1.2 Rates of Change Using Equations

The function $h(t) = -4.9t^2 + 24t + 2$ models the position of a starburst fireworks rocket fired from 2 m above the ground during a July 1st celebration. This particular rocket bursts 10 s after it is launched. The pyrotechnics engineer needs to be able to establish the rocket's speed and position at the time of detonation so that it can be choreographed to music, as well as coordinated with other fireworks in the display.

In Section 1.1, you explored strategies for determining average rate of change from a table of values or a graph. You also learned how these strategies could be used to estimate instantaneous rate of change. However, the accuracy of this estimate was limited by the precision of the data or the sketch of the tangent. In this section, you will explore how an equation can be used to calculate an increasingly accurate estimate of instantaneous rate of change.

Investigate — How can you determine instantaneous rate of change from an equation?

An outdoor hot tub holds 2700 L of water. When a valve at the bottom of the tub is opened, it takes 3 h for the water to completely drain. The volume of water in the tub is modelled by the function $V(t) = \frac{1}{12}(180 - t)^2$, where V is the volume of water in the hot tub, in litres, and t is the time, in minutes, that the valve is open. Determine the instantaneous rate of change of the volume of water at 60 min.

A: Find the Instantaneous Rate of Change at a Particular Point in a Domain

Method 1: ***Work Numerically***

1. a) What is the shape of the graph of this function?

b) Express the domain of this function in interval notation. Explain why you have selected this domain.

c) Calculate $V(60)$. What are the units of your result? Explain why calculating the volume at $t = 60$ does not tell you anything about the rate of change. What is missing?

2. Complete the following table. The first few entries are done for you.

Tangent Point P	Time Increment (min)	Second Point Q	Slope of Secant PQ
(60, 1200)	3	(63, 1140.8)	$\frac{1140.8-1200}{63-60} \doteq -19.7$
(60, 1200)	1	(61, 1180.1)	
(60, 1200)	0.1	(60.1, 1198)	
(60, 1200)	0.01		
(60, 1200)	0.001		
(60, 1200)	0.0001		

3. **a)** Why is the slope of PQ negative?

   b) How does the slope of PQ change as the time increment decreases? Explain why this makes sense.

4. **a)** Predict the slope of the tangent at P(60, 1200).

   b) **Reflect** How could you find a more accurate estimate of the slope of the tangent at P(60, 1200)?

Method 2: ***Use a Graphing Calculator***

Tools

- graphing calculator

CONNECTIONS

To see how *The Geometer's Sketchpad*® can be used to determine an instantaneous rate of change from an equation, go to *www.mcgrawhill.ca/links/calculus12* and follow the links to Section 1.2.

1. **a)** You want to find the slope of the tangent at the point where $x = 60$, so first determine the coordinates of the tangent point P(60, V(60)).

   b) Also, determine a second point on the function, Q(x, $V(x)$), that corresponds to any point in time, x.

2. Write an expression for the slope of the secant PQ.

3. For what value of x is the expression in step 2 *not* valid? Explain.

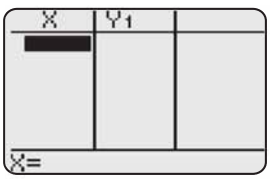
4. Simplify the expression, if possible.

5. On a graphing calculator, press Y= and enter the expression for slope from step 2 into Y1.

6. **a)** Press 2ND WINDOW to access **TABLE SETUP**.

   b) Scroll down to **Indpnt**. Select **Ask** and press ENTER.

   c) Press 2ND GRAPH to access **TABLE**.



7. a) Input values of x that are greater than but very close to 60, such as $x = 61$, $x = 60.1$, $x = 60.01$, and $x = 60.001$.

   b) Input values of x that are smaller than but very close to 60, such as $x = 59$, $x = 59.9$, $x = 59.99$, and $x = 59.999$.

   c) What do the output values for **Y1** represent? Explain.

   d) How can the accuracy of this value be improved? Justify your answer.

B: Find the Rate at *Any* Point

1. Choose a time within the domain, and complete the following table. Let $x = a$ represent the time for which you would like to calculate the instantaneous rate of change in the volume of water remaining in the hot tub. Let h represent a time increment that separates points P and Q.

Tangent Point P(a, $V(a)$)	Time Increment (min)	Second Point Q(($a+h$), $V(a+h)$)	Slope of Secant $\frac{V(a+h)-V(a)}{(a+h)-a}$
	3		
	1		
	0.1		
	0.01		
	0.001		
	0.0001		

2. a) Predict the slope of the tangent at P(a, $V(a)$).

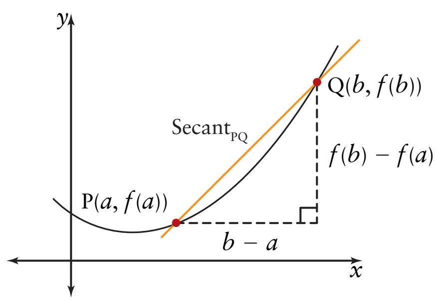
   b) Verify your prediction using a graphing calculator.

3. **Reflect** Compare the method of using an equation for estimating instantaneous rate of change to the methods used in Section 1.1. Write a brief summary to describe any similarities, differences, advantages, and disadvantages that you notice.

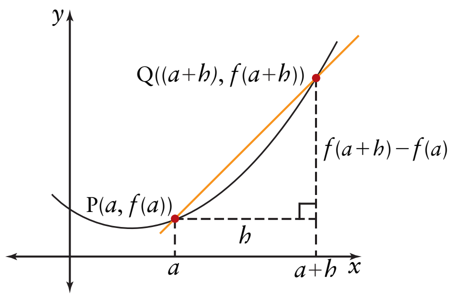
4. **Reflect** Based on your results from this Investigation, explain how the formula for slope in the table can be used to estimate the slope of a tangent to a point on a curve.

When the equation of function $y = f(x)$ is known, the average rate of change over an interval $a \le x \le b$ is determined by calculating the slope of the secant:

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$



If h represents the interval between two points on the x-axis, then the two points can be expressed in terms of a: a and $(a + h)$. The two endpoints of the secant are $(a, f(a))$ and $((a + h), f(a + h))$.



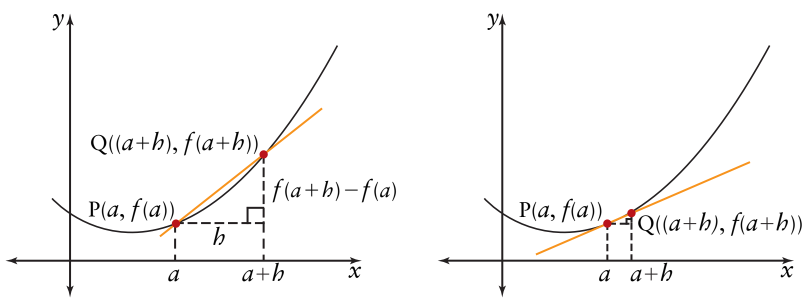
The Difference Quotient

The slope of the secant between $P(a, f(a))$ and $Q(a + h, f(a + h))$ is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \frac{f(a+h) - f(a)}{h}, \quad h \neq 0\end{aligned}$$

This expression is called the **difference quotient**.

Instantaneous rate of change refers to the rate of change at a single (or specific) instance, and is represented by the slope of the tangent at that point on the curve. As h becomes smaller, the slope of the secant becomes an increasingly closer estimate of the slope of the tangent line. The closer h is to zero, the more accurate the estimate becomes.



Example 1 Estimate the Slope of a Tangent by First Simplifying an Algebraic Expression

Ahmed is cleaning the outside of the patio windows at his aunt's apartment, which is located 90 m above the ground. Ahmed accidentally kicks a flowerpot, sending it over the edge of the balcony.

1. a) Determine an algebraic expression, in terms of a and h, that represents the average rate of change of the height above ground of the falling flowerpot. Simplify your expression.

b) Determine the average rate of change of the flowerpot's height above the ground in the interval between 1 s and 3 s after it fell from the edge of the balcony.

c) Estimate the instantaneous rate of change of the flowerpot's height at 1 s and 3 s.

2. a) Determine the equation of the tangent at $t = 1$. Sketch a graph of the curve and the tangent at $t = 1$.

b) **Use Technology** Verify your results in part a) using a graphing calculator.

Solution

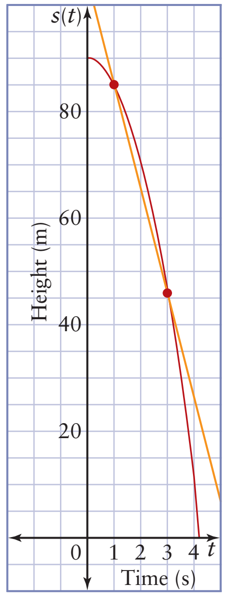
The height of a falling object can be modelled by the function $s(t) = d - 4.9t^2$, where d is the object's original height above the ground, in metres, and t is time, in seconds. The height of the flowerpot above the ground at any instant after it begins to fall is $s(t) = 90 - 4.9t^2$.

1. a) A secant represents the average rate of change over an interval. The expression for estimating the slope of the secant can be obtained by writing the difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$ for $s(t) = 90 - 4.9t^2$.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{[90 - 4.9(a+h)^2] - (90 - 4.9a^2)}{h}\\ &= \frac{90 - 4.9(a^2 + 2ah + h^2) - 90 + 4.9a^2}{h}\\ &= \frac{90 - 4.9a^2 - 9.8ah - 4.9h^2 - 90 + 4.9a^2}{h}\\ &= \frac{-9.8ah - 4.9h^2}{h}\\ &= -9.8a - 4.9h\end{aligned}$$

b) To calculate the rate of change of the flowerpot's height above the ground over the interval between 1 s and 3 s, use $a = 1$ and $h = 2$ (i.e., the 2-s interval after 1 s).

Flowerpot's Height Above the Ground



$$\frac{\Delta y}{\Delta x} = -9.8(3) - 4.9(2)$$
$$= -19.6$$

Between 1 s and 3 s, the flowerpot's average rate of change of height above the ground was −19.6 m/s. The negative result in this problem indicates that the flowerpot is moving downward.

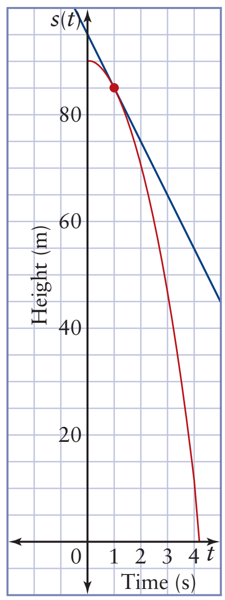
c) As the interval becomes smaller, the slope of the secant approaches the tangent at a. This value represents the instantaneous rate of change at that point.

a	h	Slope of Secant = $-9.8a - 4.9h$
1	0.01	$-9.8(1) - 4.9(0.01) = -9.75$
1	0.001	$-9.8(1) - 4.9(0.001) = -9.795$
3	0.01	$-9.8(3) - 4.9(0.01) = -29.35$
3	0.001	$-9.8(3) - 4.9(0.01) = -29.395$

From the available information, it appears that the slope of the secant is approaching −9.8 m/s at 1 s, and −29.4 m/s at 3 s.

2. a) To determine the equation of the tangent at $t = 1$, first find the tangent point by substituting into the original function.

Flowerpot's Height Above the Ground



$$s(1) = 90 - 4.9(1)^2$$
$$= 85.1$$

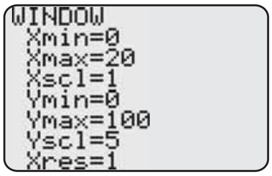
The tangent point is (1, 85.1).

From question 1, the estimated slope at this point is −9.8. Substitute the slope and the tangent point into the equation of a line formula: $y - y_1 = m(x - x_1)$.

$$s - 85.1 = -9.8(t - 1)$$
$$s = -9.8t + 94.9$$

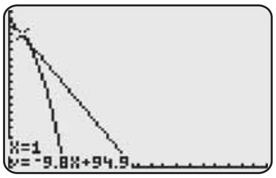
The equation of the tangent at (1, 85.1) is $s = -9.8t + 94.9$.

b) Verify the results in part a) using the Tangent operation on a graphing calculator. Change the window settings as shown before taking the steps below.



- Enter the equation **Y1** $= 90 - 4.9x^2$. Press GRAPH.
- Press 2ND PRGM.
- Choose **5:Tangent(**.
- Enter the tangent point, 1.

The graph and equation of the tangent verify the results.



Window variables:
$x \in [0, 20]$, $y \in [0, 100]$, Yscl = 5

Technology Tip

The standard window settings are $[-10, 10]$ for both the x-axis and y-axis. These window variables can be changed. To access the window settings, press WINDOW. If non-standard window settings are used for a graph in this text, the window variables will be shown beside the screen capture.

KEY CONCEPTS

- For a given function $y = f(x)$, the instantaneous rate of change at $x = a$ is estimated by calculating the slope of a secant over a very small interval, $a \leq x \leq a + h$, where h is a very small number.
- The expression $\dfrac{f(a+h) - f(a)}{h}$, $h \neq 0$ is called the difference quotient. It is used to calculate the slope of the secant between $(a, f(a))$ and $((a + h), f(a + h))$. It generates an increasingly accurate estimate of the slope of the tangent at a as the value of h comes closer to 0.
- A graphing calculator can be used to draw a tangent to a curve when the equation for the function is known.

Communicate Your Understanding

C1 Which method is better for estimating instantaneous rate of change: an equation or a table of values? Justify your response.

C2 How does changing the value of h in the difference quotient bring the slope of the secant closer to the slope of the tangent? Do you think there is a limit to how small h can be? Explain.

C3 Explain why h cannot equal zero in the difference quotient.

A Practise

1. Determine the average rate of change from $x = 1$ to $x = 4$ for each function.

   a) $y = x$ **b)** $y = x^2$ **c)** $y = x^3$ **d)** $y = 7$

2. Determine the instantaneous rate of change at $x = 2$ for each function in question 1.

3. Write a difference quotient that can be used to estimate the slope of the tangent to the function $y = f(x)$ at $x = 4$.

4. Write a difference quotient that can be used to estimate the instantaneous rate of change of $y = x^2$ at $x = -3$.

5. Write a difference quotient that can be used to obtain an algebraic expression for estimating the slope of the tangent to the function $f(x) = x^3$ at $x = 5$.

6. Write a difference quotient that can be used to estimate the slope of the tangent to $f(x) = x^3$ at $x = -1$.

7. Which statements are true for the difference quotient $\frac{4(1 + h)^3 - 4}{h}$? Justify your answer. Suggest a correction for the false statements.

   a) The equation of the function is $y = 4x^3$.

   b) The tangent point occurs at $x = 4$.

   c) The equation of the function is $y = 4x^3 - 4$.

   d) The expression is valid for $h \neq 0$.

B Connect and Apply

8. Refer to your answer to question 4. Estimate the instantaneous rate of change at $x = -3$ as follows:

   a) Substitute $h = 0.1$, 0.01, and 0.001 into the expression and evaluate.

   b) Simplify the expression, and then substitute $h = 0.1$, 0.01, and 0.001 and evaluate.

   c) Compare your answers from parts a) and b). What do you notice? Why does this make sense?

9. Refer to your answer to question 3. Suppose $f(x) = x^4$. Estimate the slope of the tangent at $x = 4$ by first simplifying the expression and then substituting $h = 0.1$, 0.01, and 0.001 and evaluating.

10. Determine the average rate of change from $x = -3$ to $x = 2$ for each function.

    a) $y = x^2 + 3x$ **b)** $y = 2x - 1$

    c) $y = 7x^2 - x^4$ **d)** $y = x - 2x^3$

11. Determine the instantaneous rate of change at $x = 2$ for each function in question 10.

12. **a)** Expand and simplify each difference quotient, and then evaluate for $a = -3$ and $h = 0.01$.

    i) $\frac{2(a + h)^2 - 2a^2}{h}$

    ii) $\frac{(a + h)^3 - a^3}{h}$

    iii) $\frac{(a + h)^4 - a^4}{h}$

    b) What does each answer represent? Explain.

13. Compare each of the following expressions to the difference quotient $\frac{f(a + h) - f(a)}{h}$, identifying

    i) the equation of $y = f(x)$

    ii) the value of a

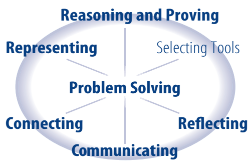
    iii) the value of h

    iv) the tangent point $(a, f(a))$

    a) $\frac{(4.01)^2 - 16}{0.01}$ **b)** $\frac{(6.0001)^3 - 216}{0.0001}$

    c) $\frac{3(-0.9)^4 - 3}{0.1}$ **d)** $\frac{-2(8.1) + 16}{0.1}$

14. Use Technology



A soccer ball is kicked into the air such that its height, in metres, after time t, in seconds, can be modelled by the function $s(t) = -4.9t^2 + 15t + 1$.

a) Write an expression that represents the average rate of change over the interval $1 \leq t \leq 1 + h$.

b) For what value of h is the expression not valid? Explain.

c) Substitute the following h-values into the expression and simplify.

i) 0.1 **ii)** 0.01

iii) 0.001 **iv)** 0.0001

d) Use your results in part c) to predict the instantaneous rate of change of the height of the soccer ball after 1 s. Explain your reasoning.

e) Interpret the instantaneous rate of change for this situation.

f) Use a graphing calculator to sketch the curve and the tangent.

15. An oil tank is being drained. The volume V, in litres, of oil remaining in the tank after time t, in minutes, is represented by the function $V(t) = 60(25 - t)^2$, $0 \leq t \leq 25$.

a) Determine the average rate of change of volume during the first 10 min, and then during the last 10 min. Compare these values, giving reasons for any similarities and differences.

b) Determine the instantaneous rate of change of volume at each of the following times.

i) $t = 5$ **ii)** $t = 10$

iii) $t = 15$ **iv)** $t = 20$

Compare these values, giving reasons for any similarities and differences.

c) Sketch a graph to represent the volume, including one secant from part a) and two tangents from part b).

16. As a snowball melts, its surface area and volume decrease. The surface area, in square centimetres, is modelled by the equation $S = 4\pi r^2$, where r is the radius, in centimetres. The volume, in cubic centimetres, is modelled by the equation $V = \frac{4}{3}\pi r^3$, where r is the radius, in centimetres.

a) Determine the average rate of change of the surface area and of the volume as the radius decreases from 25 cm to 20 cm.

b) Determine the instantaneous rate of change of the surface area and the volume when the radius is 10 cm.

c) Interpret the meaning of your answers in parts a) and b).

17.

A dead branch breaks off a tree located at the top of an 80-m-high cliff. After time t, in seconds, it has fallen a distance d, in metres, where $d(t) = 80 - 5t^2$, $0 \leq t \leq 4$.

a) Determine the average rate of change of the distance the branch falls over the interval [0, 3]. Explain what this value represents.

b) Use a simplified algebraic expression in terms of a and h to estimate the instantaneous rate of change of the distance fallen at each of the following times. Evaluate with $h = 0.001$.

i) $t = 0.5$ **ii)** $t = 1$ **iii)** $t = 1.5$

iv) $t = 2$ **v)** $t = 2.5$ **vi)** $t = 3$

c) What do the values found in part b) represent? Explain.

18. **a)** Complete the chart for $f(x) = 3x - x^2$ and a tangent at the point where $x = 4$.

Tangent Point $(a, f(a))$	Side Length Increment, h	Second Point $(a + h, f(a + h))$	Slope of Secant $\frac{f(a+h)-f(a)}{h}$
	1		
	0.1		
	0.01		
	0.001		
	0.0001		

b) What do the values in the last column indicate about the slope of the tangent? Explain.

19. The price of one share in a technology company at any time t, in years, is given by the function $P(t) = -t^2 + 16t + 3$, $0 \leq t \leq 16$.

a) Determine the average rate of change of the price of the shares between years 4 and 12.

b) Use a simplified algebraic expression, in terms of a and h, where $h = 0.1$, 0.01, and 0.001, to estimate the instantaneous rate of change of the price for each of the following years.

i) $t = 2$ **ii)** $t = 5$ **iii)** $t = 10$

iv) $t = 13$ **v)** $t = 15$

c) Graph the function.

20. **Use Technology** Two points, P(1, 1) and Q$(x, 2x - x^2)$, lie on the curve $y = 2x - x^2$.

a) Write a simplified expression for the slope of the secant PQ.

b) Calculate the slope of the secants when $x = 1.1$, 1.01, 1.001, 0.9, 0.99, 0.99, and 0.999.

c) From your calculations in part b), guess the slope of the tangent at P.

d) Use a graphing calculator to determine the equation of the tangent at P.

e) Graph the curve and the tangent.

21. **Use Technology** **a)** For the function $y = \sqrt{x}$, determine the instantaneous rate of change of y with respect to x at $x = 6$ by calculating the slopes of the secant lines when $x = 5.9$, 5.99, and 5.999, and when $x = 6.1$, 6.01, and 6.001.

b) Use a graphing calculator to graph the curve and the tangent at $x = 6$.

22. **Chapter Problem** Alicia did some research on weather phenomena. She discovered that in parts of Western Canada and the United States, chinook winds often cause sudden and dramatic increases in winter temperatures. A world record was set in Spearfish, South Dakota, on January 22, 1943, when the temperature rose from −20°C (or −4°F) at 7:30 A.M. to 7°C (45°F) at 7:32 A.M., and to 12°C (54°F) by 9:00 A.M. However, by 9:27 A.M. the temperature had returned to −20°C.

a) Draw a graph to represent this situation.

b) What does the graph tell you about the average rate of change in temperature on that day?

c) Determine the average rate of change of temperature over this entire time period.

d) Determine an equation that best fits the data.

e) Use the equation found in part d) to write an expression, in terms of a and h, that can be used to estimate the instantaneous rate of change of the temperature.

f) Use the expression in part e) to estimate the instantaneous rate of change of temperature at each time.

i) 7:32 A.M. **ii)** 8 A.M.

iii) 8:45 A.M. **iv)** 9:15 A.M.

g) Compare the values found in part c) and part f). Which value do you think best represents the impact of the chinook wind? Justify your answer.

23. As water drains out of a 2250-L hot tub, the amount of water remaining in the tub is represented by the function $V(t) = 0.1(150 - t)^2$, where V is the volume of water, in litres, remaining in the tub, and t is time, in minutes, $0 \leq t \leq 150$.

a) Determine the average rate of change of the volume of water during the first 60 min, and then during the last 30 min.

b) Use two different methods to determine the instantaneous rate of change in the volume of water after 75 min.

c) Sketch a graph of the function and the tangent at $t = 75$ min.

C Extend and Challenge

24. Use Technology For each of the following functions,

i) Determine the average rate of change of y with respect to x over the interval from $x = 9$ to $x = 16$.

ii) Estimate the instantaneous rate of change of y with respect to x at $x = 9$.

iii) Sketch a graph of the function with the secant and the tangent.

a) $y = -\sqrt{x}$ **b)** $y = 4\sqrt{x}$

c) $y = \sqrt{x} + 7$ **d)** $y = \sqrt{x - 5}$

25. Use Technology For each of the following functions,

i) Determine the average rate of change of y with respect to x over the interval from $x = 5$ to $x = 8$.

ii) Estimate the instantaneous rate of change of y with respect to x at $x = 7$.

iii) Sketch a graph of the function with the secant and the tangent.

a) $y = \frac{2}{x}$ **b)** $y = -\frac{1}{x}$

c) $y = \frac{1}{x} - 4$ **d)** $y = \frac{5}{x + 1}$

26. Use Technology For each of the following functions,

i) Determine the average rate of change of y with respect to θ over the interval from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{3}$.

ii) Estimate the instantaneous rate of change of y with respect to θ at $\theta = \frac{\pi}{4}$.

iii) Sketch a graph of the function with the secant and the tangent.

a) $y = \sin\theta$ **b)** $y = \cos\theta$ **c)** $y = \tan\theta$

27. a) Predict the average rate of change for the function $f(x) = c$, where c is any real number, for any interval $a \leq x \leq b$.

b) Support your prediction with an example.

c) Justify your prediction using a difference quotient.

d) Predict the instantaneous rate of change of $f(x) = c$ at $x = a$.

e) Justify your prediction.

28. a) Predict the average rate of change of a linear function $y = mx + b$ for any interval $a \leq x \leq b$.

b) Support your prediction with an example.

c) Justify your prediction using a difference quotient.

d) Predict the instantaneous rate of change of $y = mx + b$ at $x = a$.

e) Justify your prediction.

29. Determine the equation of the line that is perpendicular to the tangent to $y = x^5$ at $x = -2$, and which passes through the tangent point.

30. Math Contest Solve for all real values of x given that $4 - |x| = \sqrt{x^2 + 4}$.

31. Math Contest If $a - b = \sqrt{135}$ and $\log_3 a + \log_3 b = 3$, determine the value of $\log_{\sqrt{3}} (a + b)$.