# Solutions

## Introduction to Dynamical Systems and Chaos
## Homework for Unit VII: Phase Space

### Santa Fe Institute

`http://www.complexityexplorer.org`

**Beginner**

1. We are analyzing the Hénon map:

$$x_{n+1} = y_n + 1 - a x_n^2 \ , \quad y_{n+1} = b x_n \ , \tag{1}$$

with $a = 0.2$ and $b = 0.3$

(a) The orbits approach a fixed point.

(b) We need to solve the following system of equations:

$$x_n = y_n + 1 - 0.2 x_n^2 \ , \tag{2}$$
$$y_n = 0.3 x_n \ . \tag{3}$$

Plugging Eq. (3) in to Eq. (2), one gets:

$$x = 0.3x + 1 - 0.2x^2 \ . \tag{4}$$

Rearranging,

$$0.2x^2 + 0.7x - 1 = 0 \ . \tag{5}$$

This is a quadratic equation. Solving using the quadratic formula, one finds:

$$\frac{-0.7 \pm \sqrt{0.7^2 - 4(0.2)(-1)}}{2(0.2)} = \approx 1.089, -4.590 \ . \tag{6}$$

Plugging these $x$ values in to Eq. (3), one finds:

$$y \approx 0.327, -1.377 \ . \tag{7}$$

Thus, the two solutions are:

$$x = 1.089, y = 0.327 \ , \quad \text{and } x = -4.590, y = -.347 \ . \tag{8}$$

The first of these two solutions is the fixed point that is observed in the Hénon map program online.

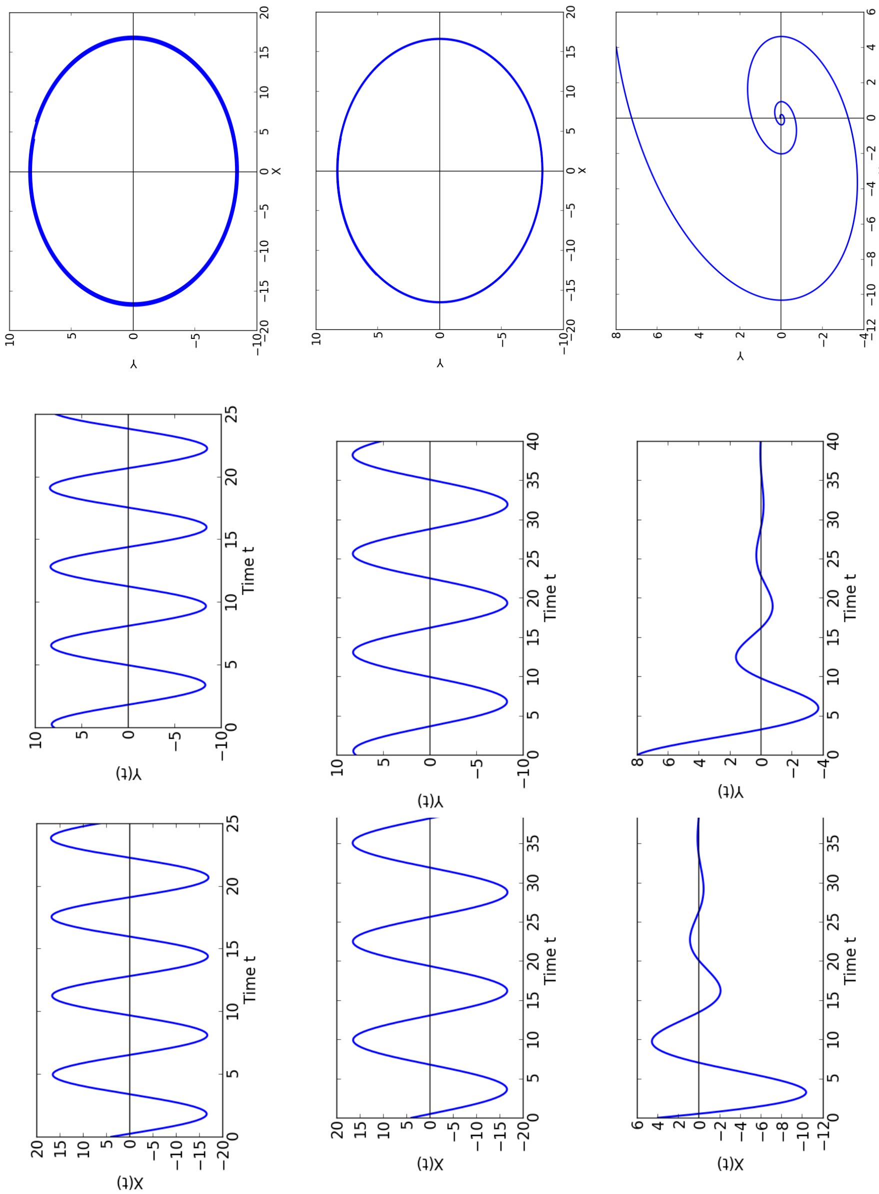


Figure 1: The solutions $x(t)$ and $y(t)$ to three different differential equations, shown with the solution curve in phase space.

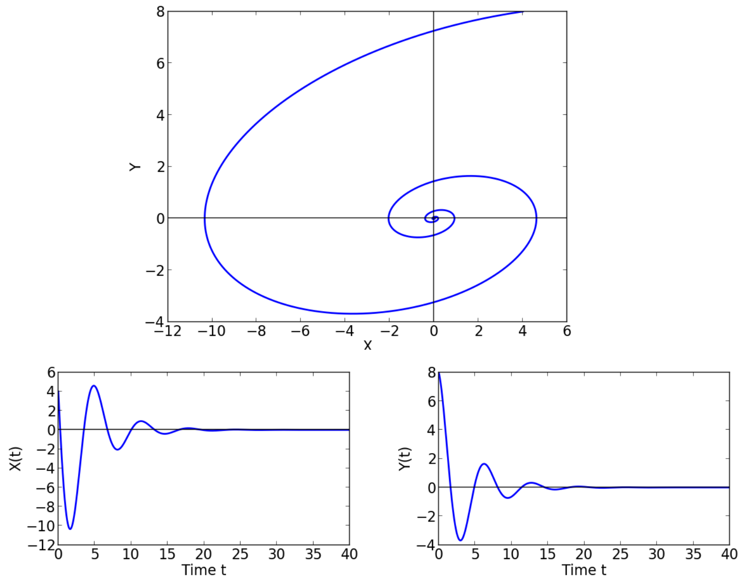


Figure 2: A solution in the phase plane.

2. The solutions are shown in Fig. 1.

3. (a) If $a = 0.8$, $b = 0.3$, orbits of the Hénon equation are pulled toward a cycle of period two.

   (b) If $a = 1.1$, $b = 0.2$, orbits of the Hénon equation are pulled toward a cycle of period four.

   (c) If $a = 1.5$, $b = 0.2$, orbits of the Hénon equation are aperiodic.

4. The solution is shown in Fig. 2.

**Intermediate**

1. We seek fixed points for the Lotka–Volterra Equations:

$$\frac{dR}{dt} = R - \frac{1}{4}RF \ , \tag{9}$$

$$\frac{dF}{dt} = 0.2RF - 0.6F \ . \tag{10}$$

Do to so, we need to find the values of $R$ and $F$ which make both of the derivatives equal to zero. We can factor these equations as follows:

$$\frac{dR}{dt} = R(1 - \frac{1}{4}F) \ , \tag{11}$$

$$\frac{dF}{dt} = F(0.2R - 0.6) \ . \tag{12}$$

This helps us see that one solution is $R = 0$, $F = 0$. This is a fixed point. There is another equilibrium point when both of the terms in parentheses are zero:

$$0 = (1 - \frac{1}{4}F) , \tag{13}$$

and

$$0 = (0.2R - 0.6) . \tag{14}$$

Solving these two equations yields $F = 4$ and $R = 3$. This is the solution that is at the center of the concentric cycles that make up the phase portrait.

**Advanced**

As usual, for the programming exercises I encourage you to post code on the course forum.

The solution for the problem analyzing the Lotka–Volterra equations can be found in a video.