Introduction to Dynamical Systems and Chaos
Homework for Unit 4: Bifurcations Part I

Santa Fe Institute.

http://www.complexityexplorer.org

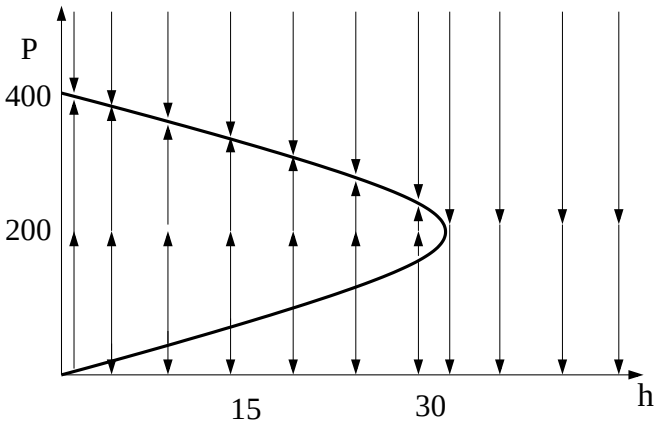

Figure 1: A bifurcation diagram.

Beginner

1. A differential equation has the bifurcation diagram shown in Fig. 1.
   (a) Sketch the phase line for $h = 12$.
   (b) Sketch the phase line for $h = 25$.
   (c) Sketch the phase line for $h = 40$.

2. Suppose a population of rabbits is described by a differential equation whose bifurcation diagram is shown in Fig. 1.
   (a) Suppose there are 400 rabbits and then hunting at a rate of $h = 20$ is allowed. What would happen to the rabbit population?
   (b) Now suppose that the hunting rate is gradually increased from 20 up to 35. What would happen to the rabbit population?

3. A bifurcation diagram for a dynamical system is shown in Fig. 2.
   (a) Suppose $h = 50$. What would happen to a population that started at $P = 25$?
   (b) Suppose $h = 10$. What would happen to a population that started at $P = 50$?
   (c) Sketch the phase line for $h = 5$.
   (d) Sketch the phase line for $h = 20$.
   (e) Sketch the phase line for $h = 40$.

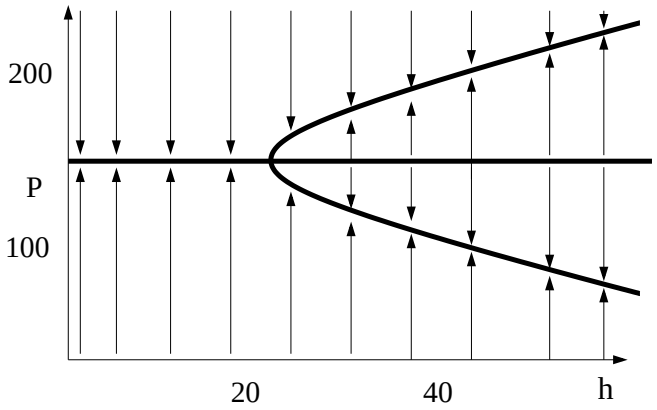

Figure 2: A bifurcation diagram.

Intermediate

These problems refer to the logistic equation with harvest,

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - h\ . \tag{1}$$

1. Use algebra to find the fixed points of the logistic equation with harvest for if $r = 3$, $K = 100$, and $h = 48$.

2. Use algebra to find the fixed points of the logistic equation with harvest as a function of r, K, and h.

3. For what value of h does the bifurcation occur? That is, at what h value does the number of fixed points change?

Advanced

1. In this problem we will analyze algebraically the differential equation I introduced in the optional lecture on hysteresis:

$$\frac{dx}{dt} = rx + x^3 - x^5\ . \tag{2}$$

   (a) Find algebraic expressions for all five fixed points as a function of r.

   (b) For what range of r values are there 3 fixed points? For what range of r values are there 5 fixed points? For what range of r values are there 1 fixed point?

2. In this problem we will construct a bifurcation diagram for the differential equation

$$\frac{dx}{dt} = rx - x^3 \,, \tag{3}$$

(a) Sketch the right-hand side of the differential equation for $r = -1$, and use this sketch to draw the phase line for $r = -1$.

(b) Repeat question (a) for $r = 0$.

(c) Repeat question (a) for $r = 1$.

(d) Use your phase lines to sketch the bifurcation diagram for the differential equation.