Fourier Series and Integrals

Fourier Series

Let $f(x)$ be a piece-wise linear function on $[-L, L]$ (This means that $f(x)$ may possess a finite number of finite discontinuities on the interval). Then $f(x)$ can be expanded in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}, \tag{1a}$$

or, equivalently,

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/L}\,, \tag{1b}$$

with

$$c_n = \begin{cases} (a_n - ib_n)/2 & n < 0, \\ (a_n + ib_n)/2 & n > 0, \\ a_0/2 & n = 0. \end{cases}$$

A useful schematic form of the Fourier series is

$$\vec{f}(x) = \sum_n (a_n \hat{c}_n + b_n \hat{s}_n). \tag{2}$$

This emphasizes that the Fourier series can be viewed as an expansion of a vector $\vec{f}$ in Hilbert space, in a basis that is spanned by the $\hat{c}_n$ (cosine waves of different periodicities) and the $\hat{s}_n$ (sine waves).

To invert the Fourier expansion, multiply Eq. (1a) by $\cos \frac{n\pi x}{L}$ or $\sin \frac{n\pi x}{L}$ and integrate over the interval. For this calculation, we need the basic orthogonality relation of the basis functions:

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \delta_{mn} L, \tag{3}$$

and similarly for the sin functions. Intuitively, for $n \neq m$, there is destructive interference of the two factors in the integrand, while for $n = m$, there is constructive interference.

Thus by multiplying Eq. (1a) by $\cos \frac{n\pi x}{L}$ and integrating, we obtain

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx = \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \frac{n\pi x}{L} \right] \cos \frac{m\pi x}{L}\, dx, \tag{4}$$

from which we find

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad\qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \tag{5}$$

The inversion of the Fourier series can be viewed as finding the projections of $\vec{f}$ along each basis direction. Schematically, therefore, the inversion can be represented as

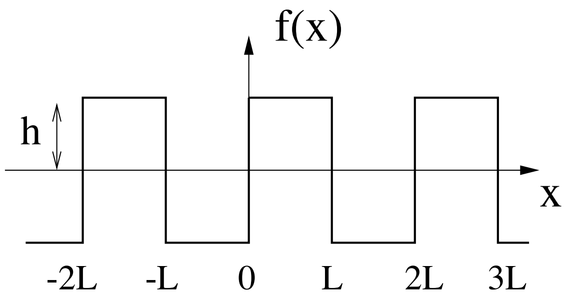
$$\vec{f} \cdot \hat{c}_m = \sum_n (a_n \hat{c}_n + b_n \hat{s}_n) \cdot \hat{c}_m = \sum a_n \delta_{nm} = a_m, \tag{6a}$$

which implies

$$a_m = \vec{f} \cdot \hat{c}_n. \tag{6b}$$

These steps parallel the calculation that led to Eq. (5). This schematic representation emphasizes that the Fourier decomposition of a function is completely analogous to the expansion of a vector in Hilbert space in an orthogonal basis. The components of the vector correspond to the Fourier amplitudes defined in Eq. (5).

Examples: 1. *Square Wave*



Clearly $f(x)$ is an odd function: $f(x) = -f(-x)$. Hence

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = 0 \qquad \forall n\,.$$

As a result, the spectral information of the square wave is entirely contained in the b_n's. These coefficients are

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2h}{L} \int_{0}^{L} \sin \frac{n\pi x}{L}\, dx = \frac{2h}{n\pi}(1 - \cos n\pi), \tag{7a}$$

from which we find

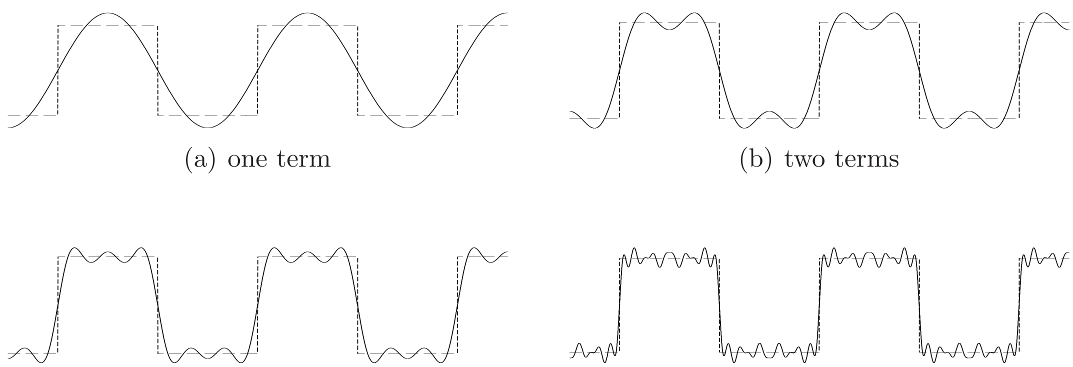
$$b_n = \begin{cases} 4h/n\pi & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \tag{7b}$$

Thus the square wave can be written as a Fourier sine series

$$f(x) = \frac{4h}{\pi} \left(\sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right). \tag{8}$$

There are a number of important general facts about this expansion worth emphasizing:

- Since $f(x)$ is odd, only *odd* powers of the *antisymmetric* basis functions appear.
- At the discontinuities of $f(x)$, the Fourier series converges to the mean of the two values of $f(x)$ on either side of the discontinuity.
- A picture of first few terms of the series demonstrates the nature of the convergence to the square wave; each successive term in the series attempts to correct for the "overshoot" present in the sum of all the previous terms



(a) one term (b) two terms

(c) three terms (d) ten terms

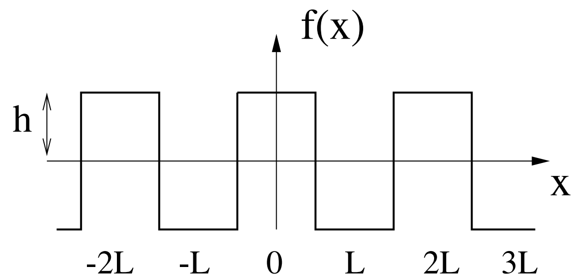
- The amplitude spectrum decays as $\frac{1}{n}$; this indicates that a square wave can be well-represented by the fundamental frequency plus the first few harmonics.
- By examining the series for particular values of x, useful summation formulae may sometimes be found. For example, setting $x = \frac{L}{2}$ in the Fourier sine series gives:

$$f(x{=}L/2) = h = \frac{4h}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right). \tag{9a}$$

  and this leads to a nice (but slowly converging) series representation for $\frac{\pi}{4}$:

$$\frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\cdots\right). \tag{9b}$$

Alternatively we may represent the square wave as an *even* function.



Now $f(x) = +f(-x)$. By symmetry, then, $b_n = 0\ \forall n$, and only the a_n's are non-zero. For these coefficients we find

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2h}{L}\left[\int_0^{L/2}\cos\frac{n\pi x}{L}\,dx - \int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{2h}{n\pi}\left[\int_0^{n\pi/2}\cos u\,du - \int_{n\pi/2}^{n\pi}\cos u\,du\right] \\
&= \frac{2h}{n\pi}\left[\sin u\,\Big|_0^{n\pi/2} - \sin u\,\Big|_{n\pi/2}^{n\pi}\right] \\
&= \begin{cases} \frac{4h}{n\pi}(-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}$$

Hence $f(x)$ can be also represented as the Fourier cosine series

$$f(x) = \frac{4h}{\pi}\left(\cos\frac{\pi x}{L} - \frac{1}{3}\cos\frac{3\pi x}{L} + \frac{1}{5}\cos\frac{5\pi x}{L}\cdots\right). \tag{10}$$

This example illustrates the use of symmetry in determining a Fourier series,

even function $\longrightarrow$ cosine series
odd function $\longrightarrow$ sine series
no symmetry $\longrightarrow$ both sine and cosine series

Thus it is always simpler to choose an origin so that $f(x)$ has a definite symmetry, so that it can be represented by either a sin or cosine series

2. *Periodic Parabola*
The periodic parabola is the periodic extension of the function x^2, in the range $[-\pi, \pi]$, to the entire real line. Given any function defined on the interval $[a, b]$, the periodic extension may be constructed in a similar fashion. In general, we can Fourier expand *any* function on a finite range; the Fourier series will converge to the periodic extension of the function.

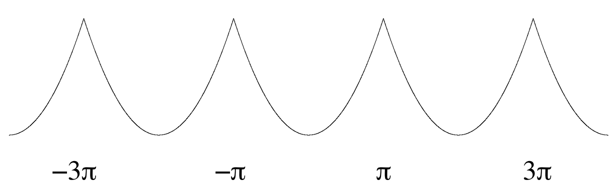


Figure 1: The periodic parabola

The Fourier expansion of the repeated parabola gives

$$\begin{aligned} b_n &= 0 \qquad \forall n\,, \\ a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}\,, \\ a_n &= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \; dx = (-1)^n \frac{4}{n^2}\,. \end{aligned}$$

Thus we can represent the repeated parabola as a Fourier cosine series

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{11}$$

Notice several interesting facts:

- The a_0 term represents the average value of the function. For this example, this average is non-zero.
- Since f is even, the Fourier series has only cosine terms.
- There are no discontinuities in f, but first derivative is discontinuous; this implies that the amplitude spectrum decays as $1/n^2$. In general, the "smoother" the function, the faster the decay of the amplitude spectrum as a function of n. Consequently, progressively fewer terms in the Fourier series are needed to represent the waveform to a fixed degree of accuracy.
- Setting $x = \pi$ in the series gives

  $$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi\,,$$

  from which we find the cool formula

  $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,. \tag{12}$$

The approach given above provides an indirect, but simple, way to sum inverse powers of the integers. Simply Fourier expand the function x^k on the interval $[-\pi, \pi]$ and then evaluate the series at $x = \pi$ from which $\sum_{n=1}^{\infty} n^{-k}$ can be computed. The sum $\sum_{n=1}^{\infty} n^{-z} \equiv \zeta(z)$, is called the *Riemann zeta function*, and by this Fourier series trick the zeta function can be evaluated for all positive integer values of z.

In summary, a Fourier series represents a spectral decomposition of a periodic waveform into a series of harmonics of various frequencies. From the amplitudes of these harmonics we can gain understanding of the physical process underlying the waveform. For example, one perceives in an obvious way the different tonal quality of an oboe and a violin. The origin of this difference are the distinct Fourier spectra of these two instruments. Even if the same note is being played, the different "shapes" of the two instruments (string for the violin; air column for the oboe) lead to very different contributions of higher harmonics. This translates into distinct respective Fourier spectra, and hence, very different musical tones.

Fourier Integrals

For non-periodic functions, we generalize the Fourier series to functions defined on $[-L, L]$ with $L \to \infty$. To accomplish this, we take Eq. (5),

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dk \, ,$$

and now let $k = n\pi/L \equiv n\, dk$. Then as $L \to \infty$, we obtain

$$A(k) \equiv a_n L = \int_{-\infty}^{\infty} f(x) \cos kx \, dx \, , \tag{13a}$$

and, by following the same procedure,

$$B(k) \equiv b_n L = \int_{-\infty}^{\infty} f(x) \sin kx \, dx \, . \tag{13b}$$

Now the original Fourier series becomes

$$\begin{aligned} f(x) &= \sum_n a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \\ &= \sum_n \frac{A(k)}{L} \cos kx + \frac{B(k)}{L} \sin kx \\ &\to \frac{1}{\pi} \int_0^{\infty} [A(k) \cos kx + B(k) \sin kx] \, dk. \end{aligned}$$

By using the exponential form of the Fourier series, we have the alternative, but more familiar and convenient Fourier integral representation of $f(x)$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx} \, dk. \tag{14}$$

Where the (arbitrary) prefactor is chosen to be $\frac{1}{\sqrt{2\pi}}$ for convenience, as then the same prefactor appears in the definition of the inverse Fourier transform. In symbolic form, the Fourier integral can be represented as

$$\vec{f} = \sum_{\text{continuous } k} f_k \hat{e}_k \, .$$

To find the expansion coefficients $f(k)$, we proceed in precisely the same manner as in the case of Fourier series. That is, multiply $f(x)$ by one of the basis functions and integrate over a suitable range. This gives

$$\int_{-\infty}^{\infty} f(x) e^{-ik'x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(k) e^{i(k-k')x}\, dx\, dk\,. \tag{15}$$

However, the integral over x is just the Dirac delta function. To see this, we write

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \lim_{L\to\infty} \int_{-L}^{L} e^{i(k-k')x}\, dx = \frac{2\sin(k-k')L}{k-k'}\,. \tag{16a}$$

The function on the right-hand side is peaked at $k = k'$, with the height of the peak equal to $2L$, and a width equal to π/L. As $L \to \infty$, the peak becomes infinitely high and narrow in such a way that the integral under the curve remains a constant. This constant equals 2π. Thus we write

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = 2\pi\delta(k-k')\,. \tag{16b}$$

Using this result in Eq. (15), we obtain

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx\,. \tag{17}$$

Schematically,

$$\vec{f} \cdot \hat{e}_{k'} = \sum_k \vec{f}_k \hat{e}_k \cdot \hat{e}_{k'} \longrightarrow f_k = \vec{f} \cdot \hat{e}_k\,.$$

These developments can be extended in a straightforward way to multidimensional functions.

The Fourier transform has a number of important features that are very useful for a variety of physical applications:

- If $f(x)$ is non-zero over some range X about $x = 0$, then $f(k)$ is non-zero in a range $1/X$ about $k = 0$. While this preceding statement is imprecise, the inverse relation between k and x is physically quite useful.

- Parseval relation: $\int_{\infty}^{\infty} |f(x)|^2 dx = \int_{\infty}^{\infty} |f(k)|^2 dk$.

- Differentiation: $\mathcal{F}[f^{(n)}(x)] = (ik)^n \mathcal{F}[f(x)]$. Here $\mathcal{F}[f(x)]$ denotes the Fourier transform of $f(x)$ and the superscript (n) denotes the n^{th} derivative. Thus the Fourier transform of the n^{th} derivative of $f(x)$ is just the Fourier transform of $f(x)$ itself times $(ik)^n$.

- Translation: $\mathcal{F}[(f(x+a)] = \frac{1}{\sqrt{2\pi}} \int f(x+a)\, e^{-ikx} dx = e^{ika} \mathcal{F}[f(x)]$. This statement is extremely useful for translationally invariant systems.

- Convolution: $\mathcal{F}(f * g) = f(k)g(k)$, where the notation $f * g$ means the convolution of f and g and is defined by

$$f * g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)\, dy\,. \tag{18}$$

In the integrand, we may think of some physical attribute being propagated from 0 to $x - y$ by the "propagator" function $f(x - y)$ and then propagated from $x - y$ to x by the propagator function $g(y)$. The intermediate point y is integrated over, so that the convolution is a function only of x. Thus we may write $f * g = F(x)$. The convolution statement $\mathcal{F}(f * g) = f(k)g(k)$ has many important physical applications.

To show the convolution statement, we rewrite (18) in terms of the respective Fourier transforms

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \\ \frac{1}{\sqrt{2\pi}} \int F(k)e^{ikx} dk &= \frac{1}{\sqrt{2\pi}} \iiint \frac{1}{\sqrt{2\pi}} f(k')e^{ik'(x-y)}\ \frac{1}{\sqrt{2\pi}} g(k'')\, e^{ik''y}\ dk'\, dk''\, dy \\ &= \frac{1}{(2\pi)^{3/2}} \iiint e^{iy(k''-k')} f(k')g(k'')\, e^{-ik'x}\ dk'\, dk''\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int f(k')g(k')e^{-ik'x} dk'\,. \end{aligned} \tag{19}$$

In going from the third to the last line, we integrate over y to give $2\pi\delta(k' - k'')$. This reduces the integral over dk' and dk'' to a single integral over dk'. Equating the integrands, we conclude that $F(k) = f(k)g(k)$. Thus *a convolution in real space reduces to multiplication in Fourier space.*

Examples 1. *Rectangular Pulse*

$$f(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2. \end{cases}$$

Then

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} e^{i\omega t} dt = \frac{T}{\sqrt{2\pi}} \frac{\sin \omega T/2}{\omega T/2}. \tag{20}$$

Notice that there is an inverse relationship between the width of $f(t)$ and the width of the corresponding Fourier transform $f(\omega)$. The Fourier spectrum is concentrated at zero frequency; these conspire to give destructive interference except in the range $|t| < \frac{T}{2}$.

2. *Damped Harmonic Wave*

$$f(t) = e^{i\omega_0 t - \alpha t} \qquad t > 0\,.$$

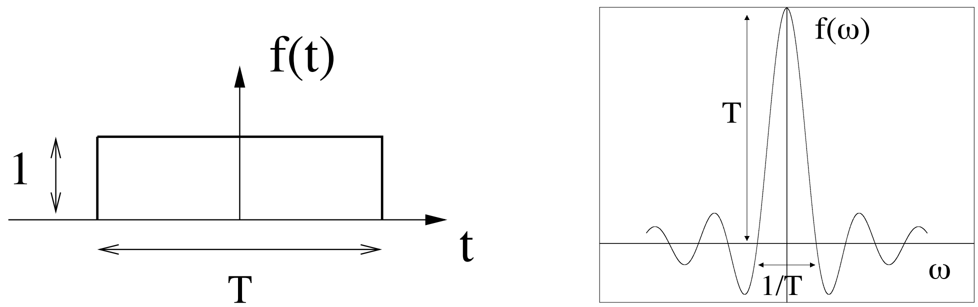


Figure 2: Rectangular pulse and its Fourier transform.

The corresponding Fourier transform is

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\alpha t + i(\omega_0 - \omega)t}\, dt = \frac{1}{\sqrt{2\pi}} \frac{1}{[\alpha - i(\omega_0 - \omega)]}\,, \tag{21a}$$

or

$$|f(\omega)| \sim [\alpha^2 + (\omega_0 - \omega)^2]^{-1/2}\,. \tag{21b}$$

This latter waveform is often called a Lorentzian. The relation between the damped harmonic wave the its Fourier transform are shown below for the case $\omega_0 = 1$ and $\alpha = \frac{1}{4}$.

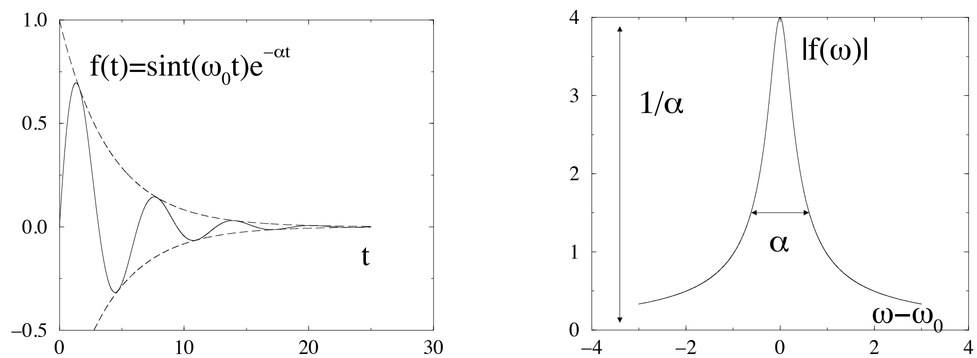


Figure 3: Damped harmonic wave and its Fourier transform.

As the damping goes to zero, the width of the Lorentzian also vanishes. This embodies the fact that a less weakly-damped waveform is less contaminated with frequency components not equal to ω_0. Clearly, as the damping goes to zero, only the fundamental frequency remains, and the Fourier transform is a delta function.

3. *Gaussian*

$$f(t) = \frac{1}{\sqrt{2\pi t_0^2}}\; e^{-t^2/2t_0^2}\,. \tag{22}$$

To calculate this Fourier transform, complete the square in the exponential:

$$\begin{aligned} f(\omega) &= \frac{1}{\sqrt{2\pi t_0^2}} \int_{-\infty}^{\infty} e^{-t^2/2t_0^2 - i\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi t_0^2}} \int_{-\infty}^{\infty} e^{-t^2/2t_0^2 - i\omega t + \omega^2 t_2^2/2 - \omega^2 t_2^2/2}\, dt \\ &= \frac{1}{\sqrt{2\pi t_0^2}} \int_{-\infty}^{\infty} e^{-(t/\sqrt{2}t_0 - i\omega t_0/\sqrt{2})^2 - \omega^2 t_2^2/2}\, dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(u - i\omega t_0/\sqrt{2})^2 - \omega^2 t_2^2/2}\, dt \\ &= e^{-\omega^2 t_0^2/2}\,. \end{aligned} \tag{23}$$

In the second-to-last line, we introduce $u = t/\sqrt{2}t_0$, and for the last line we use the fact that $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$. The crucial point is that the Fourier transform of a Gaussian is also a Gaussian! The width of the transform function equal to the inverse width of the original waveform.

4. *Finite Wave Train*

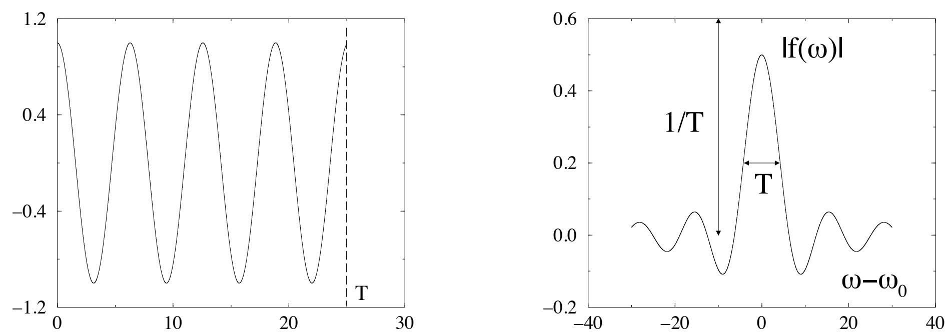


Figure 4: Finite wave train and its Fourier transform.

$$f(t) = \cos\omega_0 t = Re(e^{i\omega_0 t}) \qquad 0 < t < T\,.$$

Then

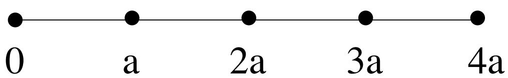
$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{i(\omega_0 - \omega)t}\, dt = \frac{1}{\sqrt{2\pi}} \frac{e^{i(\omega_0-\omega)T} - 1}{i(\omega_0 - \omega)}$$

$$= \frac{e^{i(\omega_0-\omega)T/2}}{\sqrt{2\pi}}\ \frac{2\sin(\omega_0 - \omega)T/2}{\omega_0 - \omega}\,. \tag{24}$$

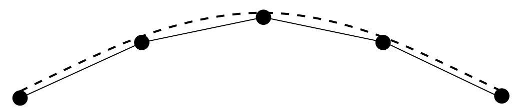
The behavior of the finite wavetrain is closely analogous to that of the damped harmonic wave. The shorter the duration of the wavetrain, the more its Fourier transform is contaminated with other frequencies beyond the fundamental.

Wavevector Quantization

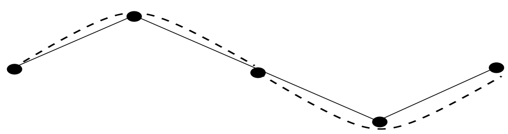
In certain applications, typically in solid state physics or other problems defined on a lattice, $f(x)$ is defined only on a discrete, periodic set points. The discreteness yields a condition of the range of possible wavevectors as well as the number of normal modes in the system. As a typical example, consider the transverse oscillations of a loaded string of length $L = 4a$ which contains 5 point masses, with both endpoints held fixed. There are a discrete set of normal mode oscillations for this system; these modes and their corresponding wavevectors $k_n = \frac{n\pi}{4a}$ are shown below. Also shown is the corresponding continuous waveform, $\sin k_n x$ (dashed).



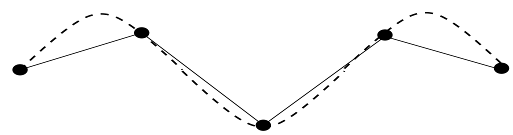
(a) $k = 0$



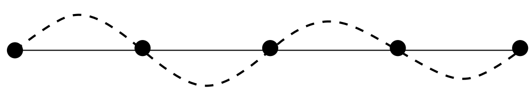
(b) $k = \pi/4a$



(c) $k = 2\pi/4a$



(d) $k = 3\pi/4a$



(e) $k = 4\pi/4a$

Notice that $k = \frac{\pi}{a}$ and $k = 0$ give identical displacement patterns for the discrete system. Therefore in defining the Fourier transform, it is redundant to consider values of k such that $k \geq \frac{\pi}{a}$. The range $0 < k < \frac{\pi}{a}$ is called the first Brillouin zone in the context of solid-state physics. Thus when performing a Fourier analysis on a discrete system (either Fourier series or Fourier integral), one must account for the restriction imposed by the system discreteness in defining the appropriate range of k values.