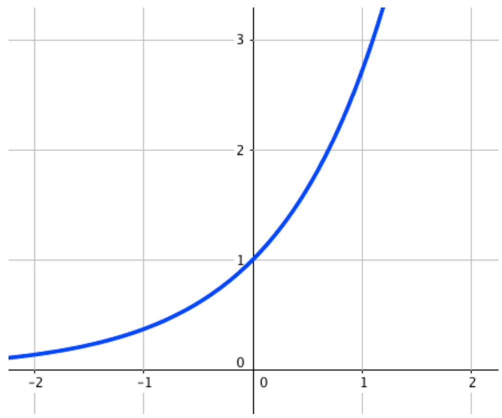
Polynomial Approximation Investigation

Consider the function $f(x) = e^x$ near $x = 0$. Since $f(0) = e^0 = 1$, the horizontal line $P_0(x) = 1$ could be used to approximate $f(x)$ near $x = 0$.

1. Sketch a graph of $P_0(x)$ to visually represent this approximation near $x = 0$.



Of course, when using $P_0(x) = 1$ to approximate $f(x) = e^x$, the approximation gets bad fast as you move away from $x = 0$.

The tangent line approximation, $P_1(x)$, is the best first-degree approximation to $f(x)$ near $x = a$ because $f(x)$ and $P_1(x)$ have the same rate of change at a. Therefore, $P_1(x)$ and $f(x)$ share a common point and a common slope.

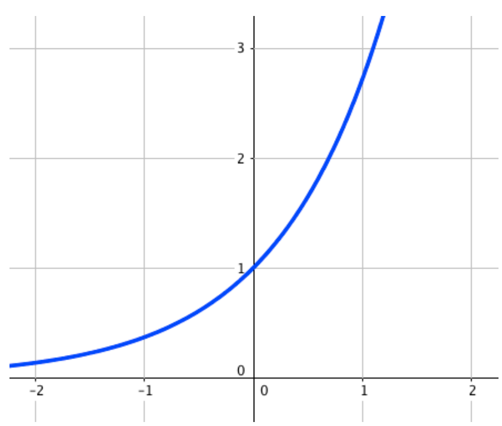
2. Write the equation of the tangent line $P_1(x)$ to $f(x) = e^x$ at $x = 0$.

Note that $P_1(x)$, the best first-degree approximation to $f(x)$ near $x = 0$, satisfies the following two conditions:

(i) $P_1(0) = f(0)$

(ii) $P_1'(0) = f'(0)$

3. Sketch a graph of $P_1(x)$ to visually represent this approximation near $x = 0$.



Does $P_1(x)$ appear to be better or worse than $P_0(x)$ at approximating $f(x) = e^x$ near $x = 0$?

For a better approximation than a linear one, let's try a second-degree approximation, $P_2(x)$. We can approximate $f(x)=e^x$ near $x=0$ with a parabola, rather than a straight line. In this case, $P_2(x)$ and $f(x)$ share a common point, a common slope, and a common concavity.

4. Write the equation of $P_2(x)=A+Bx+Cx^2$.

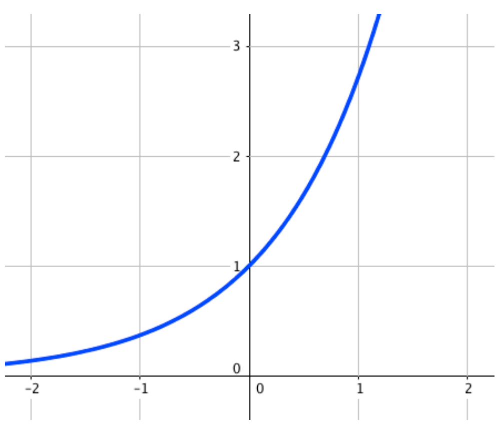
This time, to make sure that the approximation is good, we stipulate the following:

(i) $P_2(0)=f(0)$

(ii) $P_2'(0)=f'(0)$

(iii) $P_2''(0)=f''(0)$

5. Sketch a graph of $P_2(x)$ to visually represent this approximation near $x=0$.

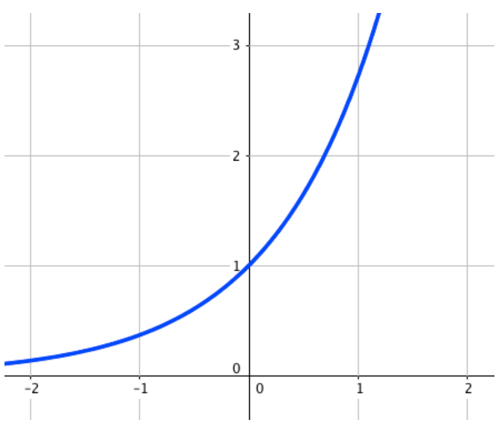


Does $P_2(x)$ appear to be better or worse than $P_1(x)$ at approximating $f(x)=e^x$ near $x=0$?

6. Write the equation of $P_3(x) = A + Bx + Cx^2 + Dx^3$ that best approximates $f(x) = e^x$ near $x = 0$.

What conditions must we satisfy in order to construct the best third-degree polynomial approximation to $f(x)$ near $x = 0$?

7. Sketch a graph of $P_3(x)$ and $f(x)$ to visually represent this approximation near $x = 0$.



8. To approximate a function by a cubic function $P_3(x)$ near $x=a$, it can be helpful to write $P_3(x)$ in the form:

$$P_3(x)=A+B(x-a)+C(x-a)^2+D(x-a)^3$$

Show that this cubic function is:

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$