding of transformations.

To see which types of functions apply to which activities, see *Activities and Functions* on page 176.

Parameters

To write a transformed function, you have to enhance the “parent” function by including additional values and operations.

These values change the generic, vanilla parent function into one that applies to this particular situation. They’re *parameters*. Parameters characterize the function, so they’re the same for every case in the data set.

Students often think of these numbers as “constants.” Calling them “parameters” may be new and confusing. So let’s talk about how to cope with parameters in this book.

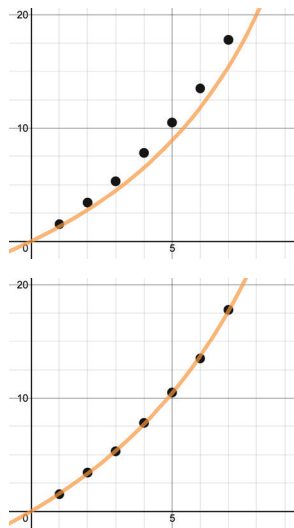
Both of the graphs below try to fit the data with the model

$$f(x) = d \frac{x}{L - x} .$$

The model has two parameters: d and L .

In the top graph, $d = 18$ and $L = 16.4$.

In the bottom one, $d = 24$. A better fit!



First coping strategy: whenever possible, express a parameter *symbolically*. Make it a letter (or a word), not a number. In the function in the margin, d , and L are parameters. Our software platforms, Desmos and Fathom, both express parameters this way.

If you use symbols for parameters, you can make your function come alive. Make *sliders* to represent the parameters. Sliders are a huge pedagogical win, because students can see, dynamically, how changing a parameter value affects the shape of the function. To fit a function, students vary the parameters by sliding them until the function fits the data.

Many students will want to edit the numbers in an expression and re-draw the function. This is slower, less dynamic, and misses “kinesthetic” understanding.

Second coping strategy: Consciously distinguish these symbolic parameters from variables. In $f(x) = A \sin x$, A and x are both letters, but A controls the shape and position of the function, whereas x goes on the axis. When you measure and plot an individual data point, you use a variable—like x . Also: even though we urge students to use letters other than x and y for variables, it sometimes helps students to think about which of the symbols is the “ x .” In contrast, a letter that could be a constant number, like A , is a parameter.

Students should be aware, though, that in some problems, parameters and variables change roles. In a problem where you calculate m and b (slope and intercept) for a linear relationship, those parameters briefly become variables.

Finally: Always look for meaning. A parameter’s value almost always means something about the situation. It could be a rate, or a special length, and area—it depends on the situation and the function. You can’t always figure it out or express it, but whenever you can, you should.

Actual Constants

Occasionally you'll come across a mathematical model with actual constants that are not parameters. In *Cardboard Circles*, for example, students weigh cardboard circles and compare the mass (grams) to the radius (centimeters). The relationship is quadratic:

$$m(r) = kr^2.$$

This is fine, but we want meaning for our parameters, and it's not obvious what k means. On the other hand, if we expressed it like this:

$$m(r) = k\pi r^2,$$

now k is the area density of the cardboard in grams per square centimeter. It will have a different—smaller—value, but it has a clear meaning. The number π , in this case, is *not a property of the particular situation*. It's a constant, not a parameter.

If we had measured diameter instead of radius, the formula would be

$$m(r) = k\pi(D/2)^2.$$

In this case, 2 is also a constant. The radius is always half the diameter, no matter what kind of cardboard we use.

Which Variable Goes on Which Axis?

Traditionally, we put the independent variable (also called “explanatory” or “predictor”) on the horizontal x -axis, and the dependent, “response” variable on the vertical y -axis.

This works well when there is one response for every predictor, that is, when you can describe the relationship as a function. For example, in *Cornbread Are Square*, the radii of the circles are the predictors so they go on x ; the areas are the responses so they go on y . Then the model is a function, specifically, $f(r) = \pi r^2$.

You can also think of it as the area depends on the radius (and not the other way around), so the area is the dependent variable.

Even so, these roles sometimes switch. Imagine an activity in which you crowd a bunch of people together and put a rope around them (this is an extension to *Bundles* on page 67). You want to know how the length of the rope depends on the number of people. If you add more people, it adds to the area they take up, and increases the circumference—the length of the rope. In this case, the circumference (and radius) depend on area. We'd use the same $A = \pi r^2$ relationship to understand the situation, but we could sensibly write it as

$$r = \sqrt{\frac{A}{\pi}} \text{ or } C = 2\sqrt{\pi A}$$

(In the activity, you might substitute $n\alpha$ for A , where the parameter α is now the “effective area” of one person.)

Having students switch the axes and talk about the resulting graph is a great way to help them become more flexible about data and the meaning of graphs. Of course, switching the axes for the *data* is one step, but how do you “flip” the function? Aha! This is a terrific chance to talk about *inverse functions*. We have noted this explicitly in a few activities, but it’s true everywhere. Also, be alert for issues of domain and range. When you have real data from geometry, restrictions happen naturally—all the time.

About Precision

How accurately do you have to draw and measure to make these activities work? Not all that accurately, but even college students may surprise you with flagrantly wrong diagrams and measurements. Angles are especially hard, but even distances can be problematic. A junior in high school should be able to measure the length of a segment to within about a millimeter, but some will still, for example, measure from the physical end of the ruler rather than from the “0” marking.

So turn this into an opportunity! Students can usually get something that kinda-sorta-mostly works, but you can challenge them to measure accurately enough that it clearly-most-sincerely works.

Does this mean they have to (gasp) re-measure as many as ten or twelve line segments? And re-enter them into a computer? Yes. They’ll live.

5. Opposite Sides of the Ruler

In this activity, you will relate the numbers on the opposite sides of your ruler to one another.

What to Do

Take a traditional U.S. student ruler—the kind with inches on one side and centimeters on the other. For at least seven spots on the ruler, record what numbers are opposite each other.

Do not convert the units! Simply record what numbers are opposite. For example, in the illustration, 8 (inches) is just about opposite 10.1 (centimeters).



How will **inches** be related to **centimeters**?

☐ Predict: What will the relationship will look like? Sketch the graph you think you will get:

- ☐ Record at least seven measurements of **centimeters** and **inches**. For each measurement, pick a spot on the ruler and record the numbers from the two sides of the ruler.
- ☐ Plot **centimeters** (on the vertical axis) against **inches**.
- ☐ Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameters.
- ☐ Use your formula to calculate the **centimeters** for some **inches** you haven't measured yet. What do you get? Check the ruler: is it right?
- ☐ Be sure you can explain why the form of your function makes sense.
- ☐ According to your formula, how many inches do you get for one meter (100 cm)? Look up the conversion between inches and meters and explain what happened.

inches	centimeters

Opposite Sides of the Ruler • Instructor Notes

In this activity, students compare the centimeter numbers to the inch numbers across their rulers. In the photo below, for example, see how 8 **inches** is kind of opposite 10 **centimeters**? That's one data point.

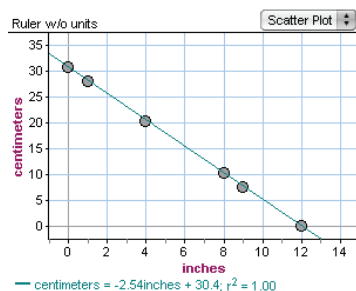
For most rulers of this type, the inch and centimeter scales run in opposite directions. This means that although the relationship will be linear, larger numbers on one scale will match up with smaller numbers on the other, giving a negative slope.

Here are some important questions for discussion:

- ❖ What goes on the y axis?
- ❖ What are the limiting cases? Why are they important?
- ❖ If you find a number outside the regular domain, such as a point with **inches = 15**, what could it mean?

The Units Issue (Fathom)

If you use Fathom, some students will get a slope of about -2.54 , some get a slope of -1.0 , and some get a slope of -0.393 . What's happening? This turns out to be an interestingly deep question.



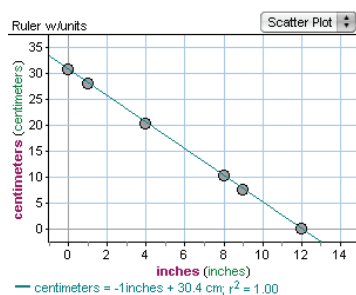
In the top illustration, the slope value means that there are 2.54 centimeters in one inch. But look at the bottom illustration. It's the same data (minus a few points) but now the slope is -1 . The data even have the same values.

How can the data have the same numbers and the computer gives you a different slope?

It's because in the bottom example, the student entered the data *with units*. Let's compute the slope in the top example between the two endpoints, which are (0, 30.5) and (12, 0): we get $(0 - 30.5)/(12 - 0)$, or 2.54. But in the bottom example, the same calculation has units, so we get

$$m = \frac{(0\text{cm} - 30.5\text{cm})}{(12\text{in} - 0\text{in})} = -\frac{2.54\text{cm}}{1\text{in}}$$

But 2.54 cm is the same as 1 inch, so the value of the fraction is 1.0, and the slope is -1 . The first example relates the *numbers*, but the second relates the *distances*. Let that sink in!



Two illustrations showing typical results. Both show least-squares lines. In the top one, students just entered numbers. In the bottom, they entered numbers with units.

8. Isosceles Angles

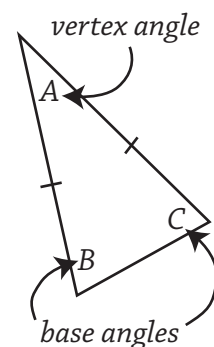
In this activity, you'll study the angles in an isosceles triangle.

What to Do

Make some isosceles triangles. Your instructor will tell you what tools to use. But if you're doing this on your own, you could just draw the triangles freehand, as accurately as you can "by eye."

You will need a protractor to measure the angles.

- ☐ Sketch some isosceles triangles, as large as is practical given your paper. Sketch a variety—some with small (acute) vertex angles, and others with large (obtuse) vertex angles. You need at least five triangles.
- ☐ Label each triangle with a number, and label the vertices *A*, *B*, and *C*, where *A* is the vertex—the point where the two like sides come together.
- ☐ Measure all the angles. Write their values in the angles, and record them in a table. The column headings should be **number**, **A**, **B**, and **C**. (The number of the triangle is **number**.)



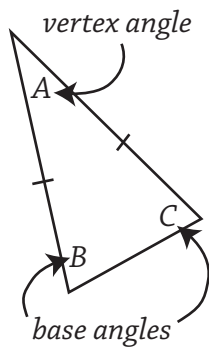
How will **A**, **B**, and **C** be related?

- ☐ Predict: What do you think the relationship will look like? Since you have three variables instead of two, give some thought to the best ways to express your prediction. It may be that you have more than one graph.
- ☐ Sketch the graph(s) you expect to get here:

- ☐ Find and explain mathematical functions that fit the points. Be sure you can explain the meanings of any parameters. For example, if your equations have any coefficients, why do they have to have the values they do?
- ☐ Test your predictions: how good were they? How well do the data support what you claimed?

Explore: if you drew your figures freehand, without a ruler, what difference would it have made if you had been able to make your triangles perfect?

Isosceles Angles • Instructor Notes



This activity is about the angles in triangles. Students create a variety of isosceles triangles. They measure A , the vertex angle, and B and C , the two base angles.

Many students know that $A + B + C = 180^\circ$, and that $B = C$. For those students, this is a perfect opportunity to make really good predictions.

The extra, open-ended challenge in this task is to deal with three variables. If you make a graph, what do you put on the axes? One good strategy is to make *two* graphs: one with B against C and another with A against B or C .

Some students may try to make a single graph that relates all three variables. This is an interesting challenge; if a student or a group seems to be getting bogged down, however, consider suggesting that they do something less ambitious (i.e., use two “normal” scatter plots). See if they want to pursue the 3-variable graph elsewhere—maybe as extra credit.

Some students may give up on graphs entirely and rely instead on formulas for their predictions. Their challenge will be to evaluate their predictions without a graph: how can they tell if their guess even has the right shape? It’s not impossible; for example, they might predict that the sum of the angles is 180° , and test the idea by adding the three angles and comparing. This is fine, but in the debriefing, ask the class what you get out of a graph (or a set of graphs) that you don’t get out of calculation.

Many students are perplexed by the graphs they get. It could be that these graphs do not immediately “look” like the relationships—at least not the way they usually think of them.

This is a great chance to connect their spatial, geometrical understanding with their understanding of relationships expressed numerically, graphically, and symbolically.

Measuring Angles

Measuring angles well seems to be a challenge even for otherwise accomplished students. If your classroom is like many, you have a motley collection of protractors, and students are inexperienced. This activity gives them some needed practice.

The biggest danger is angles slightly larger than 90° ; if students record a 95° angle as 85° , it can mess up their data. (This can be a good thing if students see that there is something wrong with the point that doesn’t fit in the graph; then they can re-measure and fix the problem.)

Isosceles Angles • More Instructor Notes

Simplifying the Task If all this open-endedness is too much for less-experienced students, you could simplify the task in two ways:

- ❖ Have students ignore angle A and study the relationship between B and C , that is, have them “discover” that the base angles of isosceles triangles are equal; or
- ❖ have students focus on the relationship between A and either B or C . Then they will find that $A = 180 - 2B$ (or some equivalent expression).

You could also make students’ lives easier (but a little less fulfilling) by giving them a template to help them make their diagrams. See the *Vertex Angles Diagram* on page 154.

Recognizing Special Numbers When many students get lines to fit their points, they will get functions like one from the next page:

$$B = -0.51A + 89.$$

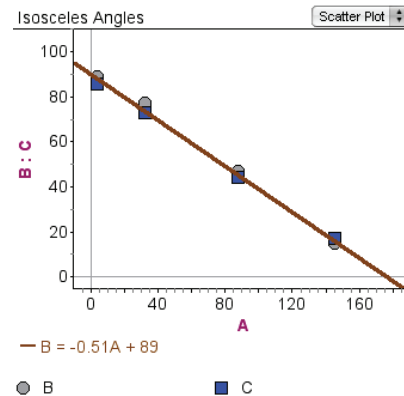
It is a sign of growing mathematical maturity to ask whether that 0.51 might *really* be 0.50, and (since it’s an angle) the 89 is really 90.

Then the student should try out the function with the special numbers and see whether it still fits well.

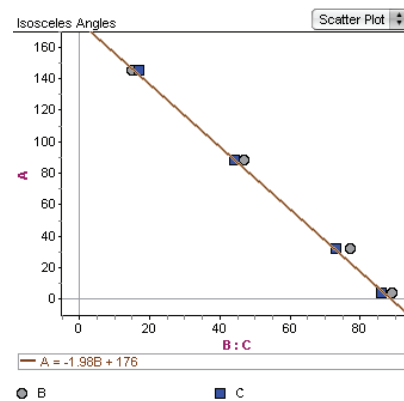
- Discussion Questions**
- ❖ What are the limiting cases in this activity? Where do they appear on the graphs?
 - ❖ What is the slope on the graph of B against C ? What significance does it have?
 - ❖ What are the y -intercepts of your graphs? What do they mean?
 - ❖ What are the x -intercepts of your graphs? What do they mean?
 - ❖ Are the constants (or parameters) in your equations close to any important numbers? If you used the important numbers instead, how far off would your lines be?
 - ❖ If you plot A against B or C , what goes on the vertical axis? What function do you get? How does the function change if you put a different variable on the vertical axis?

Isosceles Angles • Still More Instructor Notes

Results In this example, the group has put **A** on the horizontal axis and plotted *both* **B** and **C**. They have not, however, looked at limiting cases (or measured five triangles as instructed!).



Here we have taken their same graph and put **A** on the vertical axis:



Notice the function in the bottom illustration. Instead of being approximately $B = 90 - A/2$, it's $A = 180 - 2B$. Both are correct, but it's easier to see why the second one is true.

It's interesting that both models are good: they fit the points well. But we actually prefer the **A** = version in most circumstances because it communicates better, and connects better to important geometrical concepts.

It's even easier to see if you use algebra (gasp!) and rearrange it:

$$A + 2B = 180.$$

Of course! Angles add to 180. Base angles are equal.

9. Triangle Ladder

In this activity, you'll study the lengths of the "rungs" of a "triangle ladder."

What to Do

Make a triangle ladder!

- ☐ Sketch a triangle, a pretty big one, on a sheet of paper. Don't use a ruler, but draw it carefully. Label the vertices P , Q , and R . It does not have to be a special triangle (e.g., isosceles).
- ☐ Make at least five segments parallel to $\overline{PQ}$ that extend from $\overline{PR}$ to $\overline{QR}$. These are the "rungs."
- ☐ Measure and mark down the lengths of all the rungs.

The length of that **rung** will depend somehow on a measurement along the **side** of the triangle (that is, along $\overline{PR}$ or $\overline{QR}$).

- ☐ Decide what to use for your **side** distance. Write down, briefly but clearly, how to measure **side**.

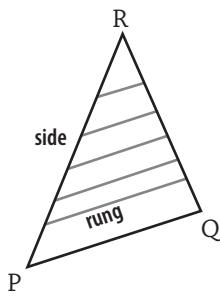
How will **rung** be related to **side**?

- ☐ Predict: What do you think the relationship will look like? Sketch the graph you think you'll get:

- ☐ Record measurements of **rung** and **side**, and plot **rung** against **side**.
- ☐ Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- ☐ Explain why the *form* of your function makes sense. If you've studied geometry, you should be able to explain it using geometrical vocabulary.
- ☐ Test your function! Make a new rung, and measure **side**. Then use your formula to predict the **rung** length. Measure and check. How did you do?
- ☐ Find someone who had a different definition of **side**, and compare your two functions. Figure out how they're related to each other.

Explore: how much did it matter that you drew the diagram freehand, without a ruler?

Triangle Ladder • Instructor Notes



In this activity, students make a “triangle ladder” like the one in the illustration. They measure the lengths of the **rungs** of the ladder, and distances along the **side**.

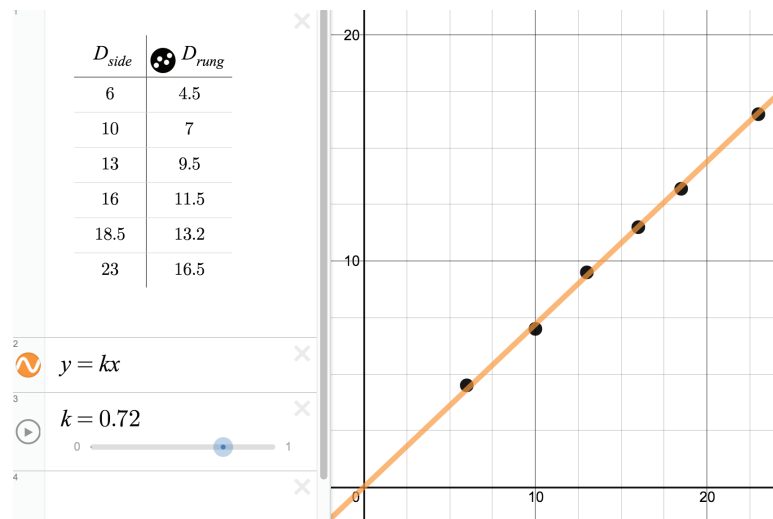
In theory, seeing the relationship is easy for any student who has studied similar triangles, but in practice it can be confusing. That’s why it’s so important to look at this utterly typical geometrical situation. Using data might give some students the perspective they need.

The problem is that it’s easiest to measure the distances *between* the ladder “rungs”—and those don’t have any particular relationship to the lengths of the rungs themselves.

Instead, students have to see that the relevant distances are the total, cumulative distances, either from the vertex or from the base. If students measure from the vertex, they get a direct proportion. If they measure from the base, they get a linear relationship with an intercept (and the intercept is the length of the base).

In this Desmos plot, the student measured **side** from the vertex R , along $\overline{RP}$ to each rung. This is plotted as the x variable, which is D_{side} in the table.

You can also see the function, $y = kx$, and the slider for k , showing a value of 0.72. Notice that the student has changed the limits of the slider from the default $[-10, 10]$ to $[0, 1]$. Presumably the bigger limits made finding a good value too “chunky.”

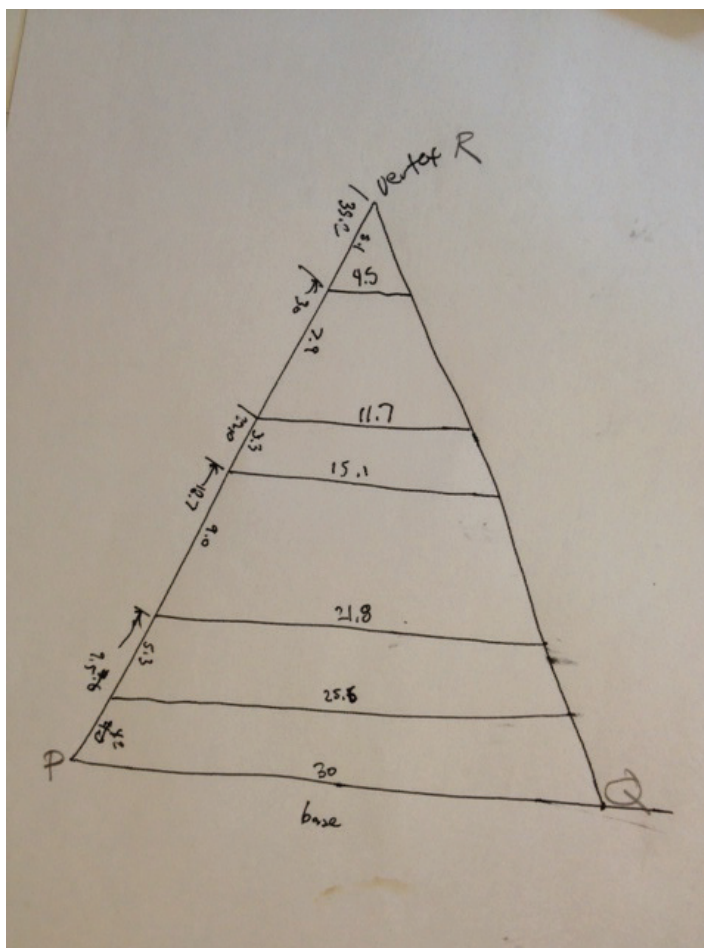


Discussion Topics

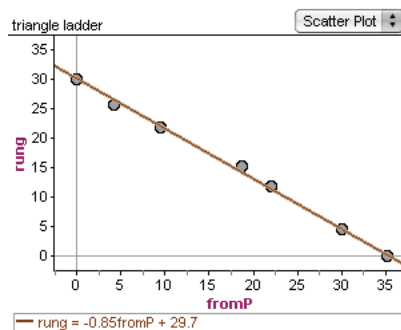
- ❖ How did you figure out what to measure for **side**?
- ❖ Why do some people have positive slopes and some have negative?
- ❖ What do the slopes mean? Could you figure out the slope simply by measuring the big triangle? How?
- ❖ If you made both graphs (one with **side** measured from the vertex, and the other measured from the base), how would they be related?
- ❖ Which way of measuring **side** gives you a slope and intercept that you can explain more easily?

Triangle Ladder • More Instructor Notes

Results Here is a student picture of such a ladder:



Below is a graph of data from this diagram. The student used **fromP** (P is a point on the base) instead of **side**, which is fine. Since they are measuring from the base instead of from the vertex, they get an intercept and a negative slope.



Paragraphs (Same Font Size) Handout

Each paragraph has the same text and font, but a different width.

Measure these paragraphs to gather data. Measure height and width. Plot the data and find a function that fits as well as possible.

Yonder, by ever-brimming goblet's rim, the warm waves blush like wine. The gold brow plumbs the blue. The diver sun—slow dived from noon—goes down; my soul mounts up! she wearies with her endless hill. Is, then, the crown too heavy that I wear? this Iron Crown of Lombardy. Yet is it bright with many a gem; I the wearer, see not its far flashings; but darkly feel that I wear that, that dazzlingly confounds. 'Tis iron—that I know—not gold. 'Tis split, too—that I feel; the jagged edge galls me so, my brain seems to beat against the solid metal; aye, steel skull, mine; the sort that needs no helmet in the most brain-battering fight!

Yonder, by ever-brimming goblet's rim, the warm waves blush like wine. The gold brow plumbs the blue. The diver sun—slow dived from noon—goes down; my soul mounts up! she wearies with her endless hill. Is, then, the crown too heavy that I wear? this Iron Crown of Lombardy. Yet is it bright with many a gem; I the wearer, see not its far flashings; but darkly feel that I wear that, that dazzlingly confounds. 'Tis iron—that I know—not gold. 'Tis split, too—that I feel; the jagged edge galls me so, my brain seems to beat against the solid metal; aye, steel skull, mine; the sort that needs no helmet in the most brain-battering fight!

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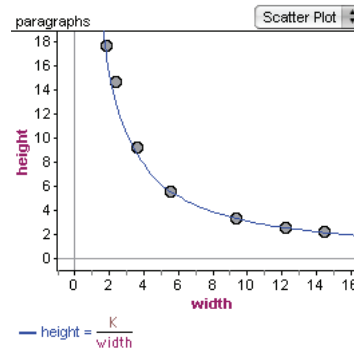
Yonder, by ever-brimming goblet's rim, the warm waves blush like wine. The gold brow plumbs the blue. The diver sun—slow dived from noon—goes down; my soul mounts up! she wearies with her endless hill. Is, then, the crown too heavy that I wear? this Iron Crown of Lombardy. Yet is it bright with many a gem; I the wearer, see not its far flashings; but darkly feel that I wear that, that dazzlingly confounds. 'Tis iron—that I know—not gold. 'Tis split, too—that I feel; the jagged edge galls me so, my brain seems to beat against the solid metal; aye, steel skull, mine; the sort that needs no helmet in the most brain-battering fight!

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Paragraphs (Same Font Size) • Instructor Notes

Students get a page with seven paragraphs, all the same text, all the same font size—but different widths.

This is a good introduction to nonlinear functions. It's about a phenomenon every student has experienced: when you squish the margins, the paragraph gets taller.



An example of pretty good data, with a function plotted. Here, K is about 31.

Materials and Procedure

This activity uses simple materials: *Paragraphs (Same Font Size) Handout* on page 55 and rulers.

Explain the situation and show students the paragraphs. Have them predict what the graph will look like before they measure. Then they can enter data and find a good function to model it.

Finally, compare their predictions to what they found and, most importantly, have a discussion about the form of the function.

The Form of the Function

It's not obvious to most students from the data that the function has this $1/x$ form. They will often suggest parabolas or exponentials.

But in fact, the *form* of the function makes sense. This hyperbola doesn't just fit the points: it arises naturally from the situation. After all, each paragraph has the same text. Therefore the letters take up the same area. If you ignore the interesting problem of the word wrap, each rectangle has the same area. And if that's the case, the *height* of the rectangle is

$$\text{height} = \frac{\text{area}}{\text{width}},$$

which is precisely the form that fits the data pretty well. And the parameter K in the illustration has a meaning: it's the area.

Paragraphs (Same Font Size) • More Instructor Notes

Understanding though Scissors Cool demo: Cut out each of the paragraphs so they're rectangles. Now superimpose them so that their lower-left corners are in the same place. Now look at the upper-right corners: they're the graph!

- Questions for Discussion**
- ❖ What decisions did you have to make about measuring the paragraphs?
 - ❖ What goes on the vertical axis? Why?
 - ❖ How was your prediction different from reality? How was it the same?

Word Wrap The amount of text in each line varies because of word wrap. This introduces some variability in the data, which is lovely because we want to give students plenty of experiences with models that don't line up perfectly with the points.

But it also lets advanced students take this to the next level. If you make a residual plot of the model, you'll see that the first two points—the tallest paragraphs—don't fit the model as well as the rest. If you fit the last five, the first two don't work so well, and if you split the difference you see a bowed trend.

Why? They should figure this out, but I'll tell you my theory: word wrap affects short lines more than long ones. And there are ways to characterize that effect, with another parameter, that make for a better model. I won't spoil your fun any further.

Units in Fathom If you use Fathom, and enter your measurements with units, you'll probably get a **#units incompatible#** error for your function (though it will still plot).

Ask: what units does **K** need so that you have **cm** on both sides? The answer, **cm²**, helps with understanding. Put the units right after the slider's value.

If you don't use units (or you use Desmos), you never see the error, but you also don't get the helpful realization that the parameter is in square centimeters.

37. Vertex Angles

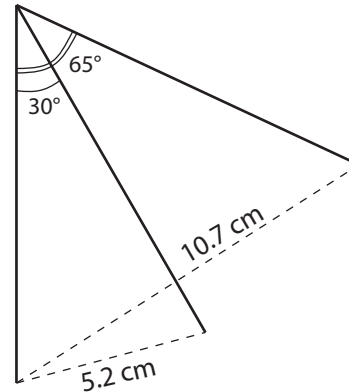
How is the vertex angle of an isosceles triangle related to the length of the base? That's what you'll be investigating in this activity.

What to Do

First, you need to prepare some triangles:

- ☐ Draw a bunch of isosceles triangles. Make sure that all of the "same" sides are the same length for all of the triangles. 10 centimeters is probably a good length. Different triangles can share sides. But be sure to get a wide range of vertex angles. (What is the possible range of vertex angles?)

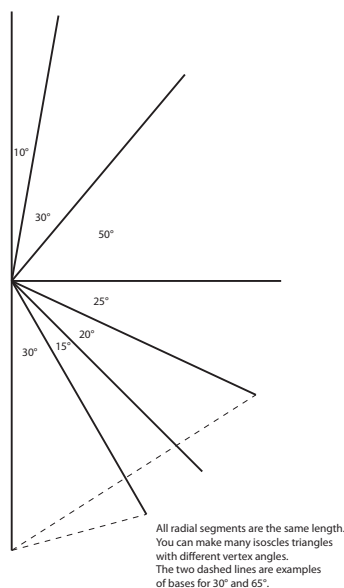
Now you are ready to measure **angle** (the size of each vertex angle) and the **length** of the opposite side (the base). The illustration shows two overlapping isosceles triangles with **angles** and **lengths** measured.



How will **length** be related to **angle**?

- ☐ Consider: What are the maximum and minimum values for **length**? What values of **angle** correspond to those **lengths**?
- ☐ Predict: What do you think the relationship will look like when you graph it? Sketch it at right:
- ☐ Record at least eight measurements of **length** and **angle**. Make sure **angle** varies widely.
- ☐ Plot **length** against **angle**. (Or **angle** against **length**. Which arrangement makes the most sense?)
- ☐ Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- ☐ Test your function! Find an **angle** you didn't use yet, and use your function to predict the **length**. Then make the triangle and check. How did you do?
- ☐ Be sure you can explain why the form of your function makes sense.
- ☐ Look back at your prediction graph. What did you get *right*? Were there any surprises?

Vertex Angles • Instructor Notes



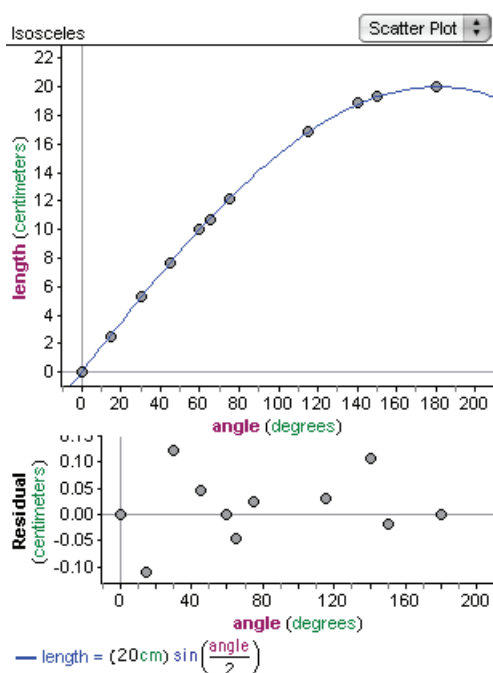
In this activity, students connect the ends of the spokes in the diagram (or one of their own construction; the only requirement is that the spokes be the same length). This makes isosceles triangles with different vertex angles. They are to study the relationship between the **angles** and the **lengths** of the “opposite” sides they have drawn.

As the diagram suggests, these triangles can overlap in order to make a wide variety of angles.

Students will need centimeter rulers. If they construct their own triangles and do not use the diagram on page 154, they will also need protractors.

Results

Here are some good results, shown with a function and a residual plot:

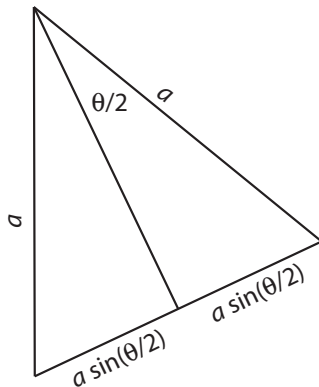


If students do not see that the data look like a sine curve, but stretched out so that you get from 0 to 1 as the angle increases from 0 to 180 instead of from 0 to 90, suggest that they fill in the rest of the circle. That is, suppose you had a vertex angle of greater than 180°—what would the length be? The length for 210°, for example, would be the same as the one for 150°.

If students fill in the graph from 180° to 360°, it may look more obviously like half a sine wave.

Vertex Angles • More Instructor Notes

Explaining the Relationship



The function arises naturally using SOH-CAH-TOA trig.

Draw the altitude of the isosceles triangle from its vertex to the base. This bisects the vertex angle, and splits the isosceles triangle into two right triangles whose hypotenuse is the length of the isosceles sides.

Now you can use trig to find the length of the base. In the illustration, the side lengths are a , and the vertex angle is θ . If the opposite side is c , we get

$$c = 2a \sin(\theta/2).$$

The amplitude of the sine (in our case, about 20 cm) is just twice the length of side a . That side length is a good candidate for a variable parameter, especially since copy machines often expand or shrink the image a little. That is, segments on the handout might not be 10 cm long.

Law of Cosines Approach

There is another approach. If students have seen the Law of Cosines, challenge them to use it, and then explain why the two functions are really the same. One explanation:

If we use the Law of Cosines, the relationship is

$$\begin{aligned} c^2 &= 2a^2 - 2a^2 \cos \theta \\ &= 2a^2(1 - \cos \theta), \text{ so} \\ c &= a\sqrt{2(1 - \cos \theta)}. \end{aligned}$$

where a is the length of the leg and c is the length of the base, the side we want. We can use that to plot out model.

But we also know from above that $c = 2a \sin(\theta/2)$.

Squaring that and setting it equal to the expression for c^2 , above, we get

$$\begin{aligned} 4a^2 \sin^2 \frac{\theta}{2} &= 2a^2(1 - \cos \theta) \\ \sin^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{2} \\ \sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} \end{aligned}$$

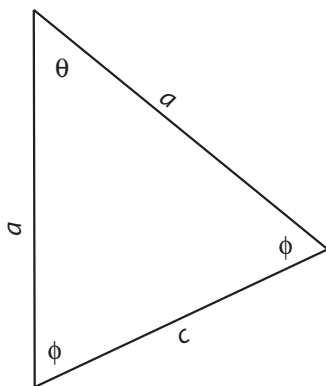
which is a half-angle formula.

Vertex Angles • Even More Instructor Notes

Law of Sines Approach

If students know about the Law of Sines, that yields a solution as well. Since the triangle is isosceles, if we know the vertex angle (and we do), we know all three angles. That means that

$$\frac{\sin \theta}{c} = \frac{\sin \phi}{a}, \text{ or } c = a \frac{\sin \theta}{\sin \phi}.$$



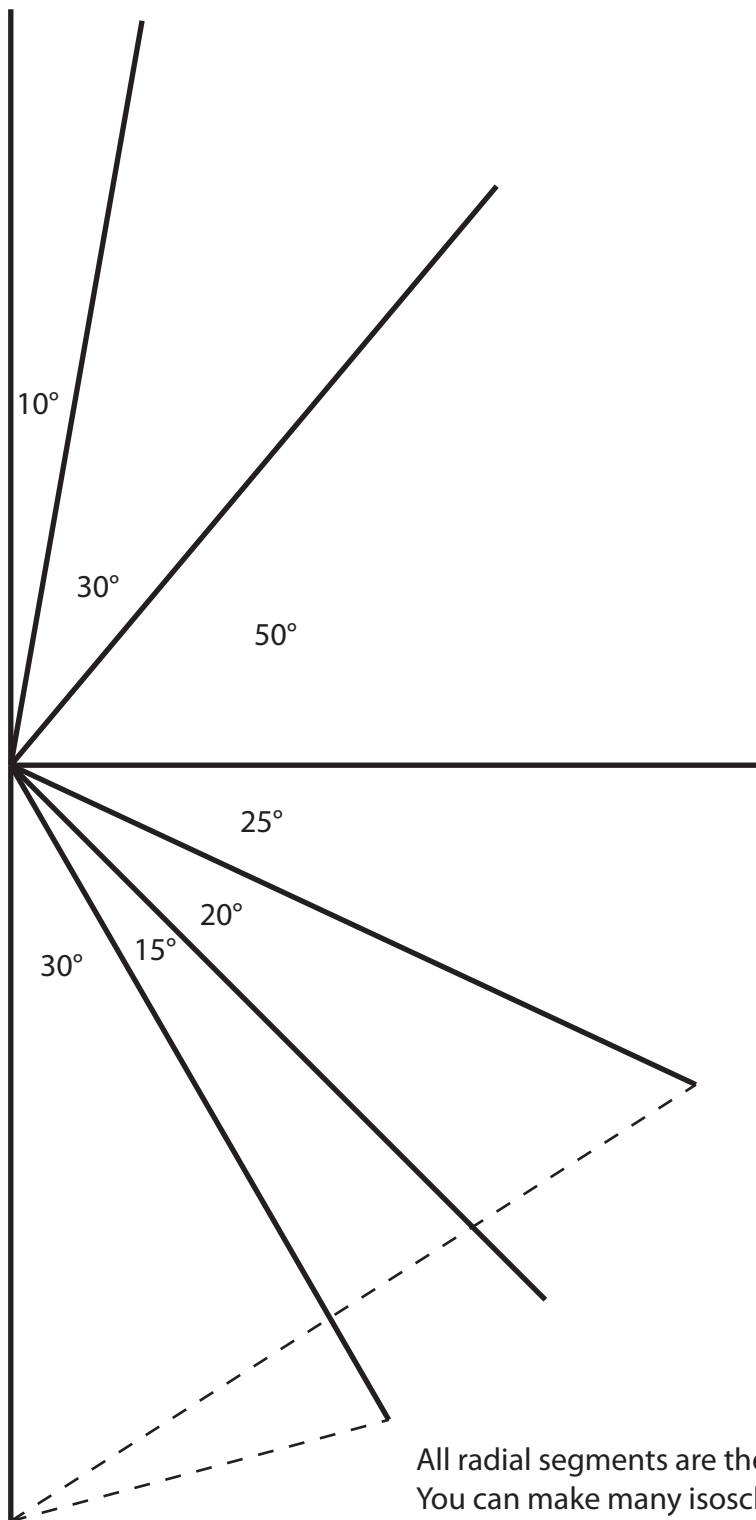
Since $\phi = (180^\circ - \theta) / 2 = 90^\circ - (\theta / 2)$, we could write

$$c = a \frac{\sin \theta}{\sin(90 - \theta / 2)} = a \frac{\sin \theta}{\cos(\theta / 2)},$$

which is ugly but effective. Students with trig-identity experience can show that this is the same as the other expressions.

It's wonderful that students can approach such a simple situation in such different ways. This is a great chance for students to tell one another about their solutions, and to work together to show that they're really all the same.

Vertex Angles Diagram



All radial segments are the same length.
 You can make many isoscles triangles
 with different vertex angles.
 The two dashed lines are examples
 of bases for 30° and 65°.

39. SAS Area

This activity is about triangle areas. You'll start with two sides, of lengths 10 cm, and connect them at various angles. You'll explore how the area of the triangle depends on the angle between the sides.

What to Do

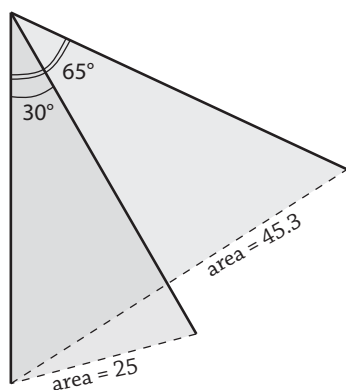
- ☐ Get a sheet of paper, a pencil, a ruler, and a protractor.
- ☐ Draw a segment 10 cm long. From one of its endpoints, and in a different direction, make another segment the same length.
- ☐ Measure the **angle** between the two segments.
- ☐ Connect the “loose” ends of the two segments to make a triangle. (It's isosceles.)
- ☐ Measure the **area** of the triangle. It's probably not a right triangle, so you'll have to decide on a *base* and make your best measurement for the triangle's *height*. Then area is $(base)(height)/2$.

How will **area** be related to **angle**?

You've measured one triangle. Soon, you'll make more, using the same procedure. Always 10-cm sides, but different angles.

- ☐ But first, predict: What will the relationship between **area** and **angle** look like? Sketch the graph you think you will find.
- ☐ Now draw more triangles, each with two 10-cm sides. Record measurements of **area** and **angle**.
- ☐ Plot **area** against **angle**.
- ☐ Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- ☐ Make a new triangle you haven't measured before. Note its **angle**. Use your formula to predict its **area**. Measure and check. How did you do?
- ☐ Make another new triangle. This time, measure its **area**. Predict its **angle**. How did you do?
- ☐ Be sure you can explain why the symbolic form of your function makes sense.

SAS Area • Instructor Notes



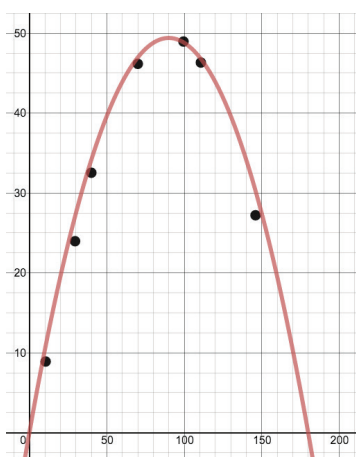
Students will need paper, rulers, and protractors.

The setup for this activity is exactly the same as for *Vertex Angles* on page 150, except that instead of finding a function to give us the *length* of the side opposite the vertex angle, students are finding the triangle's *area*.

So you can use that same handout (*Vertex Angles Diagram* on page 154) with angles and spokes, or have students draw their own. If they draw their own, you may need to encourage them to make angles greater than 90° .

In either case, you may need to remind students how to find the area of a triangle. Since most triangles will not have right triangles, students will have to pick one side for the base and then find the height by eye. This will be accurate enough for this activity and give students good practice in measurement and in estimating right angles.

Strong but Wrong



It's great if students recognize that the area *has* to be zero at 0° and 180° ; it means they're looking at extreme cases.

Those zeroes and the shape of the graph might prompt students to think that the area function has the form

$$\text{Area}(\theta) = k\theta(180^\circ - \theta),$$

which is quadratic. Now suppose the largest triangle, at 90° , has an area of 50 square centimeters. Then you can figure out k directly. You get a formula¹ of

$$\text{Area}(\theta) = \frac{50}{8100}\theta(180^\circ - \theta).$$

This is a very intelligent line of reasoning, and leads to a function that resembles the data, as you can see at left. But there is an even *better* model.

The Right Answer

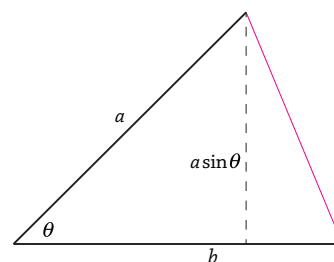


You can compute the height using SOH-CAH-TOA trig. Using the diagram at right, the area is

$$A = \frac{ab \sin \theta}{2}$$

or, in our particular case, where $a = b = 10$,

$$A = 50 \sin \theta$$

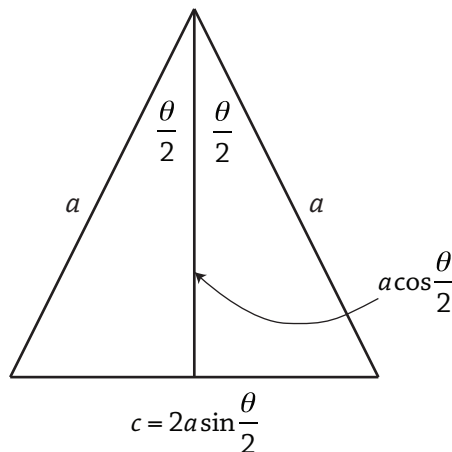


Students will have to set their technology to give trig results in "degree" mode. The value of the coefficient for the model at left is 49.5.

¹ Good question for students: where did the 8100 come from?

SAS Area • Instructor Notes

A Tortuous Path It's possible to get a good model with more effort, however. Some students feel the need to orient these isosceles triangles differently, especially if they have successfully completed *Vertex Angles* on page 150:



If they use a diagram like the one in the figure, it can lead to a different formula for the area:

$$A = \frac{bh}{2} = \frac{1}{2} \times 2a \sin \frac{\theta}{2} \times a \cos \frac{\theta}{2} = a^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2},$$

which is correct but more complicated.

Either approach works, but this is an chance to ask whether, even though both answers are correct, one might be better. What do we mean by a better formula?

First, $A = \frac{1}{2}ab \sin \theta$ is *simpler*. It uses fewer symbols. It uses the angle θ —what we measured—rather than half the angle. One trig function instead of two. And it's a lot easier to remember.

But second, *it's more general*. It works with any triangle; you can do the activity even if the sides a and b aren't the same.

This alternative area formula does, however, lead to a nice double-angle identity. Combining our two area formulas, setting $x = \theta/2$, and simplifying, we get:

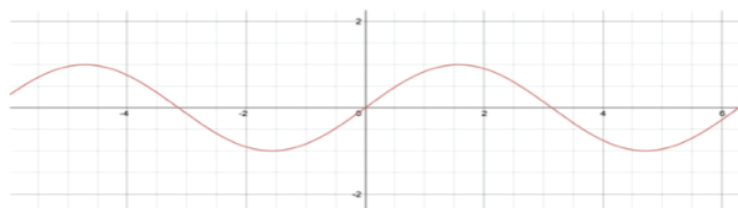
$$\sin 2x = 2 \sin x \cos x.$$

SAS Area • More Instructor Notes — and Student Work

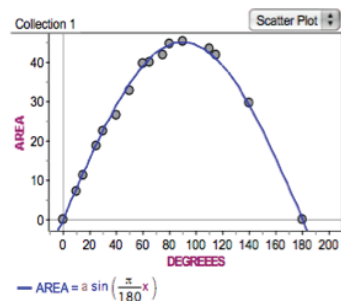
Discussion and Extension Questions

- ❖ Which **angle** gives you the largest **area**?
- ❖ What should that area be?
- ❖ What is the domain of this relationship (assuming **angle** is on the x-axis)?
- ❖ What is the range?
- ❖ Why does the limit on the domain make sense?
- ❖ How would this be different if the two sides were different lengths? (Try it and see.)
- ❖ How does your function connect to the traditional $Area = bh/2$ formula?
- ❖ Try the quadratic model. Give at least two reasons why we should prefer the one with sine.

Prediction: When looking at a triangle's area in comparison to its vertex angle I think it will increase with the angle to a certain point then switch and the area will begin to decrease while the angle continues to increase. I am not sure where this change will happen, its makes sense it would happen when the angles were all 60° because to the naked eye that looks like the largest area. I am not sure what type of function it will make, possibly a sin or cos because if you have two 10cm length segments and they are like hands of a clock(connected at a point), as they go around the area will increase till a certain point then begin to decrease and as they go around and around it will be cyclic like sin and cos.



The graph I got from my data:



I was correct in predicting that it would be a sine or cosine function. I choose sin because that involved less shifting but if I wanted to use cos I would just have to use horizontal shifts. It appears that the area stops increasing with the angle and begins to decrease while the angle continues to grow is at 90° and not 60°. This is because 90° is half way between 0° and 180° the two angles when the area is 0. So 90° will fall at the peak of the sin curve.

The function of these is triangles $45\sin\left(\frac{\pi}{180}x\right)$. Where 45 is the largest area an isosceles triangle with 9.5cm equal sides can have.

In this student work, see how they made an intelligent but wrong prediction that the biggest area would be at 60°. The student also shows understanding about transformations in explaining why to use sine and not cosine; and terrific reasoning about the area growing and then falling, repeatedly, as the angle increases.

But do they show that they understand why the function is what it is? No. There is room for growth.

Activities and Functions

This chart shows, for each activity, the “correct” functions students can use to model the data. There is much opportunity, however, for practicing other functions. For example, seeing data that are based on sinusoids, some students use quadratics. The chart also has icons indicating broad geometry topics.

Most terms in the chart are obvious, except:

- *Root Pythagorean*: The kind of square root relationship you get when you solve the Pythagorean Theorem, e.g., $f(x) = \sqrt{k^2 - x^2}$.
- *Direct Quadratic*: means $f(x) = kx^2$ here.
- *Surprise*: Experienced folks might not expect it to be this way.
- $\triangle$ triangles, $\circ$ circles, $\square$ area, $\sim$ similarity

- 1 *A Sample Activity: Hypotenuse* (p 14).
Root Pythagorean. $\triangle$
- 2 *Stack of Books* (p 22). Direct proportion.
- 3 *Rolling Rolling Rolling* (p 25).
Direct proportion. $\circ$
- 4 *Stack of Cups* (p 28). Linear.
- 5 *Opposite Sides of the Ruler* (p 31).
Linear, negative slope.
- 6 *Pick's Theorem* (p 33).
Linear, two input variables. $\triangle \square$
- 7 *Circumference* (p 38). Direct proportion. $\circ$
- 8 *Isosceles Angles* (p 41).
Linear, negative slope. $\triangle$
- 9 *Triangle Ladder* (p 45).
Direct proportion, linear. $\triangle \sim$
- 10 *ART and ACT* (p 48). Direct proportion. $\triangle \circ$
- 11 *Paragraphs (Same Font Size)* (p 54).
Inverse proportion. $\square$
- 12 *Paragraphs (Different Font Size)* (p 70).
Direct Quadratic. $\square \sim$
- 13 *Cornbread Are Square* (p 58).
Direct Quadratic. $\circ \square$
- 14 *Cardboard Squares* (p 61). Direct Quadratic. $\square$
- 15 *Cardboard Circles* (p 64).
Direct Quadratic. $\circ \square$

- 16 *Bundles* (p 67).
Direct Quadratic, square root. $\circ \square$
- 17 *Chord Star* (p 73). Inverse proportion. $\circ \sim$
- 18 *Chord Star 2* (p 76).
Quadratic, opens down. $\circ$
- 19 *Spiral 45* (p 80). Exponential. $\triangle \sim$
- 20 *Spiral 20* (p 83). Exponential. $\triangle \sim$
- 21 *Triangle Spiral* (p 87).
Surprise square root. $\triangle$
- 22 *Golden Rectangles* (p 90).
Exponential decreasing. $\sim$
- 23 *Zeno the Frog* (p 94).
Exponential, approaching from below.
- 24 *Shaded Strips* (p 97).
Exponential, approaching from below. $\square \sim$
- 25 *Do Fret* (p 100).
Exponential, approaching from below.
- 26 *Tilting Chairs* (p 104). Root Pythagorean. $\triangle$
- 27 *Shadowlands* (p 108). Surprise rational. $\triangle$
- 28 *Tinkertoy Stick Lengths* (p 111).
Complicated exponential. $\triangle$
- 29 *Chord Gap* (p 115). Rational. $\circ$
- 30 *Triangle Folding* (p 121). Surprise cubic. $\triangle \square$
- 31 *Toilet Paper Roll* (p 124). Square root. $\circ \square$
- 32 *Maximum Box* (p 131). Cubic. Volume.
- 33 *Vegetable Matter* (p 134).
Varies. Linear, cubic. $\circ$
- 34 *Making a Cone* (p 138). Root Pythagorean. $\circ$
- 35 *Filling a Cone* (p 142). Cube root. $\triangle \circ$
- 36 *Tangent* (p 147). Tangent. $\triangle$
- 37 *Vertex Angles* (p 150). Sine, cosine,
Law of Cosines square root. $\triangle$
- 38 *Sines of the Times* (p 155). Sine, Cosecant. $\triangle$
- 39 *SAS Area* (p 158). Sine, cosine. $\triangle \square$
- 40 *Small Ferris Wheel* (p 162).
Law of Cosines square root. $\triangle \circ$
- 41 *Distance to Mars* (p 167).
Law of Cosines square root. $\triangle \circ$
- 42 *Eccentric* (p 172).
Law of Cosines square root. $\triangle \circ$