Introduction to Dynamical Systems and Chaos
Homework for Unit 3: Chaos and the Butterfly Effect

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http://www.complexityexplorer.org

Beginner

1. Consider the logistic equation, $f(x) = rx(1-x)$. For each of the following r-values, describe the long-term behavior of the orbits and draw a final-state diagram.
   (a) $r = 2.5$
   (b) $r = 3.52$
   (c) $r = 3.6$
   (d) $r = 3.63$
   (e) $r = 3.8$

2. Use the program that plots two different initial conditions to explore the behavior of the logistic equation $f(x) = rx(1-x)$.
   (a) Suppose $r = 3.4$. Do you observe sensitive dependence on initial conditions?
   (b) Suppose $r = 3.7$. Do you observe sensitive dependence on initial conditions?
   (c) Suppose $r = 3.8$. Do you observe sensitive dependence on initial conditions?

Intermediate

1. Use the program that plots two different initial conditions to explore the behavior of the logistic equation $f(x) = rx(1-x)$.
   (a) Suppose $r = 3.83$. Do you observe sensitive dependence on initial conditions?

2. Suppose a fair coin is tossed many many times. (For a fair coin, heads H and tails T occur with equal probability and each outcome is independent of the previous outcomes.) You are interested in the frequency with which you see particular sequences of four tosses. What fraction of the time would you expect to see the following?
   (a) $HTTH$
   (b) $TTTT$
   (c) $HHTH$
   (d) $HTHT$

3. You are observing a very very long symbol sequence generated by the logistic equation with $r = 4.0$. (The symbols are L if $x < 0.5$ and R if $x > 0.5$.) You are interested in the frequency with which you see particular sequences of four symbols. What fraction of the time would you expect to see the following?

   (a) $RLLR$

   (b) $LLLL$

   (c) $RRLR$

   (d) $RLRL$

4. Consider the logistic equation with $r = 4.0$. Let $x_0 = 0.4$, $\epsilon = 0.1$ and $\delta = 0.3$.

   (a) Find an initial condition y_0 that is within ϵ of x_0 and which has the property that eventually its orbit is a distance δ away from the orbit of x_0. (To do this, experiment with the program that produces time series plots for two different orbits.)

   (b) Repeat part (a), but let $\epsilon = 0.01$.

   (c) Repeat part (a), but let $\epsilon = 0.001$.

5. Suppose a dynamical system has a Lyapunov exponent of $\lambda = 0.8$. Two initial conditions are 0.1 apart. Approximately how far apart would you expect them to be after two iterations? How for apart would you expect them to be after five iterations?

6. Repeat the above question for for a dynamical system that has a Lyapunov exponent of $\lambda = -0.4$.

Advanced

1. **Programming Exercises**

   (a) Write your own version of the program that the times series for two different initial conditions and also plots the difference between the two time series.

   (b) Write a program that displays the symbolic sequence for the logistic equation. Have the program calculate the frequency with which symbol sequences occur. That is, have the iterate for a long time and then calculate the frequency of LLL, LLR, LRL, and so on.

2. **Fixed Points and the Logistic Equation:** In this question you will analytically derive a few properties of the logistic equation.

   (a) Use algebra to find the fixed points for the logistic equation. That is, solve $f(x) = x$ for x. Your answer will depend on r.

(b) Check that your answer for part (a) makes sense by plugging in $r = 2.5$. Then use the program to iterate an initial condition for this orbit and you should observe that the orbit approaches the fixed point you found.

(c) Repeat the above question for for $r = 3.5$. What behavior do you see when you iterate the equation? What happened to the fixed point?

3. **SDIC and the Doubling Function:** In these questions we will look at the doubling function, $f(x) = 2x$. This function cannot be chaotic, because the orbits are not bounded. However, we will see that it has SDIC.

   (a) Suppose we have two initial conditions x_0 and y_0. Show that after one iteration, the difference between these two initial conditions has doubled. That is, show that

   $$x_1 - y_1 = 2(x_0 - y_0) . \tag{1}$$

   (b) Use this fact to argue that the doubling function has SDIC. That is, you need to show that for any initial condition x_0 and any non-zero ϵ and non-zero δ, there is an initial condition y_0 such that $|x_0 - y_0| < \epsilon$ and there is an n such that $|x_n - y_n| > \delta$.

   (c) What is the Lyapunov exponent for this function?