Solutions

Introduction to Dynamical Systems and Chaos
Homework for Unit 1: Iterated Functions

Santa Fe Institute.

http://www.complexityexplorer.org

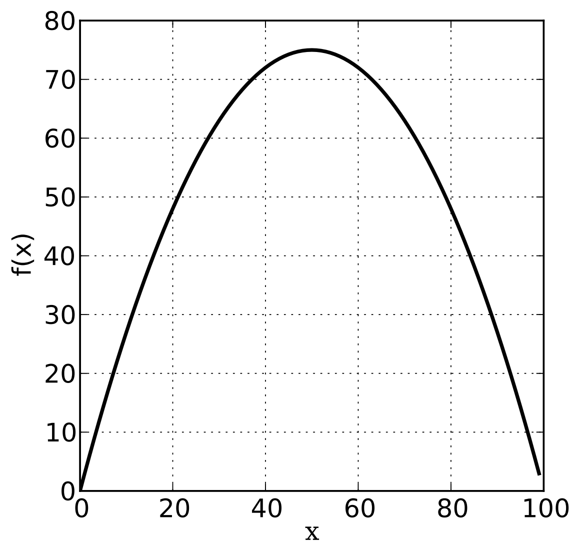

Figure 1: A graph of $f(x)$.

Beginner

1. These problems refer to the function $f(x)$ shown in Fig. 1.

   (a) What is $f(40)$? *Looking at the graph, one sees that* $f(40) \approx 73$.

   (b) What is $f(20)$? *Looking at the graph, one sees that* $f(20) \approx 50$.

   (c) Find approximate values for the first three iterates, x_1, x_2, and x_3, using the seed $x_0 = 10$. *From the graph,* $f(10) \approx 26$*, so* $x_1 = 26$*. To get* x_2*, the second iterate we apply* f *to* x_1*:* $x_2 = f(x_1) = f(26) \approx 58$*. Similarly, to get the third iterate:* $x_3 = f(x_2) = f(58) \approx 73$*. So, the orbit is approximately:* $10, 26, 58, 73$*. You may get somewhat different answers if you approximated the values from the graphs differently.*

2. Consider the function $g(x) = 4x - 12$.

(a) What are the first three iterates of the seed $x_0 = 0$? *To get the iterates, apply the function.* $x_1 = g(x_0) = g(0) = (4 \times 0) - 12 = -12$. *Next,* $x_2 = g(x_1) = g(-12) = (4 \times -12) - 12 = -60$. *Lastly,* $x_3 = g(x_2) = g(-60) = (4 \times -60) - 12 = -252$. *So the orbit is:* $0, -12, -60, -252$.

(b) What are the first three iterates of the seed $x_0 = 5$? *Applying the function to the seed, I obtain the following orbit:* $5, 8, 20, 68$.

3. A function has an attracting fixed point at 1 and a repelling fixed point at 5. Sketch this function's phase line. *The phase line is shown in Fig. 2.*

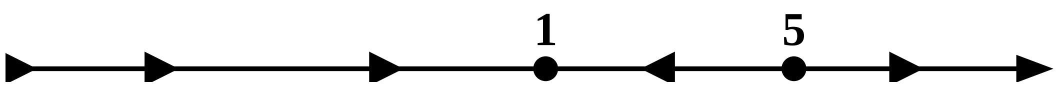

Figure 2: The solution to problem 3.

4. Consider the function $h(x) = \frac{1}{2}x + 10$. Calculate the first six iterates of h for the initial condition $x_0 = 4$, and then sketch this itinerary in a time series plot. *The orbit is:* 4, 12, 16, 18, 19, 19.5, 19.75. *The time series plot is shown in Fig. 3.*

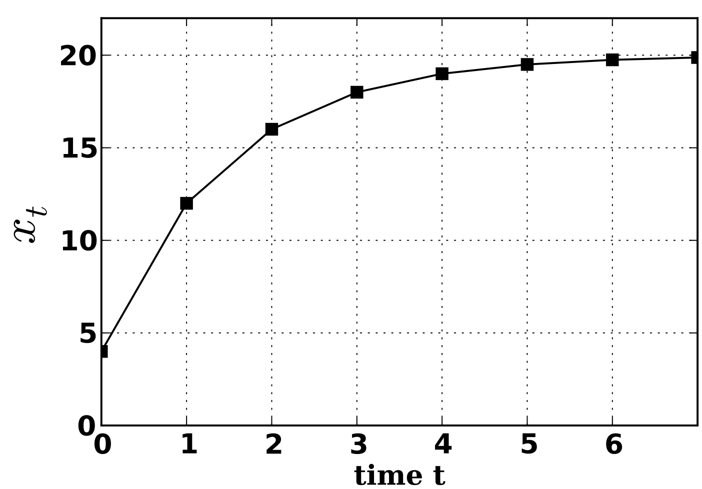

Figure 3: The solution to Exercise 4.

5. There are three fixed points for the function whose phase line is shown in Fig. 4. What is the stability of each fixed point? -2 *and* 4 *are unstable fixed points.* 1 *is a stable fixed point.*

6. For the function whose phase line is shown in Fig. 4,

(a) What is the long-term behavior of the seed $x_0 = 0$? 0 *approaches* 1.

(b) What is the long-term behavior of the seed $x_0 = 1$? 1 *is a fixed point. It remains at* 1.

(c) What is the long-term behavior of the seed $x_0 = 2$? 2 *approaches* 1.

(d) What is the long-term behavior of the seed $x_0 = 5$? 5 *approaches infinity. It keeps getting larger and larger.*

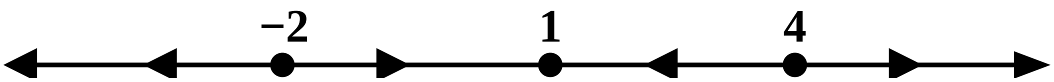

Figure 4: A phase line.

Intermediate

1. Consider the function $f(x) = x^3$.

   (a) Find all the fixed points of $f(x)$. There are three fixed points $-1, 0, and + 1$

   (b) What is the phase line for the function $f(x)$? Consider both positive and negative seeds. *The phase line is shown in Fig. 5. Note that if a negative number is cubed, it remains negative. For example,* $-2^3 = -8$.

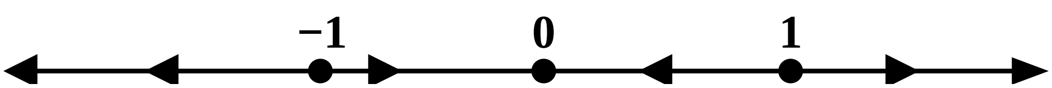

Figure 5: The solution to problem 1b.

   (c) Determine the stability of all of the fixed points of $f(x)$. 0 *is a stable fixed point and* 1 *and* -1 *are unstable fixed points.*

2. Consider the function $g(x) = x^2 - 3$.

   (a) Determine the first few iterates for the seed $x_0 = 1$. *The orbit is* 1, −2, 1, −2, ...

   (b) Make a time series plot of the orbit. How would you describe this behavior? *The time series plot is shown in Fig. 6. This behavior is periodic with period two. We will encounter behavior like this again in Unit 3.*

   (c) Is this behavior stable? Try iterating a seed close to x_0 and see what happens. *I tried using* 1.1 *as a seed. Iterating with the aid of a calculator, I obtained:* 1.1, −1.79, 0.20, −2.96, 5.75, ... *This orbit is not approaching the cycle seen in the time series plot. Thus this cyclic behavior is unstable.*

3. Find all fixed points for the function $f(x) = x+1$. *This function has no fixed points, because there is no number which remains unchanged when* 1 *is added to it. If you try to solve the*

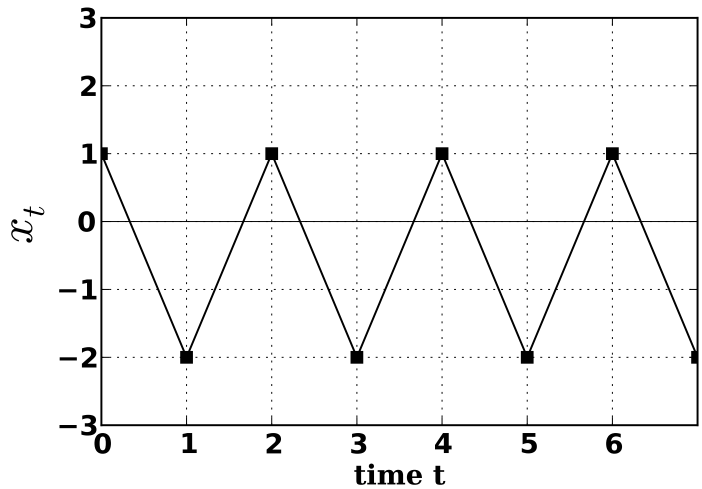

Figure 6: The solution to Exercise 2b.

fixed point equation algebraically you will end up with a contradiction:

$$f(x) = x \quad (1)$$
$$x+1 = x \quad (2)$$
$$1 = 0 \quad (3)$$
$$(4)$$

Since 1 *does not equal* 0*, we conclude that the fixed point equation does not have any solutions. Thus, the function* $f(x)$ *has no fixed points.*

Advanced

1. Determine all fixed points for the function $f(x) = \sqrt{2x}$. Which fixed point(s) are stable and which are unstable? *We need to solve the fixed point equation,* $f(x) = x$*. For this function, we have:*

$$\sqrt{2x} = x\ , \quad (5)$$

or,

$$\sqrt{2}\sqrt{x} = x\ . \quad (6)$$

So we see that one solution is $x = 0$*. Then, divide both sides of the equation by* $\sqrt{x}$ *and solve for* x*:*

$$\sqrt{2} = \frac{x}{\sqrt{x}}\ , \quad (7)$$

$$\sqrt{2} = \sqrt{x}\ . \quad (8)$$

Squaring both sides, one finds concludes that $x = 2$*. One can check to see that this really is a fixed point:* $f(2) = \sqrt{2 \times 2} = \sqrt{4} = 2$*. Thus, the two fixed points are* 0 *and* 2*. Experimenting with nearby orbits, one sees that* 0 *is unstable and* 2 *is stable.*

2. Find all fixed points for the function $f(x) = 2^x$. *We need to solve the fixed point equation $f(x) = x$. For this function, the fixed point equation is:*

$$2^x = x \,. \tag{9}$$

This equation cannot be solved algebraically; it is not possible to isolate x. One might be tempted to take the logarithm of both sides, which would yield:

$$x \log(2) = \log(x) \,. \tag{10}$$

Now the x on the left-hand side is no longer in the exponent. But the x on the right-hand side is inside a logarithm. To get the right-hand x outside the log we would have to exponentiate both sides, returning us to Eq. (9).

Instead of algebra, we can solve the fixed-point equation graphically. We are looking for an x that makes the fixed-point equation true:

$$2^x = x \,. \tag{11}$$

Moving the x to the left-hand side, this becomes:

$$2^x - x = 0 \,. \tag{12}$$

So let's make a plot of the left-hand side, $2^x - x$, and see if the graph crosses the x-axis. I used `http://www.wolframalpha.com` to make such a plot. A screenshot is shown in Fig. 7. We can see that the graph of $2^x - x$ does not cross the x-axis. Thus, we can conclude that there is no x value that makes Eq. (9) true. Hence, there is no fixed point for $f(x) = 2^x$.

3. Write a program or make a spreadsheet that will calculate orbits for the function $f(x) = 2.5x(1 - x)$. What is the long-term behavior of seeds between 0 and 1 for this function?

Answers will vary. Perhaps some students who got a program working—in a spreadsheet or in any programming language—might post a copy to the course forum.

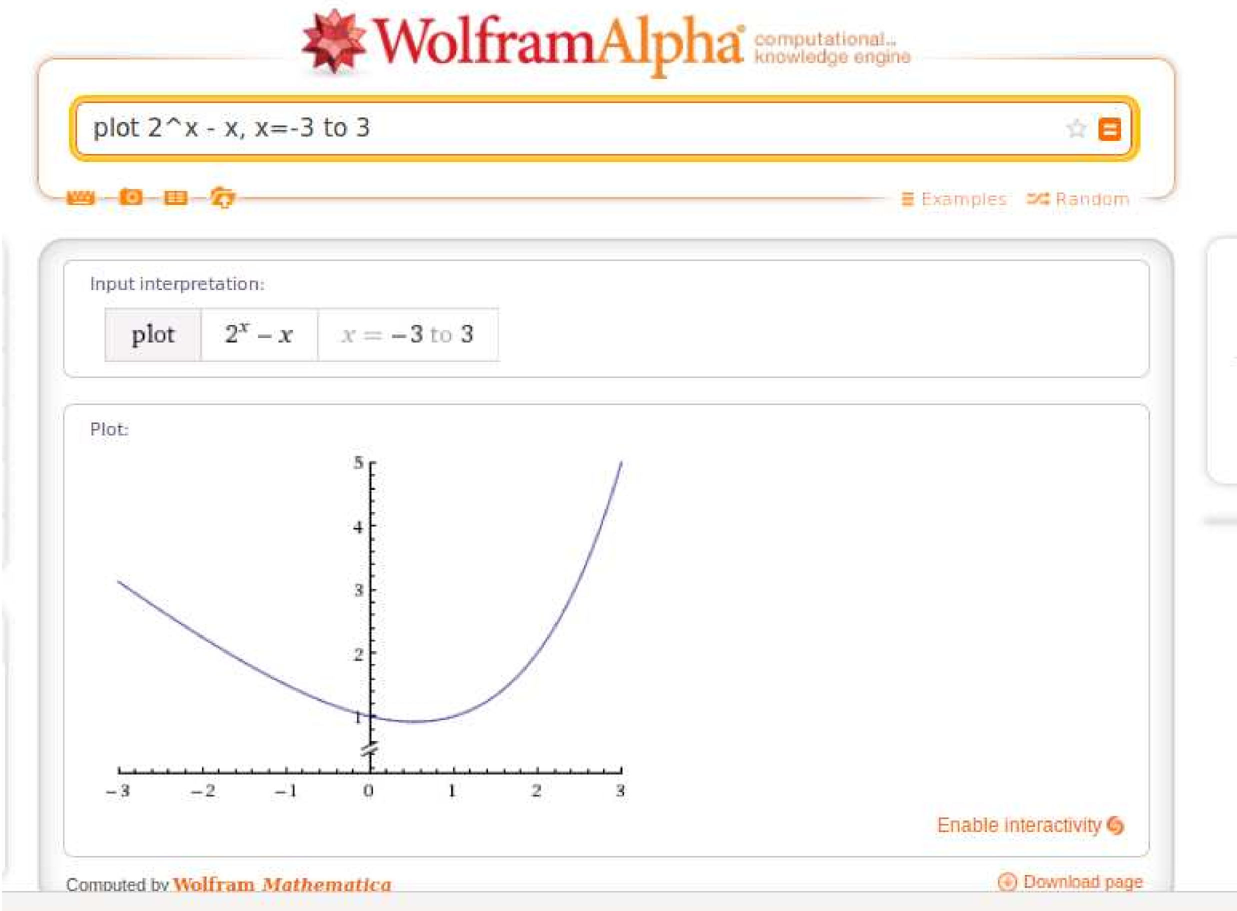

Figure 7: A screenshot of a WolframAlpha plot of $2^x - x$. Note that the plot of $2^x - x$ does not cross the x-axis. The url for the website is: `http://www.wolframalpha.com`.