Mathematics for Complex Systems

Unit 2: Introduction to Differential Equations (David Feldman)

Homework Solutions

1. Consider the differential equation that describes the temperature T of an object in a 20-degree room:

$$\frac{dT}{dt} = 0.2\,(20 - T).$$

Suppose that the initial temperature of the object is T = 10 degrees. Use Euler's method with $\Delta t = 2$ to come up with estimates for the object's temperature at t = 2 and t = 4.

Solution:

Initially, $T = 10$. We find the rate of change from the differential equation:

$$\frac{dT}{dt} = 0.2\,(20 - T) = 0.2(10) = 2\text{C}/\min.$$

So, we pretend that this rate of change is constant for 2 minutes to find the temperature at $t = 2$:

$$T(2) = 10\text{C} + 2\min\,(2\text{C}/\min) = 14\text{C}.$$

So at $t = 2$, the temperature T is 14. To determine the rate of change of the temperature at $t = 2$, plug 14 into the differential equation:

$$\frac{dT}{dt} = 0.2\,(20 - 14) = 0.2(6) = 1.2\text{C}/\min.$$

We then pretend that this rate of change is constant from $t = 2$ to $t = 4$ to find the temperature T at $t = 4$:

$$T(2) = 14\text{C} + 2\min\,(1.2\text{C}/\min) = 16.4\text{C}.$$

2. Consider the differential equation

$$\frac{dY}{dt} = -\frac{1}{2}Y.$$

Let Y (0) = 100.

Use Euler's method with Δt = 2 to determine estimates for Y (2) and Y (4).

Solution:
We are given that $Y = 100$ at $t = 0$. We can figure out the initial rate of change via the differential equation:

$$\frac{dY}{dt} = -\frac{1}{2}100 = -50$$

We pretend that this rate of change is constant from $t = 0$ to $t = 2$ to get the value of Y at $t = 2$:

$$Y(2) = 100 + (-50)(2) = 0.$$

So at $t = 2$, $Y = 0$. The differential equation then tells us the rate of change at $t = 2$. Plugging in $Y = 0$, we obtain:

$$\frac{dY}{dt} = -\frac{1}{2}0 = 0.$$

So the rate of change is 0. Thus the value of Y is not changing, and so $Y(4) = 0$.

This example illustrates that Euler's method with a too large Δt can give potentially mis-leading results. In this case the solution to this differential equation should exponentially decay, approaching a stable fixed point at $Y = 0$. However, since Δt is so large, the ap-proximate Euler solution happens to land exactly on the fixed point after two steps, so we wouldn't observe the exponential decay.

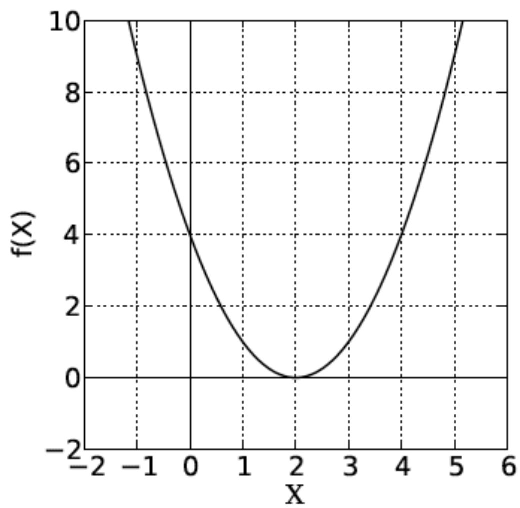
In other cases even more dramatically wrong behavior can be observed. If Δt is too large it is possible to "step over" a stable fixed point and then get pushed away by a repelling fixed point. In practice, though, this isn't too big a problem, as long as one remembers that Euler's method is only approximate at that it is important to experiment with smaller and smaller Δt values. Also, often a numerical method like Euler's would be used in combination with the qualitative techniques. That is, one can find the fixed points and the phase line first to get a qualitative picture of what the solutions have to look like and then use this as a guide when evaluating numerical solutions.

3. (Advanced): Feel free to post your solution to the class forum!

4. (Advanced): Consider again the differential equation

$$\frac{dX}{dt} = f(X)$$

where f (X) is plotted below:



Suppose the initial X value is 1. If you used Euler's method with Δt = 1 to figure out the value of X at t = 1, would your result be above or below the exact value for X (1). Why?

Solution:

X is 1 at $t = 0$ and so according to the graph of $f(X)$ shown above, the rate of change of X is approximately 1 at $t = 0$. So X is initially increasing at a rate of 1. For Euler's method we would assume that this rate of change is constant for the time interval from $t = 0$ to $t = 1$. This yields a value of $X = 2$ at $t = 1$. (X started at 1 and increased at a rate of 1 for 1 time unit: $1 + (1 \times 1) = 1$.)

However, the rate of change of X is actually not constant during this time interval. We can see from the plot above that the rate of change decreases as X increases from 1. Thus, Euler's method will overestimate the value of X at $t = 1$.

5. (Advanced): Euler's method is an approximation that becomes better and better as Δt approaches zero. Under what circumstances would Euler's method yield an exact solution without letting Δt approach zero?

Euler's method is approximate because we are pretending that the rate of change of X (or whatever is the variable of interest) is constant when it is not. However, if we had a differential equation for which the rate of change was constant, then Euler's method would yield exact solutions. Thus, equations of the form

$$\frac{dX}{dt} = k,$$

where k is a constant, are exactly solvable by Euler's method for any Δt. In practice, however, one wouldn't need Euler's method to solve for $X(t)$. A function $X(t)$ whose rate of change is constant is just a straight line. So solutions to the equation above are:

$$X(t) = X(0) + k\,t,$$

where $X(0)$ is the value of X at time $t = 0$.