Solutions

Introduction to Dynamical Systems and Chaos
Homework for Unit VI: Universality

Santa Fe Institute

http://www.complexityexplorer.org

Beginner

1. (a) I find a bifurcation from period-three to period six at $r_1 \approx 3.8414983$.
   (b) I find a bifurcation from period six to period twelve at $r_2 \approx 3.8476103$.
   (c) I find a bifurcation from period twelve to period twenty-four at $r_3 \approx 3.84903624$.
   (d) Using these r values, I find that:

$$\Delta_1 \;=\; r_2 - r_1 \;\approx\; 0.006112\;, \tag{1}$$

$$\Delta_2 \;=\; r_3 - r_2 \;\approx\; 0.001426\;, \tag{2}$$

   and

$$\delta_1 \;=\; \frac{\Delta_1}{\Delta_2} \;\approx\; 4.29\;. \tag{3}$$

2. (a) I find a bifurcation from period three to period six at $r_1 \approx 6.194766$.
   (b) I find a bifurcation from period six to period twelve at $r_2 \approx 6.205146$.
   (c) I find a bifurcation from period twelve to period twenty-four at $r_3 \approx 6.207561$.
   (d)

$$\Delta_1 \;=\; r_2 - r_1 \;= 6.205146 - 6.194766 \;=\; 0.010380\;, \tag{4}$$

$$\Delta_2 \;=\; r_3 - r_2 \;= 6.207561 - 6.205146 \;=\; 0.002415\;, \tag{5}$$

   and

$$\delta_1 \;=\; \frac{\Delta_1}{\Delta_2} \;=\; \frac{0.020380}{0.002415} \;\approx\; 4.298\;. \tag{6}$$

3. 5one to period two occurs at $r = 5$ and the bifurcation from period We are given that $r_1 = 5$ and $r_2 = 6$.

   (a) Since $\delta = \frac{\Delta_1}{\Delta_2}$, $\Delta_2 = \Delta_1/\delta$. We know that $\Delta_1 = r_2 - r_1 = 1$. Thus,

$$\Delta_2 \;=\; \frac{1}{4.669} \;\approx\; 0.214\;. \tag{7}$$

   Thus, since $\Delta_2 = r_3 - r_2$ and $r_2 = 6$, it follows that

$$r_3 \;=\; r_2 + \Delta_2 \;=\; 6 + 0.214 \;=\; 6.214\;. \tag{8}$$

(b) To determine r_4, the r value where there is a bifurcation from period eight to period sixteen, first calculate Δ_3:

$$\Delta_3 = \frac{\Delta_2}{\delta} = \frac{0.214}{4.669} \approx 0.0459 \,. \tag{9}$$

Next, since $\Delta_3 = r_4 - r_3$ and $r_3 = 6.214$,

$$r_4 = r_3 + \Delta_3 = 6.214 + 0.0459 = 6.260 \,. \tag{10}$$

Intermediate

1. Since $\delta = \Delta_1/\Delta_2$,

$$\Delta_2 = \frac{\Delta_1}{\delta} \,, \tag{11}$$

and

$$\Delta_3 = \frac{\Delta_2}{\delta} \,. \tag{12}$$

Substituting $\Delta_2 = \Delta_1/\delta$ into the above equation,

$$\Delta_3 = \left(\frac{\Delta_1}{\delta}\right)\frac{1}{\delta} = \frac{\Delta_1}{\delta^2} \,, \tag{13}$$

and, in general

$$\Delta_n = \frac{\Delta_1}{\delta^n} \,. \tag{14}$$

Since the first bifurcation occurs at 2.0 and the second at 4.0, we know that $\Delta_1 = 2$. The experimental resolution is 0.001. Let's find the n at which $\Delta_n \approx 0.001$. Plugging $\Delta_n = 0.001$ and $\Delta_1 = 2$ into Eq. (14),

$$0.001 = \frac{2}{\delta^n} \,. \tag{15}$$

Solving for n:

$$\delta^n = \frac{2}{0.001} = 2000 \,, \tag{16}$$

$$n \log(\delta) = \log(2000) \,., \tag{17}$$

$$n = \frac{\log(2000)}{\log(4.669)} \approx 4.933 \,. \tag{18}$$

So we certainly will not be able to the fifth bifurcation; the most we could see is four.

Advanced

There are no advanced problems for this unit.